1. [30] Let \( f(x, y, z) = z^2 + x^2 + xy^2 + e^{yz} \)
a) Find the gradient of \( f \) at \((1, 2, 0)\).
\[ \nabla f = (2x + y^2, 2xy + ze^{yz}, 2z + ye^{yz}) = (6, 4, 2) \text{ at } (1, 2, 0). \]
b) In which direction is \( f \) increasing most rapidly at the point \((1, 2, 0)\)? (Your answer should be a unit vector with this direction.)
\[ \frac{(6, 4, 2)}{\sqrt{36 + 16 + 4}} = \frac{(6, 4, 2)}{\sqrt{56}}. \]
c) Find \( \lim_{(x,y,z) \to (1,2,0)} f(x, y, z) \).
\[ \text{Since } f \text{ is continuous, the limit is } f(1, 2, 0) = 6. \]
d) Find an equation for the tangent plane at \((1, 2, 0)\) of the level surface \( f(x, y, z) = 6 \).
\[ 6(x - 1) + 4(y - 2) + 2z = 0 \]
e) The level surface \( f(x, y, z) = 6 \) defines \( z \) implicitly as a function of \( x \) and \( y \) near \((1, 2, 0)\). Find \( \frac{\partial z}{\partial y} \) at \( x = 1, y = 2, z = 0 \).
\[ \frac{\partial z}{\partial y} = -\frac{\partial f / \partial y}{\partial f / \partial z} = -\frac{4}{2} = -2 \]

2. [10] Find the boundary of each of the following subsets of the plane. Which of these subsets are open? Which of these subsets are closed?
a) \( A = \{(x, y) \mid x^2 + y^2 \leq 2 \} \).
\[ \text{The boundary is the circle with equation } x^2 + y^2 = 2. \text{ A is closed since it contains its boundary. It is not open.} \]
b) \( B = \{(x, y) \mid x^2 + y^2 > 1 \} \).
\[ \text{The boundary is the circle with equation } x^2 + y^2 = 1. \text{ B is open since it is disjoint from its boundary. It is not closed.} \]
c) \( C = A \cap B = \{(x, y) \mid 1 < x^2 + y^2 \leq 2 \} \).
\[ \text{The boundary is the two circles with equations } x^2 + y^2 = 1 \text{ and } x^2 + y^2 = 2. \text{ C is neither open nor closed since it does not contain all of its boundary, but does contain some points of its boundary.} \]

3. [30] Let \( R \) be the region in the first octant which lies between the cone \( z^2 = x^2 + y^2 \) and the paraboloid \( z = x^2 + y^2 \). Set up, but do not evaluate, \( \int \int \int_R x + 2y + 3z\, dV \)
a) in cartesian coordinates.
\[ \text{The cone and paraboloid intersect where } z^2 = z, \text{ so } z = 0, 1. \text{ The intersection at } z = 0 \text{ is just the origin, but at } z = 1 \text{ the intersection is an arc of the circle } 1 = x^2 + y^2. \text{ The projection of } R \text{ to the } xy \text{ plane is then the quarter disc } x^2 + y^2 \leq 1, x \geq 0, y \geq 0 \text{ in the first quadrant. The cone lies above the paraboloid as can be seen by testing a point, for example if } x = 1/4, y = 0, \text{ then } z = \sqrt{1/16} = 1/2 \text{ on the cone, but } z = 1/4 \text{ on the paraboloid. So } \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} x + 2y + 3z\, dz\, dy\, dx. \text{ Some of you used a different order of integration,} \]
for example \( dxdydz \). This requires setting up the integral as the sum or difference of two integrals, for example \( \int_0^1 \int_0^{\sqrt{\frac{z-y^2}{2}}} \int_0^{\frac{z-y^2}{2}} x + 2y + 3z \, dxdydz \). 

b) in cylindrical coordinates.

**Answer:** The cone has equation \( z = r \) and the paraboloid is \( z = r^2 \). So in the usual order, \( \int_0^{\pi/2} \int_0^r r^2 \cos \theta + 3r \, dr \, d\theta \). But if you wanted to do another order, here is a possibility, \( \int_0^1 \int_0^{\pi/2} \int_0^{\sqrt{z-y^2}} r^2 \cos \theta + 3r \, dr \, d\theta \).

c) in spherical coordinates.

**Answer:** The equation of the cone \( z = r \) is \( \rho \cos \phi = \rho \sin \phi \) which reduces to \( \phi = \pi/4 \). The equation of the paraboloid \( z = r^2 \) is \( \rho \cos \phi = \rho^2 \sin^2 \phi \) which reduces to \( \rho = \cos \phi / \sin^2 \phi \). So \( \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \phi \, \rho \sin \phi \cos \theta + 2\rho \sin \phi \sin \theta + 3\rho \cos \phi \rho^2 \sin \phi \, dp d\phi d\theta \).

4. [15] Calculate one of the following integrals where \( D \) is the region in the first quadrant bounded by the lines \( y = x, y = x + 1 \) and by the ellipses \( x^2 + 4y^2 = 4 \) and \( x^2 + 4y^2 = 16 \). You must clearly indicate which one of the three integrals you choose to evaluate. Choose wisely.

a) \( \int \int_D e^{y-x} \, dA \).

b) \( \int \int_D (x + 4y) e^{y-x} \, dA \).

c) \( \int \int_D e^{y-x} / (x + 4y) \, dA \).

**Answer:** A good coordinate change is \( u = y-x, v = x^2 + 4y^2 \). Then \( \partial(u,v)/\partial(x,y) = \det \begin{bmatrix} -1 & 1 \\ 2x & 8y \end{bmatrix} = -8y - 2x \). So \( \partial(x,y)/\partial(u,v) = -1 / (8y + 2x) \). So b) is a good choice to evaluate.

\[
\int \int_D (x + 4y) e^{y-x} \, dA = \int_0^1 \int_4^{16} (x + 4y) e^u \left| \frac{-1}{8y + 2x} \right| \, dv \, du \\
= \int_0^1 \int_4^{16} e^u / 2 \, dv \, du = \int_0^1 ve^u / 2 \big|_4^{16} \, du = \int_0^1 6e^u \, du = 6e^u \big|_0^1 = 6e - 6
\]

5. [15] A region \( D \) in space has volume 3 and centroid \( (\bar{x}, \bar{y}, \bar{z}) = (1, 2, -1) \). Calculate the following integrals:

a) \( \int \int \int_D 4 \, dV \).

**Answer:** \( \int \int \int_D 4 \, dV = 4 \int \int_D 1 \, dV = 4 \cdot 3 = 12 \).

b) \( \int \int \int_D x \, dV \).

**Answer:** \( 1 = \bar{x} = \int \int \int_D x \, dV / 3 \) so \( \int \int \int_D x \, dV = 3 \).

c) \( \int \int \int_D 3 + 2x - 4y \, dV \).

**Answer:** \( \int \int \int_D y \, dV = 3\bar{y} = 6 \) so \( \int \int \int_D 3 + 2x - 4y \, dV = 3 \cdot 3 + 2 \cdot 3 - 4 \cdot 6 = -9 \).