1. (20) Suppose \( A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -2 \end{pmatrix} \).

a) Find the row-reduced echelon form of \( A \).

\[
\begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -2 & 4 \\ 0 & 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

so the row reduced echelon form is \( \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).

b) What is rank(\( A \))?

Answer: The rank is two since there are two pivots.

c) Find a basis for the null space of \( A \).

Answer: The vectors \((x, y, z, w)^T\) in the null space satisfy \( x+z+4w = 0 \) and \( y+z-2w = 0 \) so \((x, y, z, w) = z(-1, -1, 1, 0) + w(-4, 2, 0, 1)\). So the two linearly independent vectors \((-1, -1, 1, 0)^T, (-4, 2, 0, 1)^T\) form a basis of the null space of \( A \).

d) Find a basis for the column space of \( A \).

Answer: The pivot columns of \( A \) form a basis, so \((1, 1, 0)^T, (2, 0, 1)^T\) forms a basis of the column space of \( A \).

e) Find all solutions \( X \) of \( AX = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \).

Answer: By either inspection or Gaussian elimination we see that one solution is \((1, 0, 0, 0)^T\) so all solutions are of the form \((1, 0, 0, 0)^T + z(-1, -1, 1, 0)^T + w(-4, 2, 0, 1)^T\).

2. (10) Suppose \( A \) and \( B \) are 3 \( \times \) 3 matrices and \( I \) is the 3 \( \times \) 3 identity matrix. Find \( \left( \begin{array}{cc} A & I \\ 0 & B \end{array} \right)^2 \) and \( \left( \begin{array}{cc} A & I \\ 0 & B \end{array} \right)^3 \).

Answer: \( \left( \begin{array}{cc} A & I \\ 0 & B \end{array} \right)^2 = \begin{pmatrix} A^2 & A + B \\ 0 & B^2 \end{pmatrix} \) and \( \left( \begin{array}{cc} A & I \\ 0 & B \end{array} \right)^3 = \begin{pmatrix} A^3 & A^2 + AB + B^2 \\ 0 & B^3 \end{pmatrix} \).

3. (20) Recall a square matrix \( A \) is skew symmetric if \( A^T = -A \). Show that the skew symmetric matrices are a subspace of \( \mathbb{R}^{n \times n} \). Find a basis for the 3 \( \times \) 3 skew symmetric matrices.

Answer: \( 0^T = -0 \) so 0 is skew symmetric. If \( A \) and \( B \) are skew symmetric then \((A+B)^T = A^T + B^T = -A + (-B) = -(A+B)\) so \( A + B \) is skew symmetric. If \( c \) is a scalar and \( A \) is skew symmetric, then \((cA)^T = cA^T = c(-A) = -(cA)\) so \( cA \) is skew symmetric. So the
skew symmetric matrices form a subspace of $\mathbb{R}_{n \times n}$. Many of you also correctly showed this by using the formula $\text{ent}_{ji}(A) = -\text{ent}_{ij}(A)$. Although nobody stated it this way, what these formulae give you are $n(n+1)/2$ homogeneous linear equations in the entries of the matrix, so their solutions form a subspace.

4. (35) Consider the curve $C$ parameterized by $r(t) = 4ti + 3\sin tj + 3\cos tk$, $0 \leq t \leq \pi$.
   a) Find the curvature $\kappa$ of $C$ as a function of $t$.
   Answer: $v(t) = 4i + 3\cos tj - 3\sin tk$ and $a(t) = -3\sin tj - 3\cos tk$. The speed $||v||$ is a constant, $5$. So $a_T = d(\text{speed})/dt = 0$. Then $a_N = \sqrt{||a||^2 - a_T^2} = ||a|| = 3$. $\kappa = a_N/||v||^2 = 3/25$.
   b) Find an equation of the line tangent to the curve $C$ at the point $r(\pi/2)$.
   Answer: One point on the curve is $r(\pi/2) = (2\pi, 3, 0)$ and a vector in the direction of the curve is $r'(\pi/2) = (4, 0, -3)$. So an equation is $(x, y, z) = (2\pi + 4t, 3, -3t)$.
   c) Find $\int_C 3x \, ds$.
   Answer: $\int_C 3x \, ds = \int_0^\pi 3(4t)||v(t)|| \, dt = \int_0^\pi 60t \, dt = 30\pi^2$.
   d) Find $\int_C ydx - zdy$
   Answer:
   $$\int_C ydx - zdy = \int_C (y, -z, 0) \cdot ds = \int_0^\pi (3\sin t, -3\cos t, 0) \cdot (4, 3\cos t, -3\sin t) \, dt$$
   $$= \int_0^\pi 12\sin t - 9\cos^2 t \, dt = \int_0^\pi 12\sin t - 4.5(1 + \cos(2t)) \, dt$$
   $$= -12\cos t - 4.5t + 2.25\sin(2t)]_0^\pi = 12 - 4.5\pi + 12 = 24 - 9\pi/2$$

