1. (25) Let $W_1 \subset \mathbb{R}^{2 \times 2}$ be the set of matrices with trace 0, that is, matrices of the form $\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$. Let $W_2 \subset \mathbb{R}^{2 \times 2}$ be the set of symmetric matrices, that is, matrices of the form $\begin{bmatrix} x & y \\ y & z \end{bmatrix}$.

a) Show that $W_1$ and $W_2$ are both subspaces.

\[ c \begin{bmatrix} x & y \\ z & -x \end{bmatrix} + c' \begin{bmatrix} x' & y' \\ z' & -x' \end{bmatrix} = \begin{bmatrix} cx + x' & cy + y' \\ cz + z' & -cx - x' \end{bmatrix}. \]

Since $-(cx + x') = -cx - x'$, this linear combination of elements of $W_1$ is still in $W_1$. Since $0 \in W_1$ we then know $W_1$ is a subspace.

\[ c \begin{bmatrix} x & y \\ y & z \end{bmatrix} + c' \begin{bmatrix} x' & y' \\ y' & z' \end{bmatrix} = \begin{bmatrix} cx + x' & cy + y' \\ cy + y' & cz + z' \end{bmatrix}. \]

Since the off diagonal entries are equal, this linear combination of elements of $W_2$ is still in $W_2$, and since $W_2$ is nonempty we know $W_2$ is a subspace.

b) Find the dimensions of $W_1$, $W_2$, $W_1 \cap W_2$ and $W_1 + W_2$ (and give sufficient reasons).

A basis of $W_1$ is $\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \}$ since they are linearly independent and span $W_1$ so $\dim W_1 = 3$. A basis of $W_2$ is $\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$ since they are linearly independent and span $W_2$ so $\dim W_2 = 3$. $W_1 \cap W_2$ is all matrices of the form $\begin{bmatrix} x & y \\ y & -x \end{bmatrix}$ so it has basis $\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \}$ and hence has dimension 2. $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = 3 + 3 - 2 = 4$. Alternatively, since $W_1$ is not contained in $W_2$ we know $\dim(W_1 + W_2) > \dim W_2 = 3$. But $\dim(W_1 + W_2) \leq \dim \mathbb{R}^{2 \times 2} = 4$ so we must have $\dim(W_1 + W_2) = 4$. Then $\dim(W_1 \cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1 + W_2) = 3 + 3 - 4 = 2$.

c) For 10 points extra credit, you may instead do this problem where $W_1$ is the trace 0 matrices in $\mathbb{C}^{2 \times 2}$, $W_2$ is the Hermitian matrices in $\mathbb{C}^{2 \times 2}$ and the field is $\mathbb{R}$.

$\dim W_1 = 6$ with basis $\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix} \}$. 

\( \dim W_2 = 4 \) with basis \( \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \), \( \dim(W_1 \cap W_2) = 3 \) with basis \( \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right\} \), \( \dim(W_1 + W_2) = 6 + 4 - 3 = 7 \). \( W_1 + W_2 \) is the set of matrices whose trace is real.

2. (25) Let \( T \) be the linear operator on \( \mathbb{C}^2 \) defined by

\[
T(x_1, x_2) = (-x_1 + x_2, x_1 - x_2).
\]

a) Show that \( B = \{(1, 1), (i, -i)\} \) is a basis for \( \mathbb{C}^2 \).

\( a(1, 1) + b(i, -i) = (0, 0) \) implies \( a + bi = 0 \) and \( a - bi = 0 \) so \( 2a = a + bi + a - bi = 0 \) so \( a = 0 \) so \( 0 + bi = 0 \) so \( b = 0 \). So \( \{1, 1\}, \{i, -i\} \)

is linearly independent and hence forms a basis of the 2 dimensional space \( \mathbb{C}^2 \).

b) Find the matrix \( [T]_B \) of \( T \) in the ordered basis \( \{(1, 1), (i, -i)\} \).

\( T(1, 1) = (0, 0) = 0(1, 1) + 0(i, -i) \) so \( [T(1, 1)]_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). \( T(i, -i) = (-2i, 2i) = 0(1, 1) - 2(i, -i) \) so \( [T(i, -i)]_B = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \). So \( [T]_B = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \).

3. (25) Let \( V \) be a finite dimensional vector space and let \( W \subset V \) be a subspace. Show that there is a linear transformation \( T: V \to V \) so that the

range \( R_T \) of \( T \) is \( W \) and so \( T^2 = T \). What is the dimension of the null space of \( T \)?

Choose a basis \( \{\beta_1, \ldots, \beta_k\} \) of \( W \). Extend this to a basis \( \{\beta_1, \ldots, \beta_n\} \) of \( V \). There is a unique linear transformation \( T: V \to V \) so that \( T(\beta_i) = \beta_i \) and \( i \leq k \) and \( T(\beta_i) = 0 \) for \( i > k \). Note that \( T^2(\beta_i) = T(\beta_i) \) for all \( i \) so by uniqueness we know \( T^2 = T \). The range of \( T \) is the subspace spanned by all \( \beta_i \) for \( i \leq k \), which is \( W \). \( \dim NS(W) = \dim V - \dim R_T = \dim V - \dim W \).

4. (25) Let \( B = \{\beta_1, \beta_2, \ldots, \beta_n\} \) be an ordered basis for \( V \) and let \( \{\beta_1^*, \beta_2^*, \ldots, \beta_n^*\} \) be its dual basis. Show that the \( B \) coordinates of any
\( \alpha \in V \) are given by \([\alpha]_B = \begin{bmatrix} \beta_1^*(\alpha) \\ \beta_2^*(\alpha) \\ \vdots \\ \beta_n^*(\alpha) \end{bmatrix} \).

If \( \alpha = \sum_{i=1}^n c_i \beta_i \) then \( \beta_j^*(\sum_{i=1}^n c_i \beta_i) = \sum_{i=1}^n c_i \beta_j^*(\beta_i) = c_j \) since \( \beta_j^*(\beta_i) = 0 \) for \( j \neq i \) and \( \beta_j^*(\beta_j) = 1 \). So

\[
\begin{bmatrix} \beta_1^*(\alpha) \\ \beta_2^*(\alpha) \\ \vdots \\ \beta_n^*(\alpha) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\alpha]_B.
\]