Artin Presentations
I: Gauge Theory, $3+1$ TQFT's and the Braid Groups

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Abstract

We initiate the systematic study of Artin Presentations, (discovered in 1975 by González-Acuña), which characterize the fundamental groups of *closed*, orientable 3-manifolds, and form a *discrete equivalent* of the theory of open book decompositions with planar pages of such manifolds.

We list and prove the basic properties, state some fundamental problems and describe some of the advantages of the theory: e.g., an Artin Presentation of $\pi_1(M^3)$ does not just determine the closed, orientable 3-manifold M^3, but also a *canonical*, smooth, simply-connected cobordism of it, allowing us to tap into 4-dimensional gauge theory (and $3+1$ TQFT's) in a more direct, *purely discrete*, functorial manner than others. Thus, in section 4, instead of using PDE's, we show how a canonical action of the commutator subgroup $[P_n, P_n]$ of the pure braid group P_n can be used to study the smooth structures on a closed, smooth, simply-connected 4-manifold with $b_2 = n$, in a systematic way. However, the main purpose of this first paper is to use Artin Presentations to set up simple criteria, testable with, say, MAGMA on the computer (where then no knowledge of topology is required) for finding explicit counter-examples to the so-called Weak Poincaré Conjecture: "Every homotopy 3-sphere bounds a *smooth*, compact, contractible 4-manifold," as well as: "Every irreducible **Z**-homology 3-sphere Σ, with $\pi_1(\Sigma) = I(120)$ is homeomorphic to $\Sigma(2,3,5)$" and other conjectures implied by Thurston's Geometrization Conjecture. One first philosophical goal is to convince the reader that the truth of these conjectures is at least as unlikely as that of the Andrews-Curtis Conjecture and that ultimately, Artin Presentation Theory is a non-trivial intersection of string/M theory and number theory.

§0. Introduction

Since Artin Presentation Theory is a basic new theory, containing a discrete analogue of Gauge Theory in Combinatorial Group Theory (where theories are scarce), in this introduction, we strive from the beginning to give as many examples and background references as possible, preferring to err by too many than by too few.

Let Ω_n denote the compact 2-disk, D^2, with n holes and $\partial\Omega_n$ its boundary. If $h : \Omega_n \to \Omega_n$ is a homeomorphism, we denote by $\Omega_n(h)$ the mapping torus of h, i.e., the 3-manifold with boundary obtained from $\Omega_n \times I$ by identifying $(x, 0)$ with $(h(x), 1)$ for each $x \in \Omega_n$, $I = [0, 1]$.

As is well-known, Alexander's Theorem [1], usually used in the non-intrinsic form[1]: "Every link in S^3 is a braid" can also be interpreted *intrinsically* as a general Open Book Theorem (see [65, Appendix]), and González-Acuña in 1975, [23], first proved, using fundamental work of Lickorish ([43], [44], [45]), the following basic augmented version: "Every closed, orientable 3-manifold is homeomorphic to $\Omega_n(h) \cup_{\text{id}} \partial\Omega_n \times D^2 = M^3(h)$, where $h : \Omega_n \to \Omega_n$ is some homeomorphism which restricts to the identity on $\partial\Omega_n$" (see also [48], [54], [57], [68 p. 340]).

This decomposition of any closed, orientable 3-manifold is the lowest dimensional non-trivial case of a type of decomposition very similar to Lefschetz's classical one of non-singular, complex algebraic varieties whose discrete augmentations ("Hard Lefschetz Theory = codimension 1 Hodge Theory") play a fundamental role in algebraic geometry and number theory.

The Theory of Artin Presentations substitutes the dynamic 'monodromy' homeomorphism h above by a certain type of *presentation* of the fundamental group $\pi_1(M^3)$ and seeks a discrete ('intuitionistic', 'quantic') approach to M^3-theory, more *arithmetic* than combinatorial, in the sense that all topology is compartamentalized into the pure braid group, P_n, from the beginning, and, via gauge theory, the higher number theory of the exponent sum matrix of the presentation, as well as a canonical 4-dimensional 'Torelli' action of the commutator subgroup of P_n, come into play.

Another advantage over other well-known, more ad hoc, reductions of M^3-Theory (or parts of it) to pure combinatorial group theory (see [30], [25], [58], [61], [75]) is: with Artin Presentations, one can make immediate use of the fundamental Gordon-Luecke Theorem [24] to test on the computer whether knots in homotopy 3-spheres can have exotic peripheral structures or whether they satisfy Property P, etc. (see Criterion I ahead and [65 p. 620]).

An Artin Presentation of an M^3 also determines a very special Heegaard decomposition (see §1), as well as a surgery diagram of M^3, (with framings canonically included, thus avoiding self-linking problems) and hence

[1] e.g., as a starting point of the celebrated works of V.F.R. Jones [31]

all manifold invariants computed therewith can also be computed with Artin presentations usually in a simpler way, automatically relating these invariants to $\pi_1(M^3)$. This ability of Artin Presentation Theory to simplify is illustrated, for example, by contrasting González-Acuña's discrete equivalent of the Poincaré Conjecture ([23] and §2, ahead) with the Heegaard versions due to Birman, Traub, *et al.*[9]

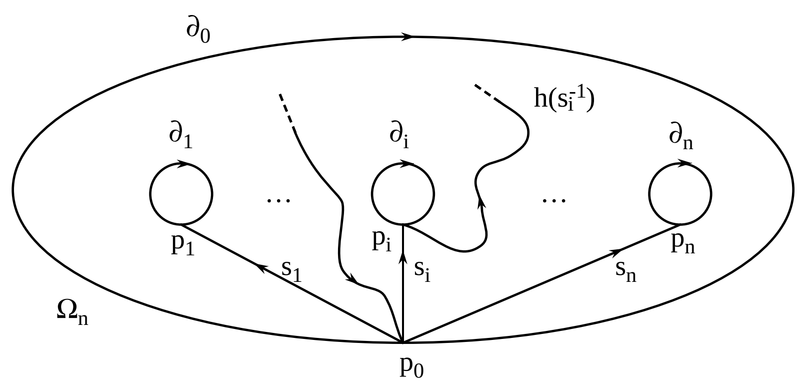
Open books with planar pages of closed, orientable 3-manifolds lead, first of all to a characterization of the fundamental groups of such manifolds by means of 'Artin Presentations' discovered and so named in 1975 by González-Acuña [23].

Let F_n denote the free group generated by $x_1, x_2, \ldots, x_n$.

Definition. *A presentation* $\langle x_1, x_2 \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle$ *is called an Artin Presentation if, in* F_n, $x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n)$.

The name is well chosen (see [3], [49, p.750] and ahead and §1). We denote the set of Artin Presentation on n generators by $\mathcal{R}_n$.

To see how they appear naturally, consider Ω_n, the 2-disk D^2 with n holes:



∂_0 and the ∂_i are the components of the boundary $\partial\Omega_n$ of Ω_n oriented clockwise; the s_i are oriented straight line segments joining p_0 to a point p_i of ∂_i; the loops $s_i \partial_i s_i^{-1}$ define generators x_i of the fundamental group $\pi_1(\Omega_n, p_0)$; by x_0 we denote the element of $\pi_1(\Omega_n, p_0)$ defined by ∂_0 and notice that $x_0 = x_1 x_2 \cdots x_n$.

Given a homeomorphism $h : \Omega_n \to \Omega_n$ such that $h|_{\partial\Omega_n} =$ identity, by $r_i = r_i(h)$ we denote the element of $\pi_1(\Omega_n, p_0)$ defined by the loop $s_i h(s_i^{-1})$, $i = 1, 2, \ldots, n$.

Then it is easy to see that the r_i define an Artin Presentation (see §1). Furthermore, given any Artin Presentation $r \in \mathcal{R}_n$, there exists an $h = h(r) : \Omega_n \to \Omega_n$, unique up to isotopy keeping $\partial\Omega_n$ fixed with $h|_{\partial\Omega_n} =$ identity such that $r_i(h) = r_i$, $i = 1, 2, \ldots, n$. This was already realized by Artin [3, p.416] and now is a particular case of a theorem of Nielsen [85, p.3] [8, chap2].

Using the HNN construction, González-Acuña showed that the Artin Presentation $r = r_i(h)$ presents the fundamental group of the open book defined by $h : \Omega_n \to \Omega_n$ (see [23], and §1, ahead).

In particular, we can say: although the fundamental group $\pi_1(M^3)$ alone does not in general determine the closed, orientable 3-manifold M^3 up to homeomorphisms, r, a presentation of a certain type of $\pi_1(M^3)$, indeed does so.

Hence, from Alexander's Theorem, considered as an Open Book Theorem, González-Acuña [23], obtained his fundamental

Theorem. *An arbitrary abstract group is isomorphic to the fundamental group of a closed, orientable 3-manifold, if and only if it has an Artin Presentation.*

This answers an important question posed in [47, p. 146] and again shows an advantage over previous Heegaard methods: In [58], Neuwirth only succeeds in finding an algorithm to detect whether certain presentations of a $\pi_1(M^3)$ come from a Heegaard decomposition.

Artin presentations also have quite a few advantages over Turaev's beautiful characterization in [77], such as Criterions I, III, and Theorem I (see ahead and also [81]).

Given an Artin Presentation $r \in \mathcal{R}_n$, we denote by $A(r)$ the exponent sum $n \times n$ matrix of r, by $\pi(r)$ the group presented by r, and by $M^3(r)$ the closed, orientable 3-manifold defined by r.

Definition. *We say r is a* Torelli *if $A(r)$ is the zero matrix.*

One has the following basic properties (see §1):

i) An arbitrary integer $n \times n$-matrix A is an $A(r)$ for some $r \in \mathcal{R}_n$ if and only if it is symmetric.

ii) $\pi(r)$ is perfect if and only if $\det A(r) = \pm 1$. $M^3(r)$ is then a $\mathbf{Z}$-homology 3-sphere and we denote it by $\Sigma^3(r)$. The knot groups, G_i, $i = 0, 1, \ldots, n$, of the knots, k_i in $\Sigma^3(r)$ and their peripheral structures m_i, ℓ_i (see [13, p.38]), defined by the boundary components ∂_i, are given by $G_0 = \langle x_1, x_2, \ldots, x_n \mid$

$r_1 = r_2 = \cdots = r_n\rangle$, $m_0 =$ any r_i, $\ell_0 = x_o m_0^{-s}$, where $B = [b_{ij}]$ denotes the inverse matrix of $A(r)$ and $s = \sum_{ij} b_{ij}$.

$$G_i = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots r_{i-1}, x_i r_i = r_i x_i, r_{i+1}, \ldots, r_n \rangle,$$

$m_i = r_i$, $\ell_i = x_i m_i^{-b_{ii}}$, $i = 1, \ldots, n$.

iii) If $r, r' \in \mathcal{R}_n$, set $y_i = r_i^{-1} x_i r_i$ and let R_i' be obtained by substituting y_j for each x_j in r_i', then $\langle x_1, x_2, \ldots, x_n \mid r_1 R_1', \ldots, r_n R_n' \rangle = r \cdot r'$ lies in $\mathcal{R}_n$ and with this composition, $\mathcal{R}_n$ becomes a group, canonically isomorphic to $P_n \times \mathbf{Z}^n$, where P_n is the the pure braid group (see [8, chap.2], [19, Th.3.4]). One has $A(r \cdot r') = A(r) + A(r')$ and so composing with a Torelli does not change the matrix A although in general the group can get killed, for example. It is easy to see that, up to the diagonal, $A(r)$ is the 'crossing matrix' of the pure braid determined by r and that r is a Torelli if and only if this pure braid lies in the commutator subgroup, $P_n' = [P_n, P_n]$, of P_n (see §1).

iv) From work of Milnor [53] (see also [34, Th. 2]), we have the absolutely fundamental 'Triality' result:

If $\det A(r) = \pm 1$ *and* $\pi(r)$ *is a finite group, then* $\pi(r)$ *is either trivial or isomorphic to the binary icosahedral group* $I(120) = \langle a, b \mid a^3 = b^5 = (ab)^2 \rangle$, *of order 120.*

In order to discuss examples, we make the following conventions. Given an Artin Presentation $r \in \mathcal{R}_n$, denote by $x_i r$ (or $x_i^{-1} r$) the Artin Presentation obtained by just changing r_i in r to $x_i r_i$ (or $x_i^{-1} r_i$). Given an integer j, $1 \leq j \leq n$, by the j-reduction of r, j-red r, we mean the Artin Presentation in $\mathcal{R}_{n-1}$ obtained by simply removing r_j in r and setting $x_j = 1$ everywhere else (and renumbering, of course). If $\overline{r} \in \mathcal{R}_m$ is another Artin Presentation where $m < n$, then given an integer k, $1 \leq k \leq n - m$, by "$\overline{r} \cdot r$ (or $r \cdot \overline{r}$) at k," we mean the Artin Presentation $\overline{\overline{r}} \cdot r$ (or $r \cdot \overline{\overline{r}}$) $\in \mathcal{R}_n$, where $\overline{\overline{r}}$ is given by

$$(\underbrace{1, \ldots, 1}_{k-1}, \overline{r}_1, \overline{r}_2, \ldots, \overline{r}_m, 1, \ldots, 1),$$

where the $\overline{r}_i$ are now written as a function of $x_k, x_{k+1}, \ldots, x_{k+m-1}$, instead of $x_1, x_2, \ldots, x_m$, respectively. We also write "$r \cdot \overline{r}$ at k" as $r \cdot \overline{r}(x_k, x_{k+1}, \ldots, x_{k+m-1})$ when convenient.

In any homology 3-sphere, we say a knot is "a trefoil" or "a 4_1", "a 10_{132}", etc., if the knot has the same group (not necessarily the same peripheral structure) of the corresponding knot in S^3 (see [13]). We say "$\Sigma^3(r)$ is a

$\Sigma(p,q,r)$" if $\pi(r)$ is isomorphic to $\pi_1\Sigma(p,q,r)$, where $\Sigma(p,q,r)$ is a Brieskorn sphere. Then (see [70]) if $\pi(r)$ is infinite, $\Sigma^3(r)$ is actually diffeomorphic to $\Sigma(p,q,r)$ up to connected sum with homotopy 3-spheres.

The following examples (see also §3), especially the ones showing the action of the Torelli, i.e., $P'_n = [P_n, P_n]$ at work, seem to show that Artin Presentation Theory is already interesting by itself in a computer-friendly way, with its own mysteries and problems.

Examples: Besides the obvious Artin Presentations $\langle x_1,\ldots,x_n \mid 1,\ldots,1\rangle$ (of F_n) and $\varepsilon_n = \langle x_1,\ldots,x_n \mid x_1,\ldots,x_n\rangle$ (of the trivial group) we have:

1) $n=2$, the only Artin Presentations are given by

$$\begin{aligned} r_1 &= x_1^{a-b}(x_1x_2)^b; \\ r_2 &= x_2^{c-b}(x_1x_2)^b. \end{aligned}$$

Here $A(r) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$; the only Torelli is $\langle x_1, x_2 \mid 1, 1\rangle$, and so $A(r)$ determines r. If say, $A(r) = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$, then $\pi(r) = I(120)$ and $M^3(r)$ is Poincaré's homology 3-sphere, $\Sigma(2,3,5)$; if $A(r) = \begin{bmatrix} -1 & -3 \\ -3 & 2 \end{bmatrix}$, $M^3(r)$ is the quotient space of $\Sigma(2,3,5)$ by a free $\mathbf{Z}_{11}$-action discovered by Seifert [71]. See also Example (5vi) of §3. The Artin presentations $\langle x_1 \mid x_1^5\rangle$ and $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ both present Z_5, but also determine the non-homeomorphic lens spaces $L(5,1)$ and $L(5,2)$, respectively.