5. (25) Let $D$ be the triangular region in the $xy$ plane bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$. Let $S$ be the portion of the surface $z = x^2 + y^2$ lying above $D$. Let $C$ be the boundary of $S$, oriented counterclockwise when viewed from above. Let $\mathbf{F}(x, y, z) = 3xyi - zj$.
   a) Set up completely, but do not evaluate, an integral giving the area of $S$.
   Answer: $dS = \sqrt{f_x^2 + f_y^2 + 1} \, dA = \sqrt{4x^2 + 4y^2 + 1} \, dA$. So the area is
   $$\int_0^1 \int_0^{1-x} \sqrt{4x^2 + 4y^2 + 1} \, dydx$$
   b) Use Stokes’ theorem to find $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$. 

Answer: \( \text{curl} \mathbf{F} = (1,0,-3x) \) so by Stokes’ theorem,

\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_S \text{curl} \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^{1-x} (1,0,-3x) \cdot (-2x,-2y,1) \, dy \, dx
\]

\[= \int_0^1 \int_0^{1-x} -5x \, dy \, dx = \int_0^1 -5x(1-x) \, dx = -5/2 + 5/3 = -5/6 \]

6. (30) Let \( D \) be the solid region above the surface \( z = x^2 + y^2 - 1 \), and below the surface \( z = 1 - x^2 - y^2 \). Let \( S \) be the boundary of \( D \) oriented pointing outward from \( D \). Let \( \mathbf{F}(x,y,z) = xi - 2yj + zk \).

a) Find \( \int_S \mathbf{F} \cdot d\mathbf{S} \).

Answer: \( \text{div} \mathbf{F} = 0 \) so \( \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_D 0 \, dV = 0 \) by Gauss’ theorem.

b) Set up completely, but do not evaluate, integrals giving the volume of \( D \) in cartesian, cylindrical, and spherical coordinates.

Answer: These surfaces intersect where \( x^2 + y^2 = 1 \) and the shadow of \( D \) in the \( xy \) plane is the disc \( x^2 + y^2 \leq 1 \). So in cartesian and cylindrical coordinates:

\[
\text{volume} = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2=1} dz \, dy \, dx
\]

\[= \int_0^{2\pi} \int_0^{1} \int_{r^2=1} r \, dz \, dr \, d\theta \]

For spherical coordinates there are different limits for \( \rho \) above and below the \( xy \) plane. Above we have \( z = 1 - x^2 - y^2 \) so \( \rho \cos \phi = 1 - \rho^2 \sin^2 \phi \), so \( \rho^2 \sin^2 \phi + \rho \cos \phi - 1 = 0 \) and by the quadratic formula, \( \rho = \frac{-\cos \phi + \sqrt{\cos^2 \phi + 4 \sin^2 \phi}}{2 \sin^2 \phi} = \frac{-\cos \phi + \sqrt{1 + 3 \sin^2 \phi}}{2 \sin^2 \phi} \). Similarly for \( \phi \geq \pi/2 \) we have \( \rho^2 \sin^2 \phi - \rho \cos \phi - 1 = 0 \) so \( \rho = \frac{\cos \phi + \sqrt{1 + 3 \sin^2 \phi}}{2 \sin^2 \phi} \). So here are a few ways to compute the volume in spherical coordinates

\[
\int_{0}^{2\pi} \int_{\pi/2}^{\pi} \int_{0}^{\rho_{max}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_{0}^{2\pi} \int_{\pi/2}^{\pi} \int_{0}^{\rho_{max}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{\rho_{max}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[2 \int_{0}^{2\pi} \int_{0}^{\pi/2} \int_{0}^{\rho_{max}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]
The last expression comes from the observation that $D$ is symmetric about the $xy$ plane so the volume of $D$ is twice the volume of the part of $D$ above the $xy$ plane.

