What is...an Artin Presentation?

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Open Books

In algebraic geometry and topology, it is natural to ask whether a variety or manifold of a certain dimension can be constructed from lower dimensional ones of a similar nature. The prototype of this sort of theorem is Lefschetz's hyperplane decomposition theorem for complex algebraic varieties. It has discrete analogues in number theory and algebraic geometry ('Hard Lefschetz theory') which are of primordial importance in modern mathematics. Roughly put, the theory of open books is a 'real' topological analogue of this in the theory of arbitrary *smooth* compact manifolds.

Let V be a compact smooth $(n-1)$-manifold with nonempty boundary ($\partial V \neq \emptyset$) and $h : V \to V$ a diffeomorphism which restricts to the identity on ∂V; in $V \times [0,1]$, identify $(x,0)$ with $(h(x),1)$ obtaining V_h, the mapping torus of h, whose boundary is $\partial V \times S^1$. Let D^2 be the 2-disk with center 0; if we paste on $\partial V \times D^2$ by means of the identity of $\partial V \times S^1$, we obtain a closed n-manifold, M^n, which in a neighborhood of the $(n-2)$-dimensional submanifold $\partial V \times 0$, $0 \in D^2$, looks like an open book: the $(n-1)$-dimensional fibers of V_h define the pages and $\partial V \times 0$ is the binding [**R**] p.615. Of course, the whole manifold looks more like a 'rolodex'.

Example 1. Let V^2 be the compact 2-disk with two holes and outer and inner boundary components $\partial_0, \partial_1, \partial_2$ and let $h : V^2 \to V^2$ be the composition $d_0^{-2} d_1^3 d_2^5$ where d_i is a clockwise Dehn twist about ∂_i; the corresponding open book M^3 is the famous Poincaré homology 3-sphere.

The first general definition of open books and the first general open book theorem were obtained in 1972 by the author: if $n > 6$ and M^n is simply connected, M^n is an open book if and only if its signature is 0.

In general, if M^n is orientable and $n > 5$ the existence of open book structures was settled by T. Lawson when n is odd: there is no obstruction, and for any $n > 3$ by F. Quinn: when n is even, the obstruction lies in the asymmetric Witt group of $\mathbb{Z}[\pi_1(M^n)]$, generalizing the Wall surgery obstruction [**R**] p.360.

When $n = 3$, in 1975 González-Acuña using fundamental work of Alexander and Lickorish rigorously proved: every closed orientable M^3 is an open book; furthermore, the page can be chosen to have a connected boundary (i.e. every such M^3 contains a fibered knot) or be as simple as possible: a compact 2-disk with holes.

Open books have had numerous applications ([**R**] Appendix); from considerably simplifying known proofs (e.g. Bing's partial solution to the Poincaré conjecture, existence of contact forms and foliations) to new applications in cobordism and SK-theory, the bordism of automorphisms, L-theory, contact, symplectic and CR structures, differential geometry, diffusion processes on manifolds, etc.

Artin Presentations

When $n = 3$ and the page, V^2, is as basic as possible (a compact 2-disk with holes) in 1975 González-Acuña discovered that the theory of open books crystallizes into a discrete purely group theoretic theory. Just as Lefschetz's theorem has discrete versions in number theory and algebraic geometry, when $n = 3$ open book theory has a discrete version in the more physically relevant discrete theory of finitely presented groups. The homeomorphism $h : V^2 \to V^2$ is substituted by a certain type of presentation (an Artin Presentation) of $\pi_1(M^3)$ leading, in fact, to a systematic new theory, called AP Theory, of compact smooth simply connected 4-manifolds [**W**], [**CW**].

Let F_n denote the free group on the generators $x_1, x_2, \ldots, x_n$.

A presentation $r = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ is called an Artin Presentation if (in F_n):

$$x_1 x_2 \cdots x_n = \left(r_1^{-1} x_1 r_1\right)\left(r_2^{-1} x_2 r_2\right) \cdots \left(r_n^{-1} x_n r_n\right).$$

An arbitrary group is isomorphic to the fundamental group of a closed *orientable 3-manifold if and only if it has an Artin Presentation.*

The set of Artin Presentations on n-generators is denoted by $\mathcal{R}_n$; the group so presented by $\pi(r)$; $A(r)$ is the exponent sum matrix of r, which is always symmetric and every symmetric integer matrix can be so obtained.

If $A(r)=0$, we say r is a Torelli.