2)[2] $n=3$;

i)

$$\begin{aligned} r_1 &= x_3^{-1}x_2x_3x_1x_3(x_2x_3x_1)^{-1}r_2; \\ r_2 &= x_3x_1(x_1x_3)^{-1}; \\ r_3 &= (x_2x_3x_1)^{-1}x_3x_1x_3^{-1}x_2x_3. \end{aligned}$$

We denote it by t_1, here $A(t_1) = 0$, i.e., t_1 is a Torelli and $\pi(t_1)$ is isomorphic to $\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$, $M^3(t_1)$ is homeomorphic to $S^1\times S^1\times S^1$.

[2] already at this stage the use of, say, MAGMA on the computer is recommended

ii) the inverse in $\mathcal{R}_3$ of (i), $t_2 = t_1^{-1}$,

$$
\begin{aligned}
r_1 &= x_1x_2(x_2x_1x_2x_3)^{-1}x_1x_2x_3(x_1x_2)^{-1}x_2;\\
r_2 &= x_1x_2x_3(x_1x_2)^{-1}x_2(x_2x_1x_2x_3)^{-1}x_1x_2r_3;\\
r_3 &= (x_1x_2x_3)^{-1}x_2x_1x_2x_3x_2^{-1}r_1.
\end{aligned}
$$

$A(t_2)$, $\pi(t_2)$ and $M^3(t_2)$ are as before.

iii) t_3:

$$
\begin{aligned}
r_1 &= x_3x_1x_2(x_3x_1)^{-1}x_1(x_1x_2)^{-1};\\
r_2 &= x_1^{-1}x_3^{-1}x_1x_3;\\
r_3 &= x_1(x_1x_3x_1x_2)^{-1}x_3x_1x_2r_2.
\end{aligned}
$$

$A(t_3)$, $\pi(t_3)$ and $M^3(t_3)$ are as before.

iv) the inverse in $\mathcal{R}_3$ of t_3, $t_4 = t_3^{-1}$:

$$
\begin{aligned}
r_1 &= x_1x_2x_3(x_1x_2x_3x_2)^{-1}x_2r_3;\\
r_2 &= (x_1x_2x_3)^{-1}x_2x_3x_2^{-1}x_1x_2x_3x_2(x_2x_3)^{-1}r_1;\\
r_3 &= (x_2x_3)^{-1}x_1x_2x_3x_2(x_1x_2x_3)^{-1}x_2x_3x_2^{-1}.
\end{aligned}
$$

$A(t_4)$, $\pi(t_4)$ and $M^3(t_4)$ are as before.

v)

$$
\begin{aligned}
r_1 &= x_1^{-1}x_3x_1x_2r_2(x_1x_2x_3)^{-1};\\
r_2 &= (x_3x_1)^{-1}x_1(x_1x_2)^{-1};\\
r_3 &= x_3x_1(x_3x_1x_2)^{-1}r_1x_1x_2x_3.
\end{aligned}
$$

$\Sigma^3(r)$ is a $\Sigma(2,3,7)$. k_0, k_1, k_2 are 4_1's whose $m\ell^{-2} = 1$ Dehn spheres are $\Sigma(2,3,7)$'s again. k_3 is a 3_1 whose $m\ell^2 = 1$ Dehn sphere has $\pi = I(120)$.

vi) By stabilizing, one can kill $\pi(r)$ (by multiplying by a Torelli), which can not be killed in the same $\mathcal{R}_n$: $\langle x_1, x_2, x_3 \mid \overline{\overline{r}}_1, \overline{\overline{r}}_2, x_3\rangle = \overline{r} \in \mathcal{R}_3$, where $\overline{\overline{r}} \in \mathcal{R}_2$ corresponds to the matrix $A = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$, has $\pi(\overline{r}) = I(120)$, but if we multiply it in $\mathcal{R}_3$ by the Torelli t_2, we obtain $\pi(\overline{r} \cdot t_2) = 1$, with MAGMA. Here the $b_{ii} = \{-3, -1, 1\}$ and $s = -7$. The knot groups G_1, $G_2 = \mathbf{Z}$, but G_0 and G_3 are those of a trefoil in S^3.

vii) If we multiply the Artin Presentation $r_1 = x_1^{-1}$, $r_2 = x_2^2x_3$, $r_3 = x_2x_3$

by the Torelli t_3 above, we obtain, setting $(x,y) = x^{-1}y^{-1}xy$:

$$r_1 = x_1^{-1}x_3x_1x_2(x_3x_1)^{-1}x_1(x_1x_2)^{-1};$$
$$r_2 = x_2(x_1,x_3)(r_1,x_1)x_2x_3;$$
$$r_3 = x_1(x_1x_2)^{-1}(x_3,x_1)r_2.$$

$\Sigma^3(r)$ is a Z-homology 3-sphere, with Casson invariant $= \pm 2$, because the Alexander polynomial of k_1 is $2t^2-3t+2$, and its Dehn sphere corresponding to $m\ell^{-1} = 1$ is simply-connected.

The knot k_3, with group $G_3 = \langle x, y \mid y^2 = (x, y^{-2})x(y^{-2}, x)\rangle$, has Alexander polynomial $= t^4 - 3t^2 + 1$, but the fundamental group of the Dehn sphere corresponding to $m\ell = 1$, is isomorphic to that of $\Sigma^3(r)$.

Somewhat more complicated counterexamples (to Property P in general) were first discovered by McCullough [51].

3) $n = 4$;

i)

$$r_1 = x_1x_2;$$
$$r_2 = x_2x_1x_2(x_2^{-1}x_1^{-1}x_2x_1x_2x_4^{-1}x_3x_4)^2;$$
$$r_3 = x_4(x_2x_1x_2)^{-1}r_2;$$
$$r_4 = x_4x_3x_4.$$

Here $A(r) = \phi_4$, the first of a sequence of unimodular, symmetric positive definite matrices considered by O'Meara [60, p.331] and $\pi(r)$ is the trivial group. Here the b_{ii} are $\{4,3,6,2\}$ and $s = 3$.

Regarding the knots, k_i, $i = 0,\ldots,4$, k_3 has the same group and peripheral structure as a trefoil of S^3, whereas the others have $G = \mathbf{Z}$. However, if we take $t_1(x_1,x_2,x_3)\cdot r$, where t_1 is the Torelli above, (MAGMA shows) we again obtain $\pi = 1$, but now the knot k_0 has the same group and peripheral structure as the knot 10_{132} of S^3, [13]. In other words, the knots of the simplest Artin Presentations can become complicated early on even if we just change them with an action of $P'_n = [P_n, P_n]$ which preserves the underlying manifold; by *ad hoc* methods, one can show that both $\Sigma^3(r)$ and $\Sigma^3(t_1\cdot r)$ are homeomorphic to S^3.

ii) Let the Torelli $T_4 \in \mathcal{R}_4$ be given by:

$$\begin{aligned} r_1 &= x_4^{-1}x_2x_3x_4x_1x_4(x_2x_3x_4x_1)^{-1}r_2;\\ r_2 &= x_4x_1x_4^{-1}x_1^{-1};\\ r_3 &= r_2;\\ r_4 &= (x_2x_3x_4x_1)^{-1}x_4x_1x_4^{-1}x_2x_3x_4. \end{aligned}$$

Then $\Sigma^3(\varepsilon_4 T_4)$ is a Brieskorn sphere $\Sigma(2,3,11)$, where G_2 is that of a trefoil in S^3 and the fundamental group of the Dehn sphere corresponding to $m\ell = 1$ is $I(120)$.

iii)

$$\begin{aligned} r_1 &= x_1^2(x_2x_3)^2x_2^{-7}r_2;\\ r_2 &= x_2^6(x_1(x_2x_3)^2)^{-1};\\ r_3 &= x_3^4(x_4^3x_2x_3x_2)^{-1}r_4;\\ r_4 &= x_4^2x_2x_3x_2(x_2x_3)^{-2}. \end{aligned}$$

Here $\Sigma^3(r)$ is a $\Sigma(2,5,7)$, G_4 is that of the torus knot $t(2,5)$ in S^3 and the Dehn sphere corresponding to $m\ell = 1$ has $\pi = I(120)$. The Alexander polynomial of the knot k_0 is irreducible and has degree $= 62$, probably caused by the fact that $A(r)$, which is that of i) with the signs changed off the diagonal, has $s = 51$. Setting $\overline{r} = x_4^{-1}r$, we have $\pi(\overline{r}) = I(120)$ where now G_3 is that of the torus knot $t(2,5)$ in S^3 and the Dehn sphere corresponding to $m\ell^{-1}$ is a $\Sigma(2,5,7)$, again.

iv) Let the Torelli $T_4' \in \mathcal{R}_4$ be given by:

$$\begin{aligned} r_1 &= x_1x_2x_3x_4(x_1x_2x_3x_4x_2x_3)^{-1}x_2x_3r_4;\\ r_2 &= (x_1x_2x_3x_4)^{-1}x_2x_3x_4(x_2x_3)^{-1}x_1x_2x_3x_4x_2x_3(x_2x_3x_4)^{-1}r_1;\\ r_3 &= r_2;\\ r_4 &= (x_2x_3x_4)^{-1}x_1x_2x_3x_4x_2x_3(x_1x_2x_3x_4)^{-1}x_2x_3x_4(x_2x_3)^{-1}. \end{aligned}$$

Then, if r is as in 3i), we again have $\pi(T_4' \cdot r) = 1$, but the non-amphicheiral $k_3 = 3_1$ of $\Sigma^3(r)$ becomes a 4_1 in $\Sigma^3(T_4' \cdot r)$; all the other knots stay trivial.

Nevertheless, at this stage, $\Sigma^3(T_4' \cdot r)$ is probably still homeomorphic to S^3.

v) Let r be given by

$$\begin{aligned} r_1 &= x_1 x_4^{-1} x_2 x_3 x_4 x_1 r_4 (x_1 x_2 x_3 x_4)^{-1}; \\ r_2 &= x_2^3 x_3^{-5} r_3; \\ r_3 &= x_3^4 (x_2 x_3 x_2)^{-1} x_4 x_1 x_4^{-1} x_1^{-1}; \\ r_4 &= x_4 (x_2 x_3 x_4 x_1)^{-1} x_4 x_1 x_4^{-1} x_2 x_3 x_4. \end{aligned}$$

Then $\det A(r) = \pm 1$ and the Alexander polynomials of the knots k_1 and k_3 are $-8t^2 + 17t - 8$ and $(t^2 - t + 1)(t^6 - t^3 + 1)$, respectively; however, both the Dehn sphere of k_1, corresponding to $m\ell = 1$, and the Dehn sphere of k_3 corresponding to $m\ell^{-1} = 1$, have $\pi = I(120)$ and are in fact $\Sigma(2,3,5)$'s (in particular implying that the Casson invariant $\lambda(r) = \pm 9$).

It is unkown whether this phenomenon can happen with S^3. The trefoil is the only known knot which has a Dehn sphere with $\pi = I(120)$.

Unlike in previous examples, here $\pi(r)$ has (up to conjugation) five distinct subgroups of index 5, with, moreover, distinct cores (of index 60).

vi) Let r be given by

$$\begin{aligned} r_1 &= x_1 x_4^{-1} x_2 x_3 x_4 x_1 x_4 (x_2^2 x_4 x_1)^{-1} r_2; \\ r_2 &= x_2^2 x_3^{-4} r_3; \\ r_3 &= x_3^3 x_2^{-1} x_4 x_1 x_4^{-1} x_1^{-1}; \\ r_4 &= x_4 (x_2 x_3 x_4 x_1)^{-1} x_4 x_1 x_4^{-1} x_2 x_3 x_4. \end{aligned}$$

Here $\pi(r)$ is infinite and abelianizes to Z_2, but the obvious 2-fold covering is a irreducible Z-homology 3-sphere, Σ^3, which has a regular 120-fold covering (with covering group $I(120)$), *which again is a Z-homology 3-sphere*, thus giving a more explicit example of this phenomenon than those first found by Luft and Sjerves in [46], (see also [11]). Furthermore, Σ^3 admits a fixed point free involution, unlike the universal case: $\Sigma(2,3,5)$ covered by S^3.

Setting $(x,y) = x^{-1}y^{-1}xy$, a (non-Artin) presentation of $\pi_1(\Sigma^3)$ is given by

$$\left\langle a,b,c,d \;\middle|\; \begin{aligned} a &= (b^{-1}, (c,d)(b,a)(c,d)); \\ b &= (a^{-1}, (d,c)(a,b)(d,c)); \\ c &= (d^{-1}, (a,b)(d,c)(a,b)); \\ d &= (c^{-1}, (b,a)(c,d)(b,a)) \end{aligned} \right\rangle$$

vii) If $r = \overline{r} \cdot t_1(x_2, x_3, x_4)$, where $\overline{r}$ denotes example 3iii) and t_1 is the Torelli from example 2i), then $\pi(x_1^{-1}x_4^{-1}r) = 1$ and the knot k_0 has Alexander polynomial $\Delta_1(t) = -t^4 + 3t^2 - 1$ and zero 2-torsion; the fundamental group of the corresponding 2-fold branched cover is presented by

$$\langle x, y \mid x = (x, y^{-1})^2(x, y)^2, y = (y, x^{-1})^2(y, x)^2 \rangle,$$

which we are unable to distinguish from $\pi_1\Sigma(2, 3, 13)$.