7. (20) Let $D$ be the region in the first quadrant bounded by $xy = 1$, $xy = 2$, $x = y$, and $x - y = 3 - xy$. Find $\int \int_D x + y \, dA$.

Answer: Let $u = xy$ and $v = x - y$. Then in the $uv$ plane the region is bounded by $u = 1$, $u = 2$, $v = 0$, and $v = 3 - u$. $\partial(u, v) / \partial(x, y) = \det \begin{pmatrix} y & x \\ 1 & -1 \end{pmatrix} = -x - y$ so the integral is

$$\int_1^2 \int_0^{3-u} (x + y) \frac{1}{-x - y} \, dv \, du = \int_1^2 \int_0^{3-u} \, dv \, du$$

$$= \int_1^2 3 - u \, du = 3u - u^2 / 2 \bigg|_1^2 = 6 - 2 - 3 + 1/2 = 3/2$$

8. (20) Let $F(x, y, z) = (3x^2 + y \sin(xy)) \mathbf{i} + (z + x \sin(xy)) \mathbf{j} + (y + 4z) \mathbf{k}$. Let $C$ be the curve parameterized by $r(t) = t^8 \mathbf{i} + t^9 (t + 1) \mathbf{j} - e^{\sin(\pi t)} \mathbf{k}$ for $0 \leq t \leq 1$. Find $\int_C F \cdot T \, ds$.

Answer: Note $F$ is conservative. Trying to solve $F = \nabla g$ we get

$$\partial g / \partial x = 3x^2 + y \sin(xy) \implies g(x, y, z) = x^3 - \cos(xy) + C(y, z)$$

$$z + x \sin(xy) = \partial g / \partial y = 0 + x \sin(xy) + \partial C(y, z) / \partial y \implies C(y, z) = yz + D(z)$$

$$y + 4z = \partial g / \partial z = y + D'(z) \implies D(z) = 2z^2 + E$$

so we may let $g(x, y, z) = x^3 - \cos(xy) + yz + 2z^2$. $C$ begins at $(0, 0, -1)$ and ends at $(1, 2, -1)$ so

$$\int_C F \cdot T \, ds = g(1, 2, -1) - g(0, 0, -1) = 1 - \cos 2 - 2 + 2 - (0 - 1 + 0 + 2) = -\cos 2$$

9. (20) True or false. (no justification required).

a) If $A$ is a square matrix and $NS(A) = \{0\}$ then $A^{-1}$ exists.

Answer: True, see page 55 or 89-90 of Cullen

b) $(AB)^T = A^T B^T$.

Answer: False, $(AB)^T = B^T A^T$.

c) $\det(AB) = \det(A) \det(B)$.

Answer: True, see page 105 of Cullen
d) Any 5 vectors in $\mathbb{R}^4$ are linearly dependent.

Answer: True, see Thm 2.7 page 84 of Cullen

e) Any 5 vectors in $\mathbb{R}^4$ span $\mathbb{R}^4$.

Answer: False, for example $(1,1,1,1), (2,2,2,2), (3,3,3,3), (4,4,4,4), (5,5,5,5)$ span a one dimensional subspace.

f) Any 3 vectors in $\mathbb{R}^4$ span a subspace of $\mathbb{R}^4$.

Answer: True, the span of any collection of vectors is a subspace. You even know in this case that they span a proper subspace of $\mathbb{R}^4$, i.e., a subspace which is definitely smaller than $\mathbb{R}^4$, since the dimension of the span is at most 3.

g) If $A$ is a $5 \times 7$ matrix then the dimension of $NS(A)$ plus the rank of $A$ is 5.

Answer: False, it is 7, see page 89 of Cullen