An r defines a unique closed orientable 3-manifold $M^3(r)$, such that $\pi_1\left(M^3(r)\right)$ is isomorphic to $\pi(r)$; $A(r)$ determines $H_*\left(M^3(r),\mathbb{Z}\right)$.

Every closed orientable 3-manifold can be so obtained.

Example 2. i)Let $r\in\mathcal{R}_2$ be given by $r_1=x_1^3(x_1x_2)^{-2}$ and $r_2=x_2^5(x_1x_2)^{-2}$, then $M^3(r)$ is the Poincaré homology 3-sphere above; $A(r)=\begin{bmatrix}1 & -2\\ -2 & 3\end{bmatrix}$.

ii) $r\in\mathcal{R}_3$ with $r_1=\left(x_3^{-1},x_1(x_1x_2)^{-1}\right)$, $r_2=x_2(x_1,x_3)$, $r_3=x_1(x_1x_2)^{-1}(x_3,x_1)r_2$ gives the Heisenberg 3-manifold (here $(x,y)=x^{-1}y^{-1}xy$).

The r can be multiplied so that $\mathcal{R}_n$ is canonically isomorphic to $P_n\times\mathbb{Z}^n$, where P_n denotes the pure braid group [**W**] p.227. The subgroup of Torelli is isomorphic to $[P_n,P_n]$, the commutator subgroup of P_n; it is a subgroup of the classical Torelli group of the closed orientable surface of genus=n.

If $\det A(r)=\pm1$, $M^3(r)$ is a $\mathbb{Z}$-homology 3-sphere $\Sigma^3(r)$ and the groups and their peripheral structures of the knots $k_i(r)$ in $\Sigma^3(r)$ defined by the components ∂_i of the binding have very simple computer friendly presentations. One need not do surgery 'by hand', avoiding self-linking problems, and here knot theory heeds the anti-skein admonitions of Penrose and Witten, without having to resort to a 'categorification' of pure braid theory.

Every link in any closed orientable M^3 is a sublink of a link so induced by an r for M^3.

If $A(r)=I$, $\mu(r)$, the Rohlin invariant of $\Sigma^3(r)$, is obtained purely group theoretically as follows. Consider the group presented by:

$$\langle x_1,\dots,x_n\mid x_1r_1=r_1x_2,\dots,x_{n-1}r_{n-1}=r_{n-1}x_n\rangle$$

which clearly abelianizes to $\mathbb{Z}$ and hence has an Alexander polynomial, whose value at -1 we denote by d_r. Then González-Acuña proved: $\mu(r)=\frac{d_r^2-1}{8}\bmod 2$.

Is every $\mathbb{Z}$-homology 3-sphere a $\Sigma^3(r)$ with $A(r)=I$?

The consequences of multiplying an r by a Torelli are very unpredictable and subtle; they always preserve $A(r)$ (in particular, $H_*\left(M^3(r),\mathbb{Z}\right)$), and usually change $M^3(r)$, $\pi(r)$, the knots $k_i(r)$; but they can also just change certain things and leave others invariant:

Example 3[1]. Consider $s\in\mathcal{R}_4$ and the Torelli $t\in\mathcal{R}_4$ given by: $s_1=(x_2x_3)^2x_2s_2$, $s_2=\left(x_1(x_2x_3)^2x_2^2\right)^{-1}$, $s_3=(x_2x_3x_2)^{-1}x_4s_4$, $s_4=x_4^{-2}x_2x_3x_2(x_2x_3)^{-2}$, and $t_1=\left(x_4^{-1},x_1(x_1x_2x_3)^{-1}\right)$, $t_2=(x_1,x_4)=t_3$, $t_4=\left(x_1^{-1},x_4x_1x_2x_3\right)t_3$. Here, $\pi(s)=1$ and all knot groups of $\Sigma^3(s)$ are isomorphic to $\mathbb{Z}$ except $k_3(s)$, whose group is isomorphic to that of the trefoil in S^3. MAGMA gives $r=t\cdot s$ as:

$$\begin{aligned}
r_1 &= \left(x_4^{-1},x_1(x_1x_2x_3)^{-1}\right)(x_4,x_1)(x_2x_3)^2x_2r_2\\
r_2 &= \left(x_2x_3x_2^2\right)^{-1}\left(\left(x_1^{-1},x_4x_1x_2x_3\right),x_4^{-1}\right)\left(x_4^{-1},x_1\right)(x_1x_2x_3)^{-1}\\
r_3 &= (x_2x_3x_2)^{-1}\left(x_4x_1x_2x_3,x_1^{-1}\right)x_4r_4\\
r_4 &= x_4^{-2}\left(x_1^{-1},x_4x_1x_2x_3\right)x_2x_3x_2(x_2x_3)^{-2}(x_1,x_4).
\end{aligned}$$

Now, we again have $\pi(r)=1$, however, the (non-amphicheiral) trefoil $k_3(s)$ of $\Sigma^3(s)$ has been transformed by the Torelli t to a (amphicheiral) figure-8 knot $k_3(r)$ in $\Sigma^3(r)$; all the other knots stay trivial.