4) $n = 5$,

i) Let $\overline{r} \in \mathcal{R}_5$ be obtained by first stabilizing the r of example 3iii) with $r_5 = x_5$ and then multiplying at 2 by the Torelli T_4' of example 3iv); then $\pi(x_1^{-1}x_4^{-2}x_5^{-2}\overline{r}) = 1$ and the knot k_1 has the same Alexander polynomial and the same i-torsion (for at least $i \leq 8$) as the knot 9_{45} of S^3; however, their knot groups differ since the abelianizations of their other index 5 subgroups are different. Using the MAGMA notation $(x, y) = x^{-1}y^{-1}xy$, $x \wedge y = y^{-1}xy$, G_1 is presented by

$$\langle x, y \mid y = (x^{-1}(y, x), x \wedge (y^{-1}x^{-1}y^4)) \rangle.$$

5) $n = 8$,

i) Set $X = x_7(x_6x_7x_8)^{-1}x_5x_6x_7x_8x_7^{-1}$,

$$\begin{aligned} r_1 &= x_1^2x_2; \\ r_2 &= x_2x_1x_2x_4^{-1}x_3x_4; \\ r_3 &= x_3x_4(x_1x_2)^{-1}r_2; \\ r_4 &= x_4x_3x_4X; \\ r_5 &= x_5x_6x_7x_8(x_3x_4x_7)^{-1}r_4; \\ r_6 &= x_6^2Xx_7; \\ r_7 &= x_7(x_6X)^{-1}r_6; \\ r_8 &= x_8^2x_7^{-1}Xx_7. \end{aligned}$$

Here $A(r) = E_8$, the well-known unimodular, even, positive definite 8×8 matrix used by Milnor to construct his exotic 7-sphere[3], and $\pi(r) = I(120)$.

ii) In (i), just change r_1 to x_1x_2. $A(r)$ is obtained from E_8 by just changing a_{11} from 2 to 1. It now has $\det A = -1$, (7 positive eigenvalues and one

[3]see, e.g., Lectures on Modern Mathematics, II, T.L. Saaty, ed. p.174.

negative one), and $\pi(r) = 1$. Here the b_{ii} are $\{-2,-3,-4,-5,-6,-2,0,-1\}$ and $s = 3$. The knot groups G_i, with their peripheral structures, are those of a trefoil in S^3 for $i = 1,\ldots,5$, all other $G_i = \mathbf{Z}$.

iii)

$$\begin{aligned} r_1 &= x_1^2 x_2; \\ r_2 &= x_2 x_1 x_2 (x_2^{-1} x_1^{-1} x_2 x_1 x_2 x_4^{-1} x_3 x_4)^2; \\ r_3 &= x_4 (x_2 x_1 x_2)^{-1} r_2; \\ r_4 &= x_4 x_3 x_4 x_6^{-1} x_5 x_6; \\ r_5 &= x_5 x_6 (x_3 x_4)^{-1} r_4; \\ r_6 &= x_6 x_5 x_6 x_8^{-1} x_7 x_8; \\ r_7 &= x_7 x_8 (x_5 x_6)^{-1} r_6; \\ r_8 &= x_8 x_7 x_8. \end{aligned}$$

Here $A(r)$ is the matrix ϕ_8 of [60, p.331] (see §1), which is congruent over $\mathbf{Z}$ to E_8, and $\pi(r) = I(120)$. We denote this Artin Presentation also by ϕ_8. Here G_8 is that of a trefoil in S^3. However, G_4 is presented by $\langle x, y, z \mid x^2 = y^3 = z^5 \rangle$, which is impossible for any knot in S^3, according to the Burde-Zieschang Theorem [13, p.77]. If we stabilize ϕ_8 with $r_9 = x_9$ and multiply by $t_2(x_7, x_8, x_9)$ (of 2ii), we obtain an irreducible Z-homology 3-sphere (with Casson invariant $\lambda = \pm 5$ or ± 7) which has a regular 120-fold cover, which again is a Z-homology 3-sphere.

iv) $x_8^{-1}\phi_8$. Now, $\pi(r) = 1$, $\det A = -1$ with one negative eigenvalue, and $b_{ii} = \{0,-2,-7,-6,-5,-4,-3,-2\}$ and $s = 3$, $G_0 = G_1 = G_2 = \mathbf{Z}$ and the other G_i and their peripheral structures are those of a trefoil in S^3.

v) If we multiply ϕ_8 at $k = 6$ by the Torelli t_2 of Example (2ii), we again obtain $\pi(r) = I(120)$ and $\pi(x_8^{-1} r) = 1$. The knot groups G_i, $i = 1,\ldots,8$ do not change, but the trivial knot k_0 of $x_8^{-1}\phi_8$ now becomes a knot with Alexander Polynomial $\Delta_1 = 2t^4 - 5t^2 + 2$ and zero 2-torsion. We are unable to distinguish the fundamental group of the corresponding 2-fold branched cover of k_0 from $\pi_1\Sigma(2,3,25)$.

vi) The 1-reduction of $x_8^{-1} x_2 \phi_8$ has $\pi(r) = 1$, A is positive definite, and $b_{ii} = \{1,5,6,7,8,9,10\}$ and $s = 4$. $G_0 = \mathbf{Z}$, but if we multiply by t_2 as in v), $\pi = 1$ again, but G_0 becomes isomorphic to

$$\langle x, y \mid y^5 = x^{-2} y^2 x^2 y^{-2} x^{-2} y^2 x^3 y^2 x^{-2} y^{-2} x^2 y^2 x^{-2} \rangle$$

with Alexander Polynomial $2t^4 - 3t^2 + 2$ and zero 2-torsion; the 2-fold branched cover is a $\Sigma(2,3,23)$.

vii) We again encounter the "non-rigidity" phenomenon of example 3i) but now for $i = 1$, in the following: $x_1^{-2}x_8^{-1}\phi_8$ has $\pi = 1$, $\det A = -1$ with one negative eigenvalue, and all $G_i = \mathbf{Z}$, $b_{ii} = \{0,0,1,2,3,4,5,6\}$; $s = 5$, but if we multiply by $t_3(x_2, x_3, x_4)$, obtaining r, then again $\pi(r) = 1$, but

$$G_1 = \langle x, y \mid x^5 = yx^2y^{-2}x^2y \rangle$$

(after setting $x = x_3x_4$, then $y = x_2x^2$) with $\Delta_1 = 2t^2 - 5t + 2$, which coincide with those of the stevedore knot 6_1 of S^3. G_0 also changes; it has $\Delta_1 = t^4 - t^3 - t^2 - t + 1$, but all other $G_i = \mathbf{Z}$, $i \geq 2$.

viii) $\pi(x_1^{-2}x_8^{-1}\phi_8) = 1$ and $G_1 = Z$, but if we multiply by $t_4(x_2, x_3, x_4)$ of example 2iv), then still $\pi = 1$, but the knot k_1 now has Alexander polynomial $\Delta_1 = t^6 - t^5 - t^4 + 3t^3 - t^2 - t + 1$ and zero 2-torsion; but differs from 10_{153}, the only such candidate in the tables of [13], [68]; the fundamental group of the corresponding 2-fold branched cover is presented by

$$\langle x, y \mid x^3 = (x^{-1}, y^{-1})^2 x^2 (y, x)^2, y^3 = (y^{-1}, x^{-1})^2 y^2 (x, y)^2 \rangle.$$

We remark that this presentation as well as those of examples 3vi), 3vii), and 4i) are obtained automatically with MAGMA on the computer, without any conscious prompting.

6) n=12;

i)

$$
\begin{aligned}
r_1 &= x_1^3(x_1x_2)^{-1};\\
r_2 &= x_2^3x_1^{-1}x_3x_4(x_3x_4x_3)^{-1}x_1(x_1x_2)^{-1};\\
r_3 &= x_3^3(x_3x_4)^{-1}x_1(x_1x_2)^{-1};\\
r_4 &= x_4^3x_3^{-1}x_5x_6(x_5x_6x_5)^{-1}x_3(x_3x_4)^{-1};\\
r_5 &= x_5^3(x_5x_6)^{-1}x_3(x_3x_4)^{-1};\\
r_6 &= x_6^3x_5^{-1}x_7x_8(x_7x_8x_7)^{-1}x_5(x_5x_6)^{-1};\\
r_7 &= x_7^3(x_7x_8)^{-1}x_5(x_5x_6)^{-1};\\
r_8 &= x_8^3(x_9x_7)^{-1}x_7(x_7x_8)^{-1};\\
r_9 &= x_9^3x_7(x_7x_8)^{-1}x_{11}x_{12}(x_{10}x_{11}x_{12}x_{11})^{-1}x_7x_8^{-2}r_8;\\
r_{10} &= x_{10}^2x_7x_8^{-2}r_8;\\
r_{11} &= x_{11}^3(x_{11}x_{12})^{-1}x_{10}^{-2}r_{10};\\
r_{12} &= x_{12}^3x_{11}^{-1}.
\end{aligned}
$$

Here $A(r)$ is the matrix of the extreme duodenary form, $2D_{12}^2$ of Coxeter-Todd ([15],[16]);

$$\pi(r) = \langle x_1, x_{12} \mid x_1^9 = x_{12}^5 = (x_1x_{12}^2)^2\rangle$$

is infinite.

ii) $x_1^{-1}r$. Now $\pi = 1$, $\det A = -1$ with one negative eigenvalue, and the b_{ii} are

$$\{-2, -3, -4, -5, -6, -7, -8, -9, -10, -2, -3, 0\},$$

and $s = -855$. For $1 \leq i \leq 9$, the G_i are those of the torus knot $t(2,5)$ of S^3, and $G_{10} = G_{11} = G_{12} = \mathbf{Z}$. G_0 is complicated: its Alexander Polynomial is irreducible and has degree 810.

iii) $x_1^{-1}x_{10}r$. Again, $\pi = 1$, A is positive definite, and the b_{ii} are

$$\{23, 22, 21, 20, 19, 18, 17, 16, 15, 2, 6, 1\},$$

and $s = 1746$. For $1 \leq i \leq 9$, the G_i are those of the torus knots $t(3,5)$ of S^3, $G_{10} = G_{12} = \mathbf{Z}$, G_{11} is that of the trefoil of S^3, and G_0 is complicated: its Alexander Polynomial is irreducible and has degree 1640.

The following computational advantages of Artin Presentations are characteristic and worth noting: If $\det A(r) = \pm 1$, i.e., the manifold determined by r is a **Z**-homology 3-sphere, then the above presentations and peripheral structures are obtained *without projecting the knots onto the 2-plane in* S^3, a somewhat arbitrary 'newtonian' procedure which in the case of knots in S^3, Reidemeister [66, p.3] observes,[4] is a curious and (we think a somewhat unnatural) intrusion of the continuum into the purely discrete combinatorial group theory involved, and leads to special invariants (e.g., the signature) for *a priori* knots in S^3, only, which do not just depend on the group of the knot.

The examples above show clearly that AP Theory's non-skein Knot Theory (for homotopy 3-spheres) diverges very fast from the usual Knot Theory of S^3*, which is ordered by means of its skein projections into the plane.*

Due to the simplicity of the above knot group presentations and their peripheral structures, Artin Presentations give a direct way to test with MAGMA on the computer how special knots in S^3 really are, as opposed to knots in arbitrary homotopy 3-spheres Σ^3, by using any theorem in M^3-Theory whose proof is valid and only known for S^3, not *a priori* for Σ^3. Here it is very important to point out that Alexander's Theorem implies that S^3 is characterized, among all homotopy 3-spheres, by its knot groups with finitely generated commutator subgroups, see Simon [73].

For example, since a prime knot in S^3 is already determined up to mirror image and orientation (see Gordon-Luecke [24], Whitten [79]) by its group, the b_{ii} in the above examples had to essentially be what they are, otherwise the respective homotopy 3-spheres would be counter-examples to the Poincaré Conjecture, since they would have contained knots with *exotic* peripheral structures.

To further illustrate our point here, consider the following simple example: MAGMA shows that the knot group, G_0, of $\Sigma(t_1 \cdot \epsilon_3)$ is isomorphic to that of the knot 5_1 in S^3; however the fundamental groups of the Dehn spheres [13, p.39] corresponding to $m\ell^{\pm 1} = 1$ of k_0 of $\Sigma(t_1 \cdot \varepsilon_3)$ have no subgroups of index 5, whereas for the knot 5_1 they do. Since 5_1 is a prime knot of S^3, if $\pi(t_1 \cdot \varepsilon_3)$ were trivial, $\Sigma(t_1 \cdot \epsilon_3)$ would be a counter-example to the Poincaré Conjecture, by a theorem of Waldhausen [13, p.38]. Alas, $\pi(t_1 \cdot \epsilon_3) = I(120)$, but how did the exotic peripheral structure arrange and regulate this?

More directly, with the theorem of Gordon and Luecke [24], we obtain our first explicit criterion for testing the Poincaré Conjecture.

[4]see also Kauffman's remark in [33, p.15], Penrose [63, p.53] and Witten [83, p.352].

Criterion I. *The Poincaré Conjecture implies that if* $\pi(r) = 1$ *and* $\pi(j\text{-red } r) = 1$*, then the group* G_j*, presented by*

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_{j-1}, x_j r_j = r_j x_j, r_{j+1}, \ldots, r_n \rangle,$$

is isomorphic to $\mathbf{Z}$.

Remark: This is not true for just any presentation of deficiency 0. Consider $H = \langle x, y \mid w_1, w_2 \rangle$, where

$$w_1 = x^3 y x^{-2} y^{-1};$$
$$w_2 = y^3 x y^{-2} x^{-1}.$$

Then $H = 1$ and the 1-reduction also presents the trivial group, but the group $H_1 = \langle x, y \mid x w_1 = w_1 x, w_2 \rangle$ is non-abelian, since, after adding the relation $x^2 = 1$, the commutator (x, y^2) has order 5.

Regarding the λ-Conjecture, (see ahead) from a theorem of Birman and Craggs [10] (see also [55, p.614]), we obtain:

Criterion II. *Let* r, $t \in \mathcal{R}_n$*, where* t *is a Torelli and both* $\pi(r)$ *and* $\pi(t \cdot r)$ *are trivial. Then the* λ*-Conjecture implies that* $\pi(t^m \cdot r) \neq I(120)$ *for all* $m \geq 2$.

Remark: In fact, here it is only necessary that the Rohlin Invariants of $\Sigma^3(r)$, $\Sigma^3(t \cdot r)$ vanish.

Example: If $r \in \mathcal{R}_3$ is given by

$$r_1 = x_1 x_2 (x_1 x_2 x_3)^{-1};$$
$$r_2 = x_2^{-1};$$
$$r_3 = (x_1 x_2)^{-1} x_2.$$

Then MAGMA shows $\overline{r} = t_1 \cdot r$ is given by

$$\overline{r}_1 = x_1 x_3^{-1} x_1^{-1};$$
$$\overline{r}_2 = x_2^{-1} x_3 x_1 x_3^{-1} x_1^{-1};$$
$$\overline{r}_3 = x_1^{-1}.$$

$\pi(r) = 1 = \pi(\overline{r})$. However, alas, $\pi(t_1^2 \cdot r)$ is infinite since, unlike $I(120)$, it has a subgroup of index 6 which abelianizes to $\mathbf{Z}_{28}$. (See also example 1 of §3).

For more criteria, see §2, ahead.

We stress that no knowledge of topology or geometry is needed for testing these criteria on the computer. We observe also that since Artin Presentations and their knots can be ordered lexicograhically (in an even more canonical way than knots in S^3), all these tests can be carried out in an orderly manner.

Let the Artin Presentation r still be such that $\det A(r) = \pm 1$, i.e., r represents a **Z**-homology 3-sphere $\Sigma(r)$.

How much influence does the number theory, arithmetic of these unimodular, symmetric integer matrices have on the topology and knot theory of $\Sigma(r)$*?*

The answer is "quite a lot," and this is (besides the immediate centrality[5] of the fundamental group) the main distinction of Artin Presentation Theory with respect to other theories.

Indeed, the second main advantage of Artin Presentation Theory, which detaches it and pushes it away from all other 3-manifold theories, comes from our discovery in [80] of the following elementary but fundamental fact: An Artin Presentation r does not just determine the closed, orientable 3-manifold $M^3(r)$, but also, *in a canonical, inductive way*, a well-defined, smooth, compact, simply-connected 4-manifold $W^4(r)$, whose boundary $\partial W^4(r) = M^3(r)$ and *whose intersection form is given by the matrix* $A(r)$: namely the cobordism constructed in Proposition 1.6 below.[6] *Thus, each Artin Presentation* r *representing a given* M^3 *literally materializes as a canonical cobordism of* M^3*.*

The purely discrete coupling $r \to A(r)$ is thus strengthened by the smooth 4-dimensional gauge theory (and $(3+1)$-TQFT's) and conversely, *Artin Presentation Theory becomes a first approximation to 4-dimensional gauge theory in the same philosophical sense as, according to Artin himself [3, p.491], braid theory is a first approximation to knot theory.*

Notice also that, given $M^3(r)$, $W^4(r)$, with its intersection form canonically related to the presentation of $\pi_1(\partial W^4(r) = M^3(r))$, a group which "almost determines" $M^3(r)$, evidently leads to a more direct (and further-

[5] undiluted by unfaithful representations

[6] $W^4(r)$ is the open book-like version of a surgery diagram cobordism corresponding to the closure of the pure braid defined by r, with surgery coefficients given by the diagonal of the matrix $A(r)$, thus avoiding the use of framings and serious self-linking problems (see e.g. Witten [83, p. 363]).

more discrete) functorial 'tapping into' gauge theory than e.g., Floer's, via the differential geometry of $M^3(r) \times \mathbf{R}$ (see also [76, p.367]).

