1. (25) For each set $H$ below determine whether or not it is a subspace. If it is not a subspace, show why it is not. If it is a subspace and is finite dimensional, write down a basis for $H$ and determine the dimension of $H$.
   a) $H$ is the set of upper triangular matrices in $M_{3 \times 3}$.
   Answer: The sum of two upper triangular matrices is upper triangular. If you multiply an upper triangular matrix by a scalar, it is still upper triangular. The 0 matrix is upper triangular. So $H$ is a subspace. An example of a basis is $B = \{ E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33} \}$ where $E_{ij}$ is the matrix which is zero in all but the entry in the $i$-th row and $j$-th column, and this entry is 1. If $A$ is an upper triangular $3 \times 3$ matrix with $ij$-th entry $a_{ij}$ then $A = a_{11}E_{11} + a_{12}E_{12} + a_{13}E_{13} + a_{22}E_{22} + a_{23}E_{23} + a_{33}E_{33}$ so $B$ spans $H$. But $B$ is also linearly independent since if $c_{11}E_{11} + c_{12}E_{12} + c_{13}E_{13} + c_{22}E_{22} + c_{23}E_{23} + c_{33}E_{33} = 0$ then all $c_{ij} = 0$. So $B$ is a basis and thus $H$ has dimension 6.

   b) $H = \text{Span}\{ v_1, v_2, v_3 \}$ where $v_1$ are nonzero vectors in a vector space $V$, $v_1 = 2v_2 - 5v_3$, and $v_2$ is not a scalar multiple of $v_3$.
   Answer: We know $H$ is a subspace because the span of a bunch of vectors is always a subspace. Since $v_1$ is a linear combination of $v_2$ and $v_3$ we know that $\text{Span}\{ v_2, v_3 \} = \text{Span}\{ v_1, v_2, v_3 \} = H$. But since $v_2$ is not a scalar multiple of $v_3$ we also know that $\{ v_2, v_3 \}$ is linearly independent. Since $B = \{ v_2, v_3 \}$ is linearly independent and spans $H$, we know it is a basis of $H$. Then $H$ has dimension 2. Actually, $\{ v_1, v_2 \}$ and $\{ v_1, v_3 \}$ and $\{ v_1 + v_2, v_2 - v_3 \}$ are other examples of bases of $H$, but the reasoning to show it is more involved.

   c) $H$ is the set of polynomials $p$ in $\mathbb{P}_3$ so that $p(0) = 1$.
   Answer: This is not a subspace for several reasons:
   1) The zero polynomial is not in $H$.
   2) If $p(0) = 1$ and $q(0) = 1$ then $(p + q)(0) = 1 + 1 = 2$ so $p + q$ is not in $H$.
   3) If $p(0) = 1$ then $cp(0) = c$ so $cp$ is not in $H$ if $c \neq 1$.

   d) $H = \left\{ \begin{bmatrix} 2s + 3t \\ s - 2t \\ 5t \end{bmatrix} : s, t \in \mathbb{R} \right\}$.
   Answer: Note that $H = \text{Span}\{ (2 1 0)^T, (3 - 2 5)^T \}$ so $H$ is a subspace. Since $\{ (2 1 0)^T, (3 - 2 5)^T \}$ is linearly independent it forms a basis of $H$ and $H$ has dimension 2.

2. (15) Let $B = \{ [1 2]^T, [0 1]^T \}$ and $C = \{ [0 1]^T, [1 0]^T \}$ be two bases of $\mathbb{R}^2$. Find the coordinate change matrix $P_{B\rightarrow C}$ from $C$ to $B$ coordinates.
   Answer: We know $P_{B\rightarrow C} = (P_{C\rightarrow B})^{-1}$. Since $c_1[1 2]^T + c_2[0 1]^T = (2c_1 + c_2)[0 1]^T + c_1[1 0]^T$ we know that $P_{C\rightarrow B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. So $P_{B\rightarrow C} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$. You could also solve
this by noting that $[0 1]^T = 0 \cdot [1 2]^T + 1 \cdot [0 1]^T$ so the first column is $[0 1]^T$, and
$[1 0]^T = 1 \cdot [1 2]^T - 2 \cdot [0 1]^T$ so the second column is $[1 -2]^T$.

3. (15) Solve the equation $\begin{bmatrix} I_5 & X \\ 0 & I_5 \end{bmatrix} \begin{bmatrix} A \\ Y \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix}$ for $X$ and $Y$ in terms of $A$, $B$, and $C$. Assume that $A$, $B$, and $C$ are invertible $5 \times 5$ matrices.

Answer: This is a chapter 2 problem.

4. (10) Determine whether or not $\{2t^2, (t-2)^2, t-1\}$ is a basis for $P_2$. If it is a basis, find the coordinate vector of $p(t) = t + 1$ relative to this basis.

Answer: Solve $a2t^2 + b(t-2)^2 + c(t-1) = 0$ and we get $(2a+b)t^2 + (-4b+c)t + (4b-c) = 0$. The solution is $c = 4b, a = -b/2$. In particular we may take $a = -1, b = 2, c = 8$ and get a nontrivial linear combination equals 0. So the vectors are linearly dependent and thus could not be a basis.

5. (30) Indicate whether each statement is true or false.
   a) rank($A$) = rank($A^T$).
   Answer: True
   b) If $A$ is invertible, then rank($A$) = rank($A^{-1}$).
   Answer: True, since the rank in each case must be $n$ if $A$ is $n \times n$.
   c) Any four vectors which span a four dimensional vector space $V$ form a basis for $V$.
   Answer: True
   d) If $v_1, v_2, v_3$ are linearly independent vectors in a four dimensional vector space $V$, then there is a vector $v_4$ so that $v_1, v_2, v_3, v_4$ is a basis for $V$.
   Answer: True
   e) Every vector space has a basis.
   Answer: False, it must be finite dimensional. (Actually this is true with a more general definition of basis which you are not likely to see in an undergraduate course. But as far as our definition of basis is concerned it is false.)
   f) The vector spaces $M_{2 \times 3}$ and $P_3$ are isomorphic.
   Answer: True, since they both have dimension 6.
   g) $\text{det}(2A) = 2\text{det}(A)$.
   Answer: False. $\text{det}(2A) = 2^n \text{det}(A)$ if $A$ is $n \times n$.
   h) $\text{det}(A) = \text{det}(A^T)$.
   Answer: True
   i) If $T: V \rightarrow W$ is a linear transformation, then the kernel of $T$ is a subspace of $V$.
   Answer: True
   j) If $T: V \rightarrow W$ is a linear transformation, then the kernel of $T$ is a subspace of $W$.
   Answer: False, (though the Range of $T$ is a subspace of $W$).

6. (10) Short answer.
a) If $A$ is an $m \times n$ matrix, then $\text{rank}(A) + \text{dim Nul}(A) = n$

b) If $B$ is obtained from $A$ by switching two rows of $A$, then $\det(B) = -\det(A)$

7. (20) Let $T: V \to W$ be a linear transformation. Given a subspace $U$ of $V$, let $T(U)$ denote the set of all images of the form $T(x)$, where $x$ is in $U$. Show that $T(U)$ is a subspace of $W$.

Answer: If $w_1$ and $w_2$ are in $T(U)$ then there are $x_1$ and $x_2$ in $U$ so that $T(x_1) = w_1$ and $T(x_2) = w_2$. Then $w_1 + w_2 = T(x_1) + T(x_2) = T(x_1 + x_2)$. But $x_1 + x_2$ is in $U$ since $U$ is a subspace. So $w_1 + w_2$ is in $T(U)$. Likewise, for any scalar $c$ we have $cw_