In AP theory with MAGMA one easily and naturally obtains, among other things, Ratcliffe's nonclassical fibered knot in the Brieskorn sphere $\Sigma^3(2,3,11)$, examples of cosmetic surgery (McCullough), and interesting examples of Luft-Sjerve $\mathbb{Z}$-homology 3-spheres.

[1]Here the use of a computer algebra system, say, MAGMA, is highly recommended.

AP theory as a 4D theory

All of the above pertains to 3-manifold theory, but the most interesting fact about AP theory is that it is actually a new systematic *general* theory of compact smooth simply connected 4-manifolds containing analogues of many of the most important features of modern physics such as Donaldson's theory, Holography, String/M-theory, Witten's puzzle, and with covering theory, Loop Quantum Gravity theory and TOEs. Indeed, a relative open book construction which uses the planarity of the page in a *canonical* way shows that an Artin Presentation, r, actually defines a unique compact smooth simply connected 4-manifold $W^4(r)$ whose boundary is $M^3(r)$ and whose quadratic form is represented by $A(r)$. This 'canonicity' is so strong that one obtains a nontrivial discrete analogue of Donaldson's theorem in AP theory [**W**] p.240, [**R**] p.621. This is a great advantage over essentially ad hoc methods such as the Kirby calculus and takes the purely group theoretic program of the Princeton school of Artin, Fox, Papakyriakopoulos, Stallings, et al., to its natural 4-dimensional metamathematical boundary.

Although all of these 4-manifolds are 2-handlebodies (e.g. smooth contractible Mazur 4-manifolds are not included) they do form a very large class indeed: every 3-manifold bounds one and every quadratic form is represented. All elliptic surfaces, in particular the Kummer surface, can be so represented by an r such that $\partial W^4(r) = M^3(r) = S^3$, thus proving also the existence of a nontrivial discrete group theoretic theory of Donaldson *invariants* [**CW**].

Is every compact smooth simply connected 4-manifold with a connected, simply connected boundary a $W^4(r)$*?*

In AP theory, a compact smooth simply connected 4-manifold is already determined by an Artin Presentation of the fundamental group of its boundary. This is a *rigorous* universal abstract analogue of the celebrated Maldacena AdS/CFT proposal of String/M-theory and is as holographic as possible.

AP theory also clarifies the so-called 'Witten puzzle' of his famous 1993 lecture ([**Wi**]): how to integrate the topology changing symmetries and transitions of String/M-theory with the already existing ones represented, e.g., by mapping class groups of the manifolds $M^3(r)$ and $W^4(r)$ which are crucial in General Relativity. If $\Gamma(r)$ denotes the mapping class group of $M^3(r)$, it may be a difficult problem to ascertain how $\Gamma(r)$ behaves with respect to the always $\mathbb{Z}$-homology preserving, but usually topology-changing, Torelli transformations, but no metamathematical mysteries arise.

It is analogies such as these, which relate AP theory intimately with String/M-theory and LQG-theory, that distinguish it from, say, Connes' theory and the purely 3D Thurston-Hamilton-Perelman Geometrization program, where none of these important phenomena, nor analogues of Donaldson's and Lefschetz's theories, appear.

In AP theory the beginning researcher can choose among many different types of important, 3D or 4D, open problems, e.g. i) find purely discrete formulas for Donaldson and Seiberg-Witten invariants similar to the formula for $\mu(r)$ above, ii) find $\Gamma(r)$ in function of r, iii) find the analogue of Floer theory in AP theory.

In a more general metamathematical vein: *in the theory of compact smooth simply connected space-times* $W^4(r)$ *(determined by the 'string' r), is AP theory the most reductive, rigorous, cone-like, background independent, non-perturbative, parameter free 'theory of everything' possible which all other such theories must include and build upon?*

References

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