For example, let D denote the set of unimodular, symmetric, integer matrices prevented by Donaldson's Theorem from representing the quadratic form of a smooth, compact simply-connected 4-manifold. From a theorem of Taubes [76, p.366], (see also [18]), showing among other things that Donaldson's Theorem holds even if we allow homotopy 3-spheres as boundaries, we obtain the non-trivial, discrete, *purely group-theoretic*:

Theorem I. *If $A(r) \in D$, then $\pi(r)$ cannot be the trivial group; in fact, $\pi(r)$ has a non-trivial representation into the Lie group $SU(2)$.*

Thus, in particular, for the groups $\pi_1(M^3)$, $\det A(r) \neq \pm 1$ is not the only abelian condition preventing $\pi_1(M^3)$ from being trivial.

Theorem I proves the existence of a discrete, purely group-theoretic Donaldson-like theory, which we suspect, among other things (see [81]) is the deeper underlying cause of the analogies of Borcherds' automorphic forms with Donaldson's polynomials (see [12, p.557]).

See Witten [82, p.353], [83, p.351], [84, p.1124] on the importance of analogues of Donaldson's Theorem appearing anywhere in mathematics or physics. Notice the philosophical similarity with Bohm-Aharonov phenomena: we obtain non-trivial homotopy where, *a priori*, intuitively, none should exist (see [4], [29]).

A healthy respect for braid theory makes it reasonable to expect the converse to hold for $n > 2$:

Conjecture I. *If $r \in \mathcal{R}_n$, $\det A(r) = \pm 1$ and $A(r) \notin D$, then there exists a Torelli $t \in \mathcal{R}_n$ such that $\pi(t \cdot r) = 1$.*

In other words, the commutator subgroup, $P'_n = [P_n, P_n]$, of the pure braid group P_n should kill these $\pi(r)$ as effectively as possible.

If true, even just up to Z-congruence, Conjecture I would embed the whole theory of compact, smooth 4-manifolds X^4, with $\pi_1(X^4) = 1 = \pi_1(\partial X^4)$, into the discrete theory of Artin presentations of the trivial group (see §4).

In any case it is hard to doubt that Conjecture I becomes true with stabilization, and (see section 4 ahead) there are enough $W^4(r)$ in AP Theory already to conclude that AP Theory contains at least a Donaldson-*like* theory.

Conjecture I declares the strongest version of Artin's Braid Philosophy: under its umbrella, M^3-Theory and compact, smooth simply-connected X^4-

Theory merge into a single *discrete*, non-classical theory, *a priori* devoid of all topology and differential geometry.

We explicitly reiterate the very important but somewhat subtle point: It is only the mere canonical, inductive 'holographic' open book construction, with the *planar* page Ω, (see Prop. 1.6 ahead) of the cobordism $W^4(r)$ which allows us to systematize, transcend and go beyond ad hoc surgery methods, the Kirby Calculus, etc. and obtain Theorem I, Conjecture I and other consequences thereof (see [81]).

Combining Theorem I with a recent theorem of Froyshov [21, p.374], whose proof uses the Seiberg-Witten equations, we obtain

Criterion III. *The Geometrization Conjecture implies: if $r \in \mathcal{R}_n$ is such that $A(r)$ is unimodular and positive definite, but not congruent over $\mathbf{Z}$ to the identity $n \times n$-matrix, nor to $E_8 \oplus (n-8)\langle 1 \rangle$, then $\pi(r)$ is infinite.*

We believe that the above new *amalgamations*, forged by gauge theory, of combinatorial group theory with such basic unpredictable number theory as integer quadratic forms and lattice theory, especially when combined with the rich, finite, regular covering theory of 3-manifolds, could bode ill for sweeping 'classical' conjectures such as Thurston's Geometrization Conjecture [69]. At the very least, it again justifies setting up criteria which can be tested on the computer in an orderly, systematic way for the following conjectures:

Weak Poincaré Conjecture. *Every homotopy 3-sphere bounds a smooth, compact, contractible 4-manifold.*

λ-Conjecture. *For every irreducible $\mathbf{Z}$-homology 3-sphere, Σ, $\pi_1(\Sigma)$ already determines the Casson Invariant, $\lambda(\Sigma)$ up to sign.*

11/8 Conjecture. *(see [20]) Any closed, smooth, simply-connected, spin 4-manifold N^4 satisfies $11\tau(N^4) \leq 8b_2(N^4)$. ($\tau$ and b_2 denote signature and second Betti number.)*

We remark that the λ-Conjecture just as the Weak Poincaré Conjecture, follow easily from Thurston's Geometrization Conjecture [69, p.482].

Similar arguments can be made *a priori* with topological quantum field theories in the sense of [5], [39], and [64]. Given r, any $3+1$ TQFT will associate to r not only a vector space $Z(r)$, but also, via $W^4(r)$, a well-defined (vacuum) vector $z(r) \in Z(r)$. Furthermore, this is done in a completely discrete fashion. Which is the most *natural* 3+1 TQFT here? Does there really exist one? What would the discrete analogue of Diff M^3 be? See Quinn's remarks in [64, p.333]. Or do they dissolve into number theory and lattice theory, loosing all their structure?

Concerning Theorem I, some of the first philosophical questions are "What is the purpose of these nontrivial groups purely from the non-topological standpoint of lattice theory and number theory?" "What do they obstruct?"

Does gauge theory, via Theorem I, disturb the conjectured *equivalence of the two basic, a priori unrelated, decision problems: the simply connectedness for $M^3(r)$ and the homeomorphism problem for S^3?*

If so, this could already disprove the Poincaré Conjecture without having to give a specific counter-example, see Haken [25, p.147].

We summarize: With Artin Presentation Theory, the *whole* theory of *closed* 3-manifolds (not just Seifert 3-manifolds) becomes a subchapter of combinatorial group theory and lattice theory in a more discrete (but non-simplicial, non-skein, non-categorical), arithmetic manner:

i) the continuum and topology have been compartamentalized, *frozen* into the pure braid group; and

ii) the very unpredictable 'actual infinite' appearing in the definition (Cayley, v.Dyck) of group presentations (i.e., 'the intersection of *all* normal subgroups such that…') has been frozen into presentations *per se.*

Some of its power is reflected in the above non-skein Knot Theory examples.

Furthermore, one seeks to extend Artin's Braid Theory to its natural boundary, encompassing smooth, compact, simply-connected 4-manifold theory as expressed with Conjecture I above as an embedding of Donaldson's Theory in Combinatorial Group Theory, in the spirit of what, in 1963, Papakyriakopoulos, [61], first did, in ad hoc fashion, for the Poincaré Conjecture.

In particular, with Artin Presentation Theory we go further and, in a sense, complete the Feynman-Atiyah-Manin-Witten program ([4],[5],[6],[82],

,[84]) of bypassing the metric in X^4-Theory: not even topology of any kind is retained, only discrete group theory, thereby raising new fundamental questions about 'quantum gravity' (see [6]), which are more number-theoretical, a la Borcherds, than 'physical'.

The philosophy of Artin Presentation Theory is to start and build on from here, always seeking to show and justify that it really is the most primitive Hard Lefschetz Theory (now enhanced via Th. I by Donaldson's Theorem), not just only because of its topological origins via Alexander's Open Book Theorem, but also (because via the Artin Presentation Theory and Floer instanton homology theory of Seifert 3-manifolds see Okonek [59]) it is also as close as possible to the original number theory pertaining to Hard Lefschetz Theory (Deligne's Weil Conjectures, etc.)

Artin Presentation Theory also has p-adic analogues via the p-adic theory of braids (Ihara [28]), which raises the question of the existence of any LOC-GLOB phenomena.

Thus, just as with Metzler's K-theoretic arguments vs. the (still unsettled but widely disbelieved) Andrews-Curtis Conjecture (see [26], [52, p.297]) as well as Reznikov's in [67], it is our contention that Artin Presentation Theory inherits so many varied phenomena from all these fields (and their covering and Galois theories when interacting with the 4-dimensional action of the Torelli $[P_n, P_n]$), especially when combined with gauge theory, TQFT's, etc, that Thurston's Geometrization Conjecture simply becomes too 'classical' and 'pristine' to be true in its entirety.

Theorem I, Criterion I and some examples were announced in an appendix to Ranicki's book [65]

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§1. Elementary Properties of Artin Presentations

Due to the fact that we are not yet using covering theory (except when using Alexander polynomials in our examples), all proofs in this paper are simple and elementary or immediate consequences of the basic theorems of 2,3,4-manifold theory.

We denote by Ω_n the 2-disk D^2 with n holes:

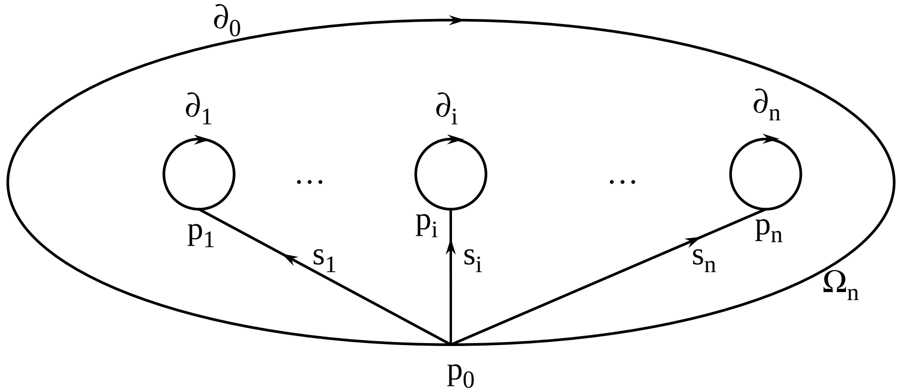

Figure 1.1

∂_0 and the ∂_i are the components of the boundary of Ω_n oriented clockwise. The s_i are oriented straight line segments joining p_0 to a point p_i of ∂_i. The loops $s_i \partial_i s_i^{-1}$ define generators x_i of the fundamental group $\pi_1(\Omega_n, p_0)$. By x_0, we denote the element of $\pi_1(\Omega_n, p_0)$ defined by ∂_0 and notice that $x_0 = x_1 x_2 \cdots x_n$. By d_i, we denote the homeomorphism $d_i : \Omega_n \to \Omega_n$ obtained by a counterclock-wise Dehn twist about ∂_i.

In the following, h will always denote a homeomorphism $h : \Omega_n \to \Omega_n$ such that $h|_{\partial\Omega} = \text{identity}$,

$$h_\# : \pi_1(\Omega_n, p_0) \to \pi_1(\Omega_n, p_0);$$
$$h_* : H_1(\Omega_n, \mathbf{Z}) \to H_1(\Omega_n, \mathbf{Z}),$$

denote the induced homomorphisms and $\eta : \pi_1(\Omega_n, p_0) \to H_1(\Omega_n, \mathbf{Z})$ the natural epimorphism.

Given h, by $r_i = r_i(h)$, we denote the element of $\pi_1(\Omega, p_0)$ defined by the loop $s_i h(s_i^{-1})$. For example,

$$r_i(d_j) = \begin{cases} 0, & \text{if } i \neq j; \\ x_i, & \text{if } i = j. \end{cases}$$

For any h, it is easy to check the following elementary properties:

Proposition 1.1.

i) $h_\#(x_i) = r_i^{-1} x_i r_i$, *in particular,* $h_* = \text{identity}$ *and in* $\pi_1(\Omega_n, p_0)$, *we have*

$$x_0 = x_1 x_2 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n). \qquad (*)$$

ii) If h' is another homeomorphism, $h' : \Omega_n \to \Omega_n$, $h'|_{\partial\Omega} =$ identity, then for the composition $h'(h)$ we have in $\pi_1(\Omega_n, p_0)$,

$$r_i(h'(h)) = r_i(h') \cdot h'_{\#}(r_i(h)).$$

We notice that $\pi_1(\Omega_n, p_0)$ is isomorphic to the free group, F_n, on n generators $x_1, x_2, \ldots, x_n$.

Given any presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ satisfying Equation $(*)$, Artin [3] had already shown implicitly that there then exists a homeomorphism $h : \Omega_n \to \Omega_n$, $h|_{\partial\Omega_n} =$ identity, unique up to isotopy keeping $\partial\Omega_n$ fixed, such that $r_i(h) = r_i$, $i = 1, \ldots, n$. (Nowadays, this is a particular case of a more general theorem of Nielsen [8, chp.2], [85, p.3].

This and their natural appearance in his fundamental Theorem 1.1 ahead, lead González-Acuña in 1975 to single these presentations out:

Definition 1.1. *A presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is called an Artin Presentation if, in F_n,*

$$x_1 \cdots x_n = (r_1^{-1} x_1 r_1)(r_2^{-1} x_2 r_2) \cdots (r_n^{-1} x_n r_n).$$

We remark that the name is well justified since similar presentations appear in Artin's characterization of link groups in S^3 (see [8, chp.2]). Furthermore, as indicated in the Introduction, they allow us to extend and augment Artin's braid theory philosophy into gauge theory and TQFTs.

We denote the set of Artin Presentations on n generators by $\mathcal{R}_n$ and point out that due to Proposition (1.1ii) and a theorem of Dehn [8, chp.2], they form a group, canonically isomorphic to a central extension of P_n by $\mathbf{Z}^n$, which by [19, Th. 3.4] is actually isomorphic to $P_n \times \mathbf{Z}^n$. The factor $\mathbf{Z}^n$ is due to the elementary Dehn twists d_i.

Given a simple, closed curve c in Ω_n, denote by $Dc : \Omega_n \to \Omega_n$ the counter-clockwise Dehn twist defined by c.

By the theorem of Dehn mentioned any of our $h : \Omega_n \to \Omega_n$, $h|_{\partial\Omega_n} =$ identity can be represented by a finite composition of Dehn twists about simple closed curves of Ω_n.

In Ω_2, the h form an abelian group generated by d_0, d_1, d_2. Hence, if $h = d_0^p d_1^q d_2^r$, then

$$r_1(h) = x_1^q (x_1 x_2)^p;$$
$$r_2(h) = x_2^r (x_1 x_2)^p,$$

and these are the only Artin presentations in $\mathcal{R}_2$.

Since $\Omega_2(h) = \Omega_2 \times S^1$, $M^3(r)$ is a Seifert space with at most 3 singular fibers (of multiplicities $|p|, |q|, |r|$, if these are > 1) or a connected sum of lens spaces.

Example 1.1. $n = 3$. Denote by c^+, c^- the two simple curves indicated in the figure

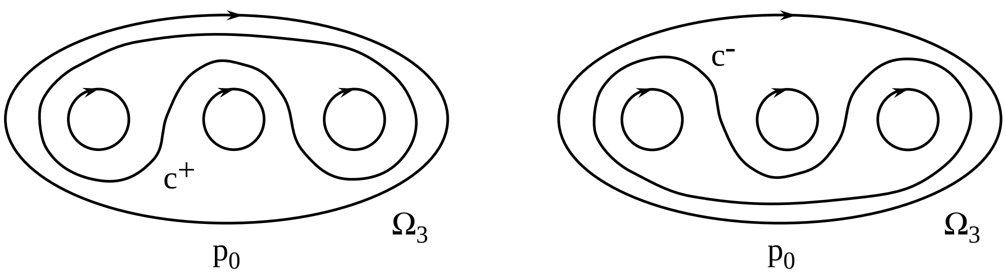

Figure 1.2

Then for Dc^+,

$$r_1 = x_1x_2x_3x_2^{-1};$$
$$r_2 = 1;$$
$$r_3 = x_2^{-1}x_1x_2x_3,$$

and for the inverse $D^{-1}c^-$ of Dc^-,

$$r_1 = x_3^{-1}x_1^{-1};$$
$$r_2 = x_3x_1(x_1x_3)^{-1};$$
$$r_3 = x_3^{-1}x_1^{-1},$$

and for the composition $D^{-1}c^-(Dc^+)$, we carefully compute using Proposition (1.1ii):

$$r_1 = x_3^{-1}x_2x_3x_1x_3(x_2x_3x_1)^{-1}r_2;$$
$$r_2 = x_3x_1(x_1x_3)^{-1};$$
$$r_3 = (x_2x_3x_1)^{-1}x_3x_1x_3^{-1}x_2x_3.$$

This is the Artin Presentation t_1 of §0. Similary, we obtain t_2, the inverse of t_1 in $\mathcal{R}_3$.

Let $\Omega(h)$ denote the mapping torus of h, i.e., the 3-manifold with boundary $(\partial\Omega) \times S^1$ obtained from $\Omega \times I$ by identifying $x \times 0$ with $h(x) \times 1$.

By $M^3(h)$, we denote the *open book* corresponding to h, that is the *closed* 3-manifold $\Omega(h) \cup_{\mathrm{id}} (\partial\Omega) \times D^2$, where id is the identity on $\partial\Omega \times S^1$.

Alexander's Existence Theorem [1] has been augmented by González-Acuña [23], using basic work of Lickorish ([43], [44], [45]), to show that any closed, orientable 3-manifold can be decomposed in this way, i.e., with a *planar* page Ω_n for some n (see also [48], [54], [57], [68]).

González-Acuña [23] discovered

Proposition 1.2. *The fundamental group of the open book $M^3(h)$, $h : \Omega_n \to \Omega_n$, can be presented by the Artin presentation*

$$\langle x_1, \ldots, x_n \mid r_1(h), \ldots, r_n(h) \rangle.$$

Hence, as a corollary, by also using the above mentioned Artin and Nielsen result, González-Acuña in 1975 obtained the fundamental

Theorem 1.1. *An arbitrary abstract group G is the fundamental group of a closed, orientable 3-manifold if and only if it has an Artin Presentation.*

Proof of Proposition 1.2. Let q_0 be the point of the mapping torus $\Omega(h)$ corresponding to $p_0 \times 0$, and let β denote the loop $p_0 \times S^1$ of $\Omega(h)$.

Then by the HNN construction (e.g., see [47]), we have

$$\pi_1(\Omega(h), q_0) \cong \langle \beta, x_1, \ldots, x_n \mid \beta x_1 \beta^{-1} = h_\#(x_1), \ldots, \beta x_n \beta^{-1} = h_\#(x_n) \rangle.$$

To compute the fundamental group of the open book $\Omega(h) \cup_{\mathrm{id}} (\partial\Omega) \times D^2$, one kills the loops $p_0 \times S^1$, $p_1 \times S^1$, $\ldots$, $p_n \times S^1$. First killing $p_0 \times S^1$, i.e., β, gives $x_i = h_\#(x_i)$, i.e., by Proposition (1.1i) $x_i = r_i^{-1} x_i r_i$. Now, by looking at the image in $\Omega(h)$ of $s_i \times I$, we see that the loop $s_i(p_i \times S^1)s_i^{-1}$ is homotopic (in $\Omega(h)$ keeping q_0 fixed) to $r_i\beta$. Therefore, the next n killing relations give $1 = r_i\beta = r_i$ (because now $\beta = 1$) which means that the relations $x_i = r_i^{-1} x_i r_i$ are redundant. Hence, $\pi_1(M^3(h), q_0)$ is presented by $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$.

Given $h : \Omega_n \to \Omega_n$, $h|_{\partial\Omega_n} =$ identity or its corresponding Artin presentation $r = r(h) \in \mathcal{R}_n$, we denote by $A(h)$ or $A(r)$ the $n \times n$ matrix of integers defined by $[a_{ij}]$ where

$$\eta(r_i(h)) = \sum_j a_{ij} \eta(x_j),$$

η denoting the natural epimorphism $\pi_1(\Omega_n, p_0) \to H_1(\Omega_n, \mathbf{Z})$. In other words, $A(r)$ is the exponent sum matrix of the presentation r and one immediately has

Proposition 1.3.

i) $M^3(r)$ *is a* **Z***-homology 3-sphere,* $\Sigma^3(r)$, *if and only if* $\det A(r) = \pm 1$.

ii) $M^3(r)$ *is a rational homology 3-sphere if and only if* $\det A(r) \neq 0$.

Furthermore, in case i) we have

Proposition 1.4. *The knot groups* G_i *and peripheral structures* m_i, ℓ_i *of the knots,* k_i, *in* $\Sigma^3(r)$, *defined by the boundary components,* ∂_i, *of* Ω_n *are given as in* §*0.*

Proof. The complement c_i in $\Sigma^3(r)$ of the knot k_i is obtained by simply not performing the open book identification at ∂_i. We have $x_i = \sum_{j=1}^{n} b_{ij} r_j$, in $H_1(c_i, \mathbf{Z}) = \mathbf{Z}$, r_i is a generator and by Proposition 1.2, $r_j = 0$ for $j \neq i$. Thus, if $\ell_i = x_i m_i^{\nu_i} = 0$, we have $(b_{ii} + \nu_i) r_i = 0$, i.e., $-\nu_i = b_{ii}$. Similarly, if $i = 0$.

Observe that if $\det A(r) = \pm 1$, then $A^{-1}(r)$ is the linking matrix of the knots k_i, $1 \leq i \leq n$, in $\Sigma^3(r)$.

In F_n, if $1 \leq k < \ell \leq n$, define

$$X_{k\ell} = \begin{cases} 1, & \text{if } \ell = k+1; \\ x_{k+1} x_{k+2} \cdots x_{\ell-1}, & \text{if } \ell > k+1. \end{cases}$$

Then

i) the Artin presentation $r \in \mathcal{R}_n$ defined by $r_i = 1$, if $i \neq k, \ell$, $r_k = X_{k\ell} x_\ell X_{k\ell}^{-1}$, $r_\ell = x_\ell^{-1} X_{k\ell}^{-1} x_k X_{k\ell} x_\ell$ has as $A(r)$ the symmetric $n \times n$-matrix $[a_{ij}]$, where $a_{k\ell} = a_{\ell k} = 1$, all other $a_{ij} = 0$.

ii) Similarly, the Artin presentation $r \in \mathcal{R}_n$ defined by $r_i = 1$ if $i \neq k, \ell$, $r_k = x_k X_{k\ell} x_\ell^{-1} X_{k\ell}^{-1} x_k^{-1}$ and $r_\ell = X_{k\ell}^{-1} x_k^{-1} X_{k\ell}$ has as $A(r)$ the symmetric $n \times n$-matrix $[a_{ij}]$, where $a_{k\ell} = a_{\ell k} = -1$, all other $a_{ij} = 0$.

Any diagonal matrix can easily be represented as an $A(r)$ by using the elementary Dehn twists d_i above.

Proposition 1.5. *An arbitrary integer* $n \times n$*-matrix* A *is equal to a* $A(r)$ *for some* $r \in \mathcal{R}_n$ *if and only if it is symmetric.*

Proof. From Proposition (1.1ii) above $A(r \cdot r') = A(r) + A(r')$ and so by the examples just given, the symmetry of A is sufficient for it to be $A(r)$ for some $r \in R_n$.

ii) Given $h : \Omega \to \Omega$ with $h|_{\partial\Omega} =$ identity, we denote by $H = H(h) : \partial(\Omega \times I) \to \partial(\Omega \times I)$ the orientation preserving homeomorphism obtained by setting $H = h$ on $\Omega \times \{0\}$ in the obvious way, and equal to the identity elsewhere on $\partial(\Omega \times I)$.

By imagining (in the definition of $M^3(h)$) that $(\partial\Omega) \times I$ shrinks to $\partial\Omega \times \{0\}$, it is easy to see that the open book $M^3(h)$ has the Heegaard decomposition $\Omega \times I \cup_{H(h)} \Omega \times I$.

All homology groups are taken with respect to $\mathbf{Z}$, and, since no confusion arises, we denote the elements in $H_1(\Omega)$, $H_1(\partial(\Omega \times I))$, $H_1(\Omega \times I)$, corresponding to $\eta(x_i)$, $\eta(r_i)$, and the canonical homomorphisms induced by inclusions, by the same letters. Thus, we consider $H_1(\partial(\Omega \times I))$ as being generated by the basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$:

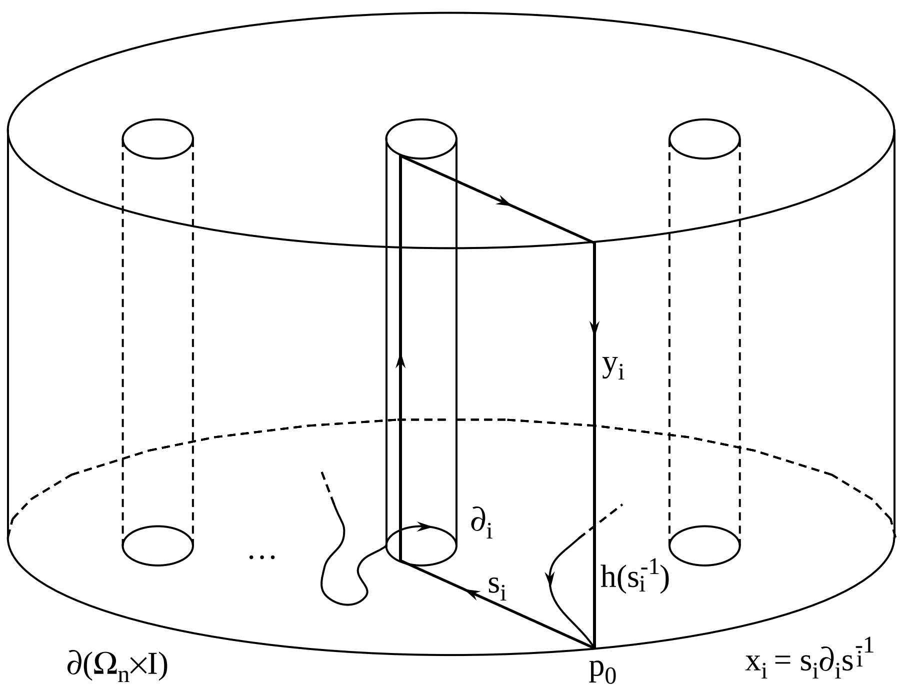

Figure 1.3

Since, already on the arc level $H(y_i) = h(s_i) \cdot s_i^{-1} \cdot y_i$ and $h_* =$ identity, the $(2n \times 2n)$-matrix of $h_* : H_1(\partial(\Omega \times I)) \to H_1(\partial(\Omega \times I))$ with respect to this basis is

$$\begin{bmatrix} I & 0 \\ -A(h) & I \end{bmatrix};$$

furthermore, from the sympletic condition

$$\begin{bmatrix} I & 0 \\ -A & I \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & -A^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & A^T - A \end{bmatrix}$$

we see that $A(h)$ is always a symmetric matrix for any h.

Definition. *We say r 'is a Torelli' if $A(r) = 0$. E.g., t_1 of Example 1.1, above.*

The name is justified, since then the associated homeomorphism $H(r)$: $\partial(\Omega_n \times I) \to \partial(\Omega_n \times I)$ induces the identity on $H_1(\partial(\Omega_n \times I), \mathbf{Z})$, i.e., is a Torelli in the usual sense.

It is easy to see that r is a Torelli if and only if it corresponds to an element in the commutator $[P_n, P_n]$ of P_n due to the fact that A is symmetric and the abelianization of P_n is $\mathbf{Z}^{\binom{n}{2}}$. [13, p.162]

Two $r \in \mathcal{R}_n$ with the same matrix $A(r)$ differ only by composition in $\mathcal{R}_n$ with a Torelli.

The symmetry of the matrix $A(r)$ lead us in 1992 [80] to the following elementary but fundamental discovery, which relates Artin Presentation Theory with gauge theory and TQFTs, *thus augmenting Artin's philosophy of braid theory as a first approximation to knot theory in S^3.*

Proposition 1.6. *An Artin presentation does not just determine the closed, orientable 3-manifold $M^3(r)$, but also a canonical smooth, simply-connected cobordism, $W^4(r)$, $\partial W^4(r) = M^3(r)$, of it,* whose quadratic form is determined by the matrix $A(r)$.

Proof. Let $h : \Omega \to \Omega$, $h|_{\partial\Omega} =$ identity, correspond to r. We obtain $W^4(r)$ as follows: Embed Ω in S^2, set $c\Omega =$ *closure* of $S^2 - \Omega$ in S^2. Extend h to S^2, then to all of D^3, by means of a diffeomorphism[7] $H : D^3 \to D^3$. The mapping torus of H, $W(H)$, has $c\Omega \times S^1 \subset \partial W(H)$, pasting $c\Omega \times D^2$ onto $W(H)$ via the canonical identification gives $W^4(r)$.

The proof now follows easily by the v. Kampen Theorem and the Mayer-Vietoris sequence after interpreting the entries of $A(r)$ as intersection numbers in Figure 1.1 as in the proof of Proposition (1.5ii). See §0 (Theorem I) and §4 for the philosophical implications of this result.

Let $\det A(r) = \pm 1$ and $B = [b_{ij}]$ be its inverse and set $s = \sum_{i,j} = b_{ij}$.

[7] H always exists, see J. Munkres *Elementary Differential Topology*, Princeton U. Press, 1963.

Proposition 1.7. *In $W^4(r)$, there exist closed, connected, orientable, smoothly embedded surfaces Σ_i, $i = 0, 1, \ldots, n$, with self-intersection numbers $\Sigma_0 \cdot \Sigma_0 = s$ and $\Sigma_i \cdot \Sigma_i = b_{ii}$.*

Proof. For each i, top off the Seifert surface of the knot k_i in $\Sigma^3(r)$ with the corresponding component of $c\Omega$ (of Proposition 1.6) and use Lefschetz duality.

See §4 for examples of $W^4(r)$.

The one-to-one correspondence of $\mathcal{R}_n$ and the isotopy classes of the $h : \Omega_n \to \Omega_n$ immediately implies that conjugate elements in $\mathcal{R}_n$ determine the same 3-manifolds; similarly, $M^3(r^{-1})$ and $M^3(r)$ are the same up to orientation.

Furthermore, it allows us to define the following elementary operations on Artin Presentations:

1. j-reduction.

   Given $r \in \mathcal{R}_n$ and j, $1 \leq j \leq n$, we denote by j-red $r \in \mathcal{R}_{n-1}$, the Artin Presentation obtained by ommitting r_j and setting $x_j = 1$ everywhere else in the presentation r (and renumbering, of course). The j-red r corresponds to covering the j-th hole in $h : \Omega_n \to \Omega_n$, and then we call the knot determined by $p_j \times S^1$ (in the open book construction) a monodromy knot of $M^3(j\text{-red } r)$ and observe that it has the same knot group as the knot k_j of $M^3(r)$, for the simple reason that the knot complements are the same.

2. Stabilization.

   Given $r \in R_n$, $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$, we can stabilize and obtain an $\bar{r} \in R_{n+m}$ with

   $$\bar{r} = \langle x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \mid r_1, \ldots, r_n, x_{n+1}, \ldots, x_{n+m} \rangle$$

   for every $m \geq 1$, without changing $M^3(r)$ since this change corresponds simply to connected sum with S^3. Stabilization allows 'more room' for changing by a Torelli (see Example (2vi) of §0).

3. Order Reversing.

   Given $r \in \mathcal{R}_n$, let $r^* \in \mathcal{R}_n$ be obtained by defining $(r^*)_i$ by, in each r_{n+1-i}, substituting each x_j by x_{n+1-j}. Then $r^{**} = r$ and $r \to r^*$ defines an involutory isomorphism $\mathcal{R}_n \to \mathcal{R}_n$ such that $(r^{-1})^* = (r^*)^{-1}$. One has $A(r^*) = -\overline{A}(r)$, where $\overline{A}$ denotes the flip about the antidiagonal of the matrix A and $\pi(r^*)$ is isomorphic to $\pi(r)$.

However, the $(r^*)_i$ *or* $(r^{-1})_i^*$ *can be non-trivial in* $\pi(r)$*:*

Consider t_1 and t_3 of §0, then $t_3 = t_1^*$ and if $r = \varepsilon_3 t_3$, then $\pi(r) = I(120)$ and each of the $(r^*)_i$ has order 10 in $\pi(r)$.

If $h(r)$ corresponds to r, then $\rho h(r)\rho$ corresponds to r^* where $\rho : \Omega_n \to \Omega_n$ denotes the flip about the y-axis, when Ω_n is conveniently situated.

4. j-fusing.

   Given two Artin Presentations $r \in \mathcal{R}_m$, $r' \in R_n$ and j, $1 \leq j \leq m$, we obtain an Artin Presentation, $(r, r')_j \in R_{m+n-1}$ by simply inserting the $h' : \Omega_n \to \Omega_n$ corresponding to r' in the j-th hole of $h : \Omega_m \to \Omega_m$ and renumbering.

Example: $n = 10$, the fusion $r = (x_8^{-1}\phi_8, \varepsilon_3)_8$ gives a Brieskorn sphere $\Sigma(2, 3, 7)$. The 10×10-matrix $A(r)$ is even with signature 8 and $\det A(r) = -1$. $x_8^{-1}r$ has $\pi = 1$ and $b_{ii} = \{0, -2, -7, -6, -5, -4, -3, 0, 1, 1\}$.

In example 3vi) of §0,

$$r = \left(\varepsilon_3 t_1, \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}\right)_2 .$$

For more fusion examples, see §3.

We end this section by showing how to obtain the Artin Presentation of §0 Example (5iii) for the following 8×8 matrix (see [60, p.331]).

$$\phi_8 = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 4 & 2 & & & & & \\ & 2 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{bmatrix}.$$

One has $\det \phi_8 = +1$, ϕ_8 is even and congruent over $\mathbf{Z}$ to E_8, the matrix used by Milnor to find his exotic spheres.

Denote by $D_i : \Omega_8 \to \Omega_8$, $i = 1, \ldots, 6$, the counterclockwise Dehn twist about the curve γ_i shown in Figure 1.4,

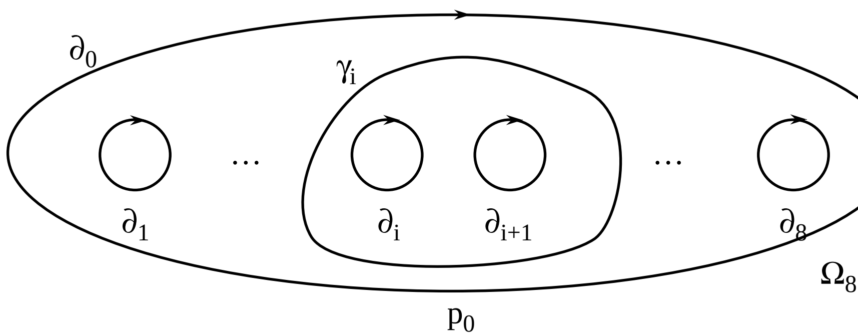

Figure 1.4

i.e., γ_i only contains the i-th and $(i+1)$-th hole. Let d_i be as before and set $d = d_1 d_2 \cdots d_8$. Then if $\overline{h} = d^{-1} D_1 D_3 D_5 D_7 : \Omega_8 \to \Omega_8$ and $\overline{\overline{h}} = d_1 d_2^{-1} d_3^{-1} d_8 d^{-1} D_2^2 D_4 D_6$, then $h = d_2^2 d^2 \overline{h}(\overline{\overline{h}})$ has $A(h) = \phi_8$ and the Artin Presentation $\overline{r}$, $\overline{\overline{r}}$ corresponding respectively to $\overline{h}$ and $\overline{\overline{h}}$ are easy to compute:

$$
\begin{aligned}
\overline{h} : \overline{r}_1 &= x_2; & \qquad \overline{\overline{h}} : \overline{\overline{r}}_1 &= 1;\\
\overline{r}_2 &= x_2^{-1} x_1 x_2; & \overline{\overline{r}}_2 &= x_2^{-1} x_3 x_2 x_3;\\
\overline{r}_3 &= x_4; & \overline{\overline{r}}_3 &= x_3^{-2} (x_2 x_3)^2;\\
\overline{r}_4 &= x_4^{-1} x_3 x_4; & \overline{\overline{r}}_4 &= x_5;\\
\overline{r}_5 &= x_6; & \overline{\overline{r}}_5 &= x_5^{-1} x_4 x_5;\\
\overline{r}_6 &= x_6^{-1} x_5 x_6; & \overline{\overline{r}}_6 &= x_7;\\
\overline{r}_7 &= x_8; & \overline{\overline{r}}_7 &= x_7^{-1} x_6 x_7;\\
\overline{r}_8 &= x_8^{-1} x_7 x_8; & \overline{\overline{r}}_8 &= 1;
\end{aligned}
$$

Now, using Proposition (1.1ii), we obtain the following Artin Presentation for $\overline{h}(\overline{\overline{h}})$:

$$\begin{aligned}
&R_1 = x_2;\\
&R_2 = \overline{r}_2\overline{r}_2^{-1}x_2^{-1}\overline{r}_2\overline{r}_3^{-1}x_3\overline{r}_3\overline{r}_2^{-1}x_2\overline{r}_2\overline{r}_3^{-1}x_3\overline{r}_3,\\
& = x_2^{-2}x_1x_2x_4^{-1}x_3x_4x_2^{-1}x_1^{-1}x_2x_1x_2x_4^{-1}x_3x_4;\\
\text{similarly}:\quad &R_3 = x_3^{-2}x_4(x_2^{-1}x_1^{-1}x_2x_1x_2x_4^{-1}x_3x_4)^2;\\
&R_4 = x_4^{-1}x_3x_4x_6^{-1}x_5x_6;\\
&R_5 = x_5^{-1}x_6x_4^{-1}x_3^{-1}x_4x_3x_4x_6^{-1}x_5x_6;\\
&R_6 = x_6^{-1}x_5x_6x_8^{-1}x_7x_8;\\
&R_7 = x_7^{-1}x_8x_6^{-1}x_5^{-1}x_6x_5x_6x_8^{-1}x_7x_8;\\
&R_8 = x_8^{-1}x_7x_8;
\end{aligned}$$

from which the one for h easily follows.

Similarly, by using the inverse of $\overline{h}$ and $\overline{\overline{h}}$, we obtain the following as the Artin Presentation of h^{-1}:

$$\begin{aligned}
r_1 &= x_1^{-1}(x_2x_3)^2x_2^{-1}(x_2x_3)^{-2}x_1^{-1};\\
r_2 &= x_2^{-2}(x_2x_3)^{-2}x_1^{-1};\\
r_3 &= x_3(x_2x_3)^{-2}x_4x_5r_4;\\
r_4 &= (x_4x_5x_4)^{-1}(x_2x_3)^2x_3^{-1}(x_2x_3)^{-2};\\
r_5 &= (x_4x_5)^{-1}x_6x_7r_6;\\
r_6 &= (x_6x_7x_6)^{-1}x_4(x_4x_5)^{-1};\\
r_7 &= (x_6x_7)^{-1}x_8r_8;\\
r_8 &= x_8^{-2}x_6(x_6x_7)^{-1}.
\end{aligned}$$

Now $A(r) = -\phi_8$.

Of course, once we know the inverse, we do not need to prove anything anymore since then with MAGMA, we can just check it.

r^* of the previous r, i.e., $(\phi_8^{-1})^*$ is given by

$$
\begin{aligned}
r_1 &= x_1^2 x_3^{-1} x_2 x_3;\\
r_2 &= x_2 x_3 x_1 x_3^{-1} x_2 x_3;\\
r_3 &= x_3 x_2 x_3 x_5^{-1} x_4 x_5;\\
r_4 &= x_4 x_5 (x_2 x_3)^{-1} r_3;\\
r_5 &= x_5 x_4 x_5 (x_6 x_7)^{-2} x_6 (x_6 x_7)^2;\\
r_6 &= x_6^{-1} (x_6 x_7)^2 (x_4 x_5)^{-1} r_5;\\
r_7 &= x_7^2 (x_6 x_7)^2 x_8;\\
r_8 &= x_8 (x_6 x_7)^{-2} x_7^{-1} r_7.
\end{aligned}
$$

Its matrix $A(r)$ is the flip about the antidiagonal of ϕ_8.

Similarly, if we multiply $\overline{h}$ and $\overline{\overline{h}}$ in a different order we get another r with $A(r) = \phi_8$,

$$
\begin{aligned}
r_1 &= x_1^2 (x_3 x_2 x_3)^{-1} (x_2 x_3)^2;\\
r_2 &= x_2 (x_2 x_3)^2 x_1^{-1} r_1;\\
r_3 &= (x_2 x_3)^2 x_5^{-1} x_4 x_5;\\
r_4 &= x_4 x_5 (x_2 x_3)^{-2} x_3 r_3;\\
r_5 &= x_5 x_4 x_5 x_7^{-1} x_6 x_7;\\
r_6 &= x_6 x_7 (x_4 x_5)^{-1} r_5;\\
r_7 &= x_7 x_6 x_7 x_8;\\
r_8 &= x_8 (x_6 x_7)^{-1} r_7.
\end{aligned}
$$

The group $\pi(r)$ for all these examples is $I(120)$, i.e., as small as possible, due to Theorem I of §0.

Remark: It seems that tinkering judiciously with Artin Presentations whose $A(r)$ are the important, basic matrices of lattice theory (e.g., ϕ_8, E_8, Coxeter-Todd's extremal form, and, as for now, beyond our capacities: the 24×24-Leech matrices!) seem to give the most interesting examples. For more examples, see §3.

§2. The Criteria

In this section, we assume $\det A(r) = \pm 1$, i.e., $M^3(r)$ is a **Z**-homology 3-sphere, $\Sigma^3(r)$, and exploit the simplicity of the presentations and peripheral structures above to set up criteria, easily testable by computer (where then no other knowledge is required) for various important conjectures implied by Thurston's Geometrization Conjecture.

Furthermore, due to the canonical ordering, with a starting point, of the set of Artin Presentations, these tests can and should be done in an orderly fashion. Thus, even failure of finding counter-examples will give the positive result of empirical evidence for the conjectures in question.

As already indicated in §0, a very general criterion to keep in mind at all times is the following: By theorems of Whitten [79] and Gordon-Luecke [24], any prime knot in S^3 is determined up to it's mirror image and orientation already by its knot group. Hence if we succeed in recognizing a $G_i(r)$ of a $\Sigma^3(r)$ with $\pi(r) = 1$, as that of a prime knot in S^3, *the test is to check the peripheral structure*: if it is different than that of the prime knot in S^3 which has that same group, then the homotopy 3-sphere $\Sigma^3(r)$ is certainly not homeomorphic to S^3, by a theorem of Waldhausen [13, p.38].

In fact, instead of the Gordon-Luecke Theorem invoked above, any theorem concerning the 3-sphere S^3 whose proof uses more than its simply-connectness can be used to set up a criterion. Thus, González-Acuña in 1975 used Waldhausen's Uniqueness Theorem for Heegaard decompositions of S^3, [78], to find the following purely algebraic equivalent of the so-called Poincaré Conjecture:

Let the group $\overline{S}_n$ of $2n$ generators and one relation be presented by

$$\langle x_1, x_2, \ldots, x_n, y_1, \ldots, y_n \mid x_1 x_2 \cdots x_n = (y_1 x_1 y_1^{-1})(y_2 x_2 y_2^{-1}) \cdots (y_n x_n y_n^{-1})\rangle$$

and let $N \subset \bar{S}_n$ be the normal closure of the $y_1, y_2, \ldots, y_n$ in $\bar{S}_n$.

If $r \in \mathcal{R}_n$ is an Artin Presentation, define the auto-isomorphism $f_r : \bar{S}_n \to \bar{S}_n$ by $f_r(x_i) = r_i^{-1} x_i r_i$, $f_r(y_i) = y_i r_i$, $i = 1, \ldots, n$.

Call two Artin Presentations r, r', equivalent, $r \approx r'$, if there exist auto-isomorphisms E_1, E_2 of $\overline{S}_n$ such that $E_1 f_r = f_{r'} E_2$ and $E_1(N) = N = E_2(N)$.

If $r \approx r'$, $M^3(r)$ and $M^3(r')$ are homeomorphic. Using Waldhausen's Theorem on the unicity of Heegaard decompositions of S^3, González-Acuña [23] obtained: The Poincaré Conjecture is true if and only if the following conjecture is true for all n.

Conjecture. *If $r \in \mathcal{R}_n$ is an Artin Presentation of the trivial group, then r is equivalent to the Artin Presentation $\varepsilon_n = \langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$.*

This is the Artin Presentation theory version of Birman's more complicated Heegaard version in [9].

The proofs of all the following criteria are immediate consequences of the well-known basic theorems of M^3-Theory. Thus, the proof of Criterion I, see §0, follows from the Gordon-Luecke Theorem and the observation that then $\Sigma^3(r)$ is a Dehn sphere corresponding to the monodromy knot of $\Sigma^3(j\text{-red } r)$, which has the same group G_j.

Remark: González-Acuña (unpublished) has shown that Criterion I is in fact *equivalent* to the following strong form of the Property P Conjecture: non-trivial surgery on a non-trivial knot in a homotopy 3-sphere never yields another homotopy 3-sphere.

Criterion II follows immediately from a theorem of Birman-Craggs [10] applied to $H(r) : \partial(\Omega_n \times I) \to \partial(\Omega_n \times I)$ of §1: The Rohlin invariant of $\Sigma^3(t^m r)$ would vanish and hence the Casson Invariant would be even.

Using the relation with gauge theory provided by Proposition 1.6 allows us to again test the λ-Conjecture and the following two:

Weak Poincaré Conjecture. *Every homotopy 3-sphere bounds a smooth, compact, contractible 4-manifold*

and (see [20], [22]) the

11/8-Conjecture. *Any closed, smooth, simply-connected,* spin *4-manifold N^4 satisifies*

$$11\tau(N^4) \leq 8b_2(N^4);$$

where τ and b_2 denote signature and second Betti number.

The simplest criterion (an immediate consequence of Rohlin's Theorem [35]) is:

Criterion IV. *If the λ-Conjecture is true, then for any Artin Presentation r such that $A(r)$ is a unimodular,* even *matrix with signature $\tau(r) \equiv 0 \pmod{16}$, we have $\pi(r) \neq I(120)$. Furthermore, if $A(r)$ is also definite, then (by Theorem I) $\pi(r) \neq 1$, i.e., $\pi(r)$ cannot be of finite order.*

In a similar vein, we have

Criterion V. *Assume $r \in \mathcal{R}_n$ is such that the $n \times n$ matrix $A(r)$ is unimodular,* even *and* $11|\tau(r)| > 8n$. *Then, the 11/8-Conjecture together with the Weak Poincaré Conjecture imply $\pi(r) \neq 1$.*[8] *($\tau(r)$ denotes the signature of $A(r)$.)*

Of course, instead of this inequality, one can substitute the negation of already proven ones, e.g., $5|\tau(r)|+8 > 4n$ (see [22]) and obtain unconditional criteria for the Weak Poincaré Conjecture.

Criterion VI. *Let $r \in \mathcal{R}_n$ and assume $\pi(r) = 1$. Let $A_i(r)$, $i = 1, \ldots, n$, denote the matrix obtained from $A(r)$ by substituting the i-th column by the diagonal, and assume* $\det A_i(r) \equiv 1 \pmod 2$ *for each $i = 1, \ldots, n$. Assume also $\sum_{i,j} b_{ij} \not\equiv \tau(r) \pmod{16}$ where $B = [b_{ij}]$ is $A^{-1}(r)$. Then, if the Poincaré Conjecture is true, the group $G_0 = \langle x_1, \ldots, x_n \mid r_1 = r_2 = \cdots = r_n \rangle$ is not isomorphic to $\mathbf{Z}$.*

This follows easily from applying a Theorem of Kervaire-Milnor [35] to the 4-manifold $W^4(r)$, the condition on the $A_i(r)$ implying that x_0 defines a characteristic element in $H_2(W^4(r), \mathbf{Z})$, which would be represented by an embedded S^2 by Proposition 1.7.

Notice that $\pi(r) = 1$ is a necessary condition for $G_0(r) = \mathbf{Z}$ and the condition on $A(r)$ is satisfied if $A(r) \equiv I \pmod 2$.

Example: $r = \bar{r} \cdot t_2$ of Example (2vi) of §0 has $\pi(r) = 1$, $A \equiv I \pmod 2$, $\sum_{i,j} b_{ij} = -7$, and signature $A(r) = 1$. Hence, if $G_0(r)$ were equal to $\mathbf{Z}$, we would have a counter-example to the Poincaré Conjecture. Of course, this is not the case. Here $G_0(r) = \langle a, b \mid a^2 = b^3 \rangle \neq \mathbf{Z}$.

Observe the hypothesis of all these criteria are invariant under multiplication with Torelli and suitable stabilizations. Similar criteria can be given by using the work of Hsiang-Szczarba [27] and others.

Finally, in this section, consider the following (at this time probably not really feasible) criteria:

Given $r \in \mathcal{R}_n$, the corresponding $h(r) : \Omega_n \to \Omega_n$ has a finite decomposition into Dehn twists [8] (and other types of decompositions [7]). If $\Sigma^3(r) = S^3$, then a moment of reflection convinces (since we are in S^3 where we can "see") that there should exist a reasonable bound on the number of the planar crossings of the knots $k_i(r)$ (pick $k_0(r)$, say) in function of

[8] When testing this criterion, one should, of course, due to Theorem I of §0, make sure that $A \notin D$.

the above decomposition. Translating this into a bound on the number of generators and word length of the r_i creates, *a priori*, the possibility of the existence of such an r with $\pi(r) = 1$, but with $k_0(r)$ having some strange properties which the knots in S^3, up to that crossing number bound, do not have. This would show that $\Sigma^3(r) \neq S^3$.

Why should the canonical ordering of the $k_0(r)$ in any $\Sigma^3(r)$ with $\pi(r) = 1$ keep pace with the "newtonian" ordering by crossings numbers in S^3?

Why should they have the same rate of complexity?

§3. More Examples

1. $n = 5$. Let $r \in \mathcal{R}_5$ be given by the stabilization

$$\langle x_1, \ldots, x_5 \mid r_1, \ldots, r_4, x_5 \rangle$$

where $r \in \mathcal{R}_4$ is as in Example 3 of §0. Then both r and $r \cdot t_1(x_3, x_4, x_5)$ have $\pi = 1$, but $\pi(r \cdot t_1^2(x_3, x_4, x_5))$ is presented by

$$\pi = \langle y, z \mid y^{-3}z^{-2}y^3 = z^{-2}y^3z^2, y^3z^5y^3 = z^2y^4z^2 \rangle$$

which is similar to

$$\pi_1\Sigma(2, 3, 35) = \langle b, c \mid b^{18} = c^{35}, b^2 = cbc \rangle$$

in that their subgroups of lowest index m have the same abelianizations up to $m \leq 12$ or more. Their Casson invariants are the same: ± 6.

The only reason that we are able to ascertain that π is not isomorphic to $\pi\Sigma(2, 3, 35)$ is because the cores (both of index 60) of their unique subgroups of index 5 both have a unique subgroup of index 2 which abelianizes to $\mathbf{Z}_3 \oplus \mathbf{Z}_{21}^{11}$ in the case of the former and to $\mathbf{Z}_7^{11}$ in the case of the latter. (In both cases, these subgroups are normal and factor onto $I(120)$.)

2. $n = 8$. Let $r = \phi_8 \cdot t_1(x_1, x_2, x_3)$. Then, again, $\pi(r) = I(120)$. Both $x_8^{-1}r$ and $x_1^{-2}x_8^{-1}r$ have $\pi = 1$ with Alexander Polynomial

$$\Delta_1(k_0) = t^4 - t^3 - t^2 - t + 1.$$

However, the groups G_0 are different. Although they have the same $2, 3$-torsion, both have one extra subgroup of index 3, which abelianizes to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z} \oplus \mathbf{Z}$ for the first and to $\mathbf{Z}_2 \oplus \mathbf{Z} \oplus \mathbf{Z}$ for the second.

3. $n = 8$. Let $r = \phi_8 t_3(x_2, x_3, x_4)$. Then $x_1^{-2} x_4^{-1} x_8^{-1} r$ has $\pi = 1$ and $\Delta_1(k_1) = t^4 - 2t^3 + 3t^2 - 2t + 1$ and $\Delta_2(k_1) = t^2 - t + 1$.

4. $n = 10$. The fusion

$$r = \left(\left(\phi_8, \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}\right)_8, \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\right)_7$$

has $\pi(r) = I(120) = \pi(10\text{-red } r)$ and $G_{10} = \mathbf{Z} * I(120) = \pi(x_{10}^{-1} r)$. Also, $\pi(x_7^{-1} r) = 1$.

5. n=12. i) Let r be as in 6.i) of §0; then again we have the "unrigidity" phenomenon in the following. If we multiply $x_1^{-1} r$ at 10 by the Torelli t_2, then we still have $\pi = 1$, but the torus knots $G_1, \ldots, G_9$ change to ones with Alexander Polynomial

$$(t^2 - t + 1)(t^8 - t^7 - t^6 + t^5 + t^4 + t^3 - t^2 - t + 1)$$

and are not torus knots anymore.

ii) Similarly, if we multiply $x_1^{-1} x_{10} r$ at 10 by t_2, we still have $\pi = 1$, but the torus knots G_i, $i = 1, \ldots, 9$ change to knots with Alexander Polynomial

$$t^{14} - 2t^{13} + t^{12} - t^{11} + t^{10} - t^8 + t^7 - t^6 + t^4 - t^3 + t^2 - 2t + 1,$$

again showing they are not torus knots anymore. The trefoil knot G_{11}, though, is again a trefoil with the same peripheral structure, it seems.

iii) Tinkering with $r \in \mathcal{R}_{12}$ of Example (5i) gives:

1. π of the $(4, 12)$-reduction of $x_1^{-1} x_{10} r$ is of order 1320 and presented by $\langle x_5, x_{11} \mid x_5^5 = x_{11}^2 = (x_5 x_{11})^3 \rangle$. The regular covering (of index 11) corresponding to the commutator has $\pi = I(120)$.

2. Similarly, the 7-reduction of $x_1^{-1} x_{10} r$ has $\pi = \langle x_8, x_{12} \mid x_8^2 = x_{12}^5 = (x_8 x_{12}^2)^3 \rangle$ of order $2040 = 17 \cdot 120$. The 17-cover corresponding to $[\pi, \pi]$ has $\pi = I(120)$.

6. $n = 16$. Let r_0 denote $x_8^{-1} \phi_8$ and set $r = x_{16}\Big((r_0, (r_0^*)^{-1})_2, \varepsilon_2\Big)_2 \in \mathcal{R}_{16}$. Then $A(r)$ is even and positive definite, but, alas, $\pi(r)$ has 36 subgroups of index 7 (more than usual) and hence $\pi(r) \neq I(120)$. $\pi(x_{16}^{-1} r)$ is also infinite, but G_1 is that of a trefoil (in S^3), $b_{11} = 0$, and the Dehn sphere Σ corresponding to $m\ell^{-2} = 1$ has $\pi = I(120)$. The Dehn sphere corresponding to $m\ell^{-1} = 1$ has $\pi = 1$, and hence, the Casson invariant of $\Sigma^3(x_{16}^{-1}r)$ is $= \pm 1$, probably not sufficient to show $\lambda(\Sigma) \neq \pm 1$.

7. $n = 17$. Let

$$r_0 = \left(\phi_8, \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}\right)_8 \in \mathcal{R}_9$$

and consider $r = (r_0, (r_0^{-1})^*)_2$, then $\pi(r) = 1$ and $\det A = +1$ with 4 negative eigenvalues. G_2 is that of the knot 4_1 in S^3, G_6, G_7 that of 3_1, and G_8 that of 5_1, and all other $G_i = \mathbf{Z}$.

$$b_{ii} = \left\{ \begin{matrix} -1, -4, -1, -3, -4, -6, -7, -11, -3, -1, \\ -23, -22, -21, -20, -19, -1, -19 \end{matrix} \right\}, s = 2.$$

As presentations of the 2 Dehn's of k_2 corresponding to $m\ell = 1$, $m\ell^{-1} = 1$, MAGMA gives, respectively

$$\langle x_1, x_3 \mid x_3^3 = x_1 x_3 x_1^{-2} x_3 x_1, x_3 = x_1 (x_1 x_3)^{-1} x_3 x_1 x_3 (x_3 x_1)^{-1} x_1 \rangle$$

and

$$\langle x_1, x_3 \mid x_3 = x_1 x_3 x_1^{-2} x_3 x_1, x_3 = x_1^{-1} x_3 x_1^3 x_3 x_1^{-1} \rangle,$$

which, presumably, are isomorphic, since otherwise, 4_1 in S^3 being amphicheiral, $\Sigma^3(r) \neq S^3$.

Let $r_0 \in \mathcal{R}_3$ be given by $x_1^3(x_1x_2)^{-2}$, $x_2^5(x_1x_2)^{-2}$, x_3 and let r' be the $(1, 3, 9, 10, 16)$-reduction of r. Then $\overline{r} = (r', r_0)_1 \in \mathcal{R}_{14}$ has $\pi = I(120)$ and A is positive definite $\not\approx I$ over $\mathbf{Z}$.

8. $n = 18$. Let R be the 3-reduction of r of Example 7. Then the matrix of $r = ((R, \varepsilon_2)_2, \varepsilon_2)_{17}$ is even with signature $= 16$ (one negative eigenvalue). However, $\pi(r)$ has several subgroups of index 5 and hence, alas, $\pi(r) \neq I(120)$. For $\pi(x_{18}^{-1}r)$, MAGMA gives exactly the same presentation as for π of the $m\ell^{-1}$ Dehn sphere of k_2 of Example 7, which presumably is $\Sigma(2, 3, 7)$. $\frac{1}{2}\Delta''(1)$ of k_{18} of r is 17, but for $r \cdot t_3(x_{16}, x_{17}, x_{18})$, it switches to -8.

Another $r' \in \mathcal{R}_{18}$ with $A(r)$ even of index 16 is $r' = x_{18}(16\text{-red } r, \varepsilon_3)_2$, where $r \in R_{17}$ is that of Example 7, but again, alas, $\pi(r')$ has 7 subgroups of index 5 and hence is infinite.

9. $n = 20$. Let $r_0 \in \mathcal{R}_3$ be given by $x_1^3(x_1x_2)^{-2}$, $x_2^5(x_1x_2)^{-2}$, x_3, and let r be as in Example 8. Then $(x_1^{-2}r, r_0)_{18} \in \mathcal{R}_{20}$ has an even A with signature 16 (2 negative eigenvalues). However, π is infinite.

10. For $n \geq 3$, set $X_n = x_2 \cdots x_{n-1}$, $(x, y) = x^{-1}y^{-1}xy$, and denote by $T_n \in \mathcal{R}_n$ the Torelli:

$$r_1 = x_n^{-1} X_n x_n (x_1^{-1}, x_n^{-1}) X_n^{-1} (x_n^{-1}, x_1^{-1});$$
$$r_2 = r_3 = \cdots r_{n-1} = (x_n^{-1}, x_1^{-1});$$
$$r_n = (X_n x_n x_1)^{-1} (x_n^{-1}, x_1^{-1}) x_1 X_n x_n.$$

Then, if $r(n) = \varepsilon_n T_n$, G_2 is that of a trefoil in S^3, where the Dehn spheres corresponding to $m\ell^{n-2}$ and $m\ell^{n-3}$, respectively, have $\pi = 1$ and $\pi = I(120)$. In particular, the Casson invariant of $\Sigma^3(r(n))$ is $= \pm(n-2)$.

§4. The 4-manifolds $W^4(r)$

We saw that, although AP Theory starts off as the lowest dimensional Lefschetz Decomposition theory, it really becomes, via the manifolds $W^4(r)$, a first approximation to compact, smooth, simply-connected 4-manifold theory, at least in the same philosophical sense Artin himself considered the Theory of Braids as a first approximation to the knot theory in S^3.

Our compact, smooth, simply-connected 4-manifolds $W^4(r)$, constructed in Proposition 1.6, are sufficiently general in the sense that:
i) Every closed, orientable 3-manifold bounds one of them.
ii) Any integer symmetric matrix is realized as an $A(r)$. In particular, any integer quadratic form can be represented as the intersection form of an $W^4(r)$. (See Proposition 1.5 of §1.)

On the other hand, they are also sufficiently 'minimal' and 'transparent' so that:
iii) The matrix $A(r)$ is also the exponent sum matrix of r, a presentation of the fundamental group of the boundary of $W^4(r)$.
iv) If $\det A(r) = \pm 1$, i.e., if the boundary of $W^4(r)$ is a **Z**-homology 3-sphere $\Sigma^3(r)$, then by Proposition 1.7, the theory of closed, orientable, smoothly embedded surfaces in $W^4(r)$ has a very sharp approximation by means of the theory of Seifert surfaces of the knots k_i in $\Sigma^3(r)$.

Furthermore, once the $n \times n$-matrix A is given and we have constructed some $W^4(r)$ with $A(r) = A$, the commutator subgroup $P_n' = [P_n, P_n]$ of the pure braid group P_n operates on the r via multiplication by Torelli *to give all such $W^4(r)$ in a systematic and complete manner.*

This, and the reduction to knot theory in the boundary $\partial W^4(r)$ of the smooth surface theory[9] in $W^4(r)$, has considerable advantages in the still somewhat *ad hoc* theory of examples in smooth, simply-connected 4-manifold theory.

For example, suppose we want to construct a closed, smooth, simply-connected 4-manifold homeomorphic to $\mathbf{C}P^2\#\mathbf{C}P^2\#\overline{\mathbf{C}P}^2$ with a smoothly embedded 2-sphere S^2 whose self-intersection number, $S^2\cdot S^2$, is $=-3$.

Start with any unimodular symmetric 3×3-matrix A with two positive eigenvalues whose inverse matrix $B=[b_{ij}]$ has -3 in its diagonal, e.g.

$$A=\begin{bmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $b_{11}=-3$. Now construct an $r\in\mathcal{R}_3$ with $A(r)=A$, say

$$\begin{aligned} r_1 &= x_1^3(x_1x_2)^{-2};\\ r_2 &= x_2^5(x_1x_2)^{-2};\\ r_3 &= x_3, \end{aligned}$$

and check the group $\pi(r)$ with MAGMA: Here $\pi(r)=I(120)$, i.e., $W^4(r)$ does not give a closed manifold. However, bringing in multiplication by Torelli, i.e., the action of $P_3'=[P_3,P_3]$, after a few attempts, MAGMA gives $\pi(t\cdot r)=1$ and $G_1(t\cdot r)=\mathbf{Z}$, where $t=t_2$ ot Example 2ii) of §0. Now, the knot k_1 of $\Sigma^3(t\cdot r)=S^3$ will bound a smooth disk which (see Proposition 1.7) can be 'topped off' to give a smooth embedded 2-sphere with $S^2\cdot S^2=-3$.

We have done this in a systematic manner by letting the group P_3' operate.

Remark: In this example, we also have a smoothly embedded 2-sphere with $S^2\cdot S^2=-1$ and two smoothly embedded torii (corresponding to the trefoils k_3, k_0) with self-intersection numbers $+1$ and -7, respectively.

Since at least in S^3 the Alexander polynomial of a knot is related to its genus and iv) relates this genus to that of smooth surfaces in $W^4(r)$, it is natural to conjecture, due to the basic work of Kronheimer, Mrowka[10] and others, that at least when $\partial W^4(r)=S^3$, the Donaldson and Seiberg-Witten invariants of the smooth 4-manifold $W^4(r)$ should be related or even determined by the Jones or Alexander polynomials of the knots and links formed by the $k_i=k_i(r)$ in $\Sigma^3(r)=S^3$.

[9] at least that part of it which is relevant for describing elements of $H_2(W^4(r),\mathbf{Z})$.

[10] see the reviews of their work in MR 96e:57019,96b:57038 by Kotschick and Bauer.

Consider example 3i) of §0. Here the sum of the generators corresponding to x_0 of $H_2(W^4(r), \mathbf{Z})$ is represented by a smoothly embedded S^2, since k_0 is trivial. However, if we multiply r by the Torelli $t = t_1(x_1, x_2, x_3)$, then the knot becomes 10_{132} of genus 2 (for knots in S^3 of less than or equal 10 crossings, the genus is $\frac{1}{2}$ of the degree of the Alexander polynomial [13, p.311]).

This should mean that the smooth, closed, simply-connected 4-manifolds $W^4(r)$ and $W^4(t \cdot r)$, both homeomorphic to $4\mathbf{C}P^2$, are not diffeomorphic.

Similarly with Examples 4ii) and 4iv) of §0, which are both homeomorphic to $7\mathbf{C}P^2 \# \overline{\mathbf{C}P}^2$. Here the latter has an extra trefoil.

Does AP Theory also 'discreticize' the theory of all closed, smooth, simply-connected 4-manifolds?

Let such a X^4 be given and set $b_2(X^4) = n$ and assume $n > 2$.

i) Choose an $n \times n$-matrix A representing the quadratic form of X^4. (Here we could choose a tri-diagonal matrix) and construct some $r \in \mathcal{R}_n$ with $A(r) = A$ (Proposition 1.5). We will have $\pi(r) \neq 1$, most probably and so $W^4(r)$ cannot be considered a closed 4-manifold.

ii) Now (in a perfect world, we could assume unlimited computer proficiency) let the group $P'_n = [P_n, P_n]$, i.e., the Torelli operate on r. If there exists a $t \in \mathcal{R}_n$ such that $\pi(t \cdot r) = 1$, then the Weak Poincaré Conjecture would imply by Freedman's basic Theorem, that we have at least found a closed smooth $W^4(t \cdot r)$ homeomorphic to X^4. (Here it is actually enough that $\Sigma^3(t \cdot r)$ bound a smooth, contractible, compact 4-manifold, but "$\pi(t \cdot r) = 1$ + WPC" is a (computer approachable) sufficient way of saying this)

iii) Now, let the Torelli operate some more to obtain the actual smooth structure of X^4, i.e., *the canonical action of $P'_n = [P_n, P_n]$ takes center stage and substitutes for ad hoc PDE techniques or unsystematic Tietze-like 'move' procedures such as the Kirby Calculus.*

Hence, assuming Conjecture I of §0, the falsity of the Weak Poincaré Conjecture reveals itself as the only obstruction to incorporating the whole theory of *closed*, smooth, simply-connected 4-manifolds into discrete combinatorial group theory, just as this was done with the whole theory of closed, orientable 3-manifolds.

In particular, the Kronheimer-Mrowka surface theory would become 'just' Seifert surface theory of knots in S^3 or some other homotopy 3-sphere.

If in step ii), we never get $\pi(t \cdot r) = 1$, i.e., if Conjecture I were false, then for these matrices A, we would have a stronger type of Theorem I, which we would use to butress Haken's question in §0.

Furthermore, these matrices then would be candidates for augmenting the explicitly known subset of D in Theorem I and lead to questions about the existence of a sensible converse to this theorem.

We hope that at least some of the purely algebraic and number-theoretic techniques used in the higher dimensional study of non-simply-connected open books (work of Quinn, Ranicki, see [65]) percolate down to the more crystallic setting of AP Theory.

§5. Questions, Problems, Conjectures

1. A first fundamental question is: Consider the (now somewhat unnatural) equivalence relation on Artin Presentations given by

$$r \sim r', \qquad \text{if } M^3(r) \text{ is homeomorphic to } M^3(r').$$

How can one describe it in the discrete context of Artin Presentation Theory? Due to the purity of the braids involved, the Kirby Calculus does not apply, but see Simon [72, §5].

For irreducible $M^3(r)$, can this be written as, "There exists an isomorphism $i : \pi(r) \to \pi(r')$ preserving certain characteristic subsets," as suggested by Waldhausen's Theorem on sufficiently large manifolds and in the theory of high-dimensional manifolds? Or does it have to be in an entirely different form due to the relations with gauge theory and TQFT?

What philosophical hints do we get from González-Acuña's Criterion in §2?

We venture the

Conjecture. *The use of the canonical order with starting point of Artin Presentations is* already *a prerequisite to start a sensible description of* $\sim$.

2. Concerning the λ-Conjecture, we venture the straightforward, testable by computer,

Conjecture. *Given any integer n, there exists an Artin Presentation r with $\pi(r) = I(120)$ and Casson Invariant $\lambda(r) = n$. In fact, we conjecture that there exists an r with $\pi(r) = I(120)$, $\lambda(r) \neq \pm 1$, and $A(r)$ congruent to some 24×24-Leech Matrix [15].*

As remarked before, this would contradict Thurston's 'Finite' Geometrization Conjecture [69, p.482].

Although, we do not yet have a general formula for the Casson invariant $\lambda(r)$ of a $\Sigma^3(r)$, it can be computed in many instances, with MAGMA, by finding simply-connected Dehn spheres (or other Dehn spheres whose λ is known) and the Alexander polynomials of the knots k_i of $\Sigma(r)$ (see, e.g., examples 2 vii) and 3v) of §0, examples 6, 10 of §3).

3. Can one axiomatize the Casson Invariant $\lambda(r) = \lambda(M^3(r))$ purely in the context of Artin Presentation Theory without using representation theory? (See [42]).

4. Let $\mathfrak{a}_0$, $\mathfrak{a}$ denote the sets of matrices such that there exist an r such that $A(r) = A$ where $\pi(r) = 1$, respectively, $I(120)$. Similarly, let $\mathfrak{a}_{00}$ denote the set of $A(r)$ where $\Sigma^3(r) = S^3$. Can one characterize these sets at least stably?

Is any **Z**-homology 3-sphere homeomorphic to an $\Sigma^3(r)$ where $A(r) = I$? The analogue is true for Heegaard decompositions (see, e.g. [55]). For such $\Sigma^3(r)$, a formula of González-Acuña [14, p. 66] for the Rohlin Invariant $\mu(r) = \mu(\Sigma^3(r))$ is particularly simple:

$$\mu(r) = \frac{1}{8}(d^2 - 1) \pmod 2,$$

where $d = \Delta(-1)$, Δ being the Alexander Polynomial of the group presented by

$$\langle x_1, \ldots, x_n \mid x_1 r_1 = r_1 x_2, x_2 r_2 = r_2 x_3, \ldots, x_{n-1} r_{n-1} = r_{n-1} x_n \rangle.$$

What is the analogous formula for the Casson invariant $\lambda(r)$?

In the case $n = 3$ without any restrictions on $A(r)$, a formula has been announced by Armas-Sanabria [2].

5. The Gassner representation of P_n (see [8, p.218, problem 20]) is *unfaithful* if and only if there exists a non-trivial Artin Presentation $r \in \mathcal{R}_n$ such that each $r_i \in F_n''$ (= the second commutator subgroup of F_n). *Can $M^3(r)$ of such an r have a geometric structure in the sense of Thurston* [69]? Would the corresponding spin cobordism $W^4(r)$ have any strange properties?

6. In the context of Artin Presentation Theory, does Bott's Question in his review of [4] concerning the existence of a conceptual definition ("Alexanderfication") of the Jones Polynomial still make sense? Or does the lack of topology make it moot?

7. What is the most natural Artin Presentation of a Seifert 3-manifold and how is it related to its Floer instanton homology? Can any symmetric, integer $n \times n$ matrix be realized as $A(r)$ where $r \in \mathcal{R}_n$ and $M^3(r)$ is a Seifert 3-manifold? Do Seifert 3-manifolds thus function as "intrinsic conductors" between lattice theory and other types of number theory?

8. Given r, what information for $M^3(r)$ do the liftings of the corresponding $h(r) : \Omega \to \Omega$ to covering spaces of Ω and their Nielsen-Thurston dynamics, especially the characteristic one corresponding to the $r_i \in \pi_1(\Omega)$, give? We recall Papakyriakopoulos' Remark [61, p.254] concerning the existence of regular coverings, which are non-planar, as a crucial obstruction to proving the Poincaré Conjecture.

9. We think Theorem I, in the sense that it "provides non-trivial homotopy where none was expected," should be considered as a number-theoretic analogue of the important Bohm-Aharonov Effect concerning quantum interference which accounts for the phase change of an electron wave function and reveals, among other important things, the connection-like behaviour of the Maxwell Field. (see [4], [29], [50])

Notice that in particular, Theorem I says: If A is any unimodular, symmetric, positive definite integer $n \times n$ matrix which is not congruent over $\mathbf{Z}$ to the identity $n \times n$ matrix I, $A \not\approx I$, then for any Artin Presentation r such that $A(r) \approx A$, the group $\pi(r)$ has to be non-trivial.

How do these intrinsic, non-trivial dynamics affect the lattice theory and the Hecke Θ-Function [15] (see also [40]) of the matrix A? Do these non-trivial groups $\pi(r)$ help obstruct the incorporation and integration of Θ-theory into the Langlands' Program (see [56, p.ix])? Do they cause some of the other difficulties in Hecke modular theory?

For any $r \in \mathcal{R}_n$, what is the relation between the above Θ function and the ones appearing in [40] and the other invariants there in of the mapping torus of $H(r)$ (see §1).

Similar questions can be asked for the Maass ζ-functions associated to the quadratic forms defined by the matrix A.

10. How does the general Kronheimer-Mrowka theory [38] discretize for our manifolds $W^4(r)$ in function of r alone? Do we obtain any new "Bohm-Aharonov-like" theorems like Theorem I above? Does it lead to a more crystallic , Onsager-like statistical mechanics (see [41]) than V. Jones' [32] as well as to new criteria?

11. If $\Sigma^3(r) = S^3$, $W^4(r)$ can be considered a *closed*, simply-connected, smooth 4-manifold. How can one characterize them? Due to their "topological" minimality, should they have complex or other type of structures? In function of r, when are they symplectic?

Are their Donaldson and Seiberg-Witten invariants in function of the Alexander or Jones polynomials of the knots $k_i(r)$ of S^3?

In this purely algebraic setting, what becomes of the 'Duality' between the Donaldson and Seiberg-Witten theories?

12. Is the Jones polynomial (Witten [83]) of the pair $[M^3(r), L(r)]$ in function of the Jones polynomial corresponding to the pure braid determined by r and the matrix $A(r)$?

13. One of the advantages of AP Theory as an M^3-Theory is that besides the bypassing of the difficult combinatorics of Skein theory, Vassiliev theory, etc, it also bypasses the use of closed Heegaard surfaces (whose genus can become very troublesome and can lead to dead ends, see, e.g. [25, p.145]) as well as the use of the spaces $\mathfrak{M}_{g,n}$, [37], whose difficult theory can also be extra baggage if one is interested only in M^3-Theory.

In other words, AP Theory 'clears the arena' so to speak: For example, the following difficult problem, which seems to be an analogue of the one discussed by Witten and Kontsevich (see [37, p.118]), can be stated in a more basic, precise form.

Given an Artin presentation r, denote by $\Gamma(r)$ the group of components of Diff $M^3(r)$, i.e. the mapping class group of $M^3(r)$. (This group is completely determined by r and acts on the vector space associated to $M^3(r)$ by a $3+1$ TQFT, see [4, p.13]).

Problem: In a purely algebraic fashion, describe $\Gamma(r)$ in function of r.

A solution (probably too difficult at this stage) would be to find a presentation of $\Gamma(r)$ in function of the Artin presentation r.

Nevertheless, the strongest statement versus the Poincaré Conjecture we can make: "There exists an r with $\pi(r) = 1$ and $\Gamma(r) \neq 1$," is *a priori* entirely natural in Artin Presentation Theory.

14. Concerning "physics" (see [4], [33], [50])

i) Due to the lack of topology (and hence metrics), is Theorem I—for being Bohm-Aharonov-like—related philosophically to concepts such as "energy of the vacuum", "topological charge", and "Bohm Quantum Potential"? Is the latter, in our case, for each n, induced by the 4-dimensional Torelli action of $[P_n, P_n]$ on the $W^4(r)$?

ii) Does it make sense to call an Artin Presentation r a "string" and the associated group $\pi(r)$ as its "vibration"? (See [81]). This seems in line with

Heisenberg's Remark [62, p.103] that perhaps ultimately, *matter* is just *order* plus *symmetry.*

iii) Does the group $I(120)$ play the role in Artin Presentation Theory that the Monster plays in conformal field theory (see, e.g., [17])?

iv) Is it just a curiosity that in Gauss's theory of the electrodynamic force (see [74, p.707] there exist exactly 120 possible electrodynamic force equations? Here it is relevant to recall that besides his famous work on integer quadratic forms, and his pre-Maxwell physical theories, Gauss was the first to take knots seriously as a mathematical concept.

v) Should one call the above conjecture in Question 2, "charging the integer n"?

Finally, we recall that, as is well-known, Poincaré himself, after discovering his famous **Z**-homology 3-sphere, never really conjectured or gave any indications in favor of the so-called "Poincaré Conjecture". His relevant paper in *Oeuvres*, t. 6, p. 498 ends with the sentence, '*Mais cette question nous entrainerait trop loin.*' In fact, in his later years, he was a great booster of the theory of quanta, e.g., beating P. Ehrenfest (see [36, p.251]) in showing that discreteness is actually necessary in order to prove Planck's Law rigorously, and furthermore, discussing quanta of action and of time itself. [*Oeuvres*, t. 9, p. 666]. Thus, the above arguments certainly do not contradict, rather, they are more in line with his intuition and philosophy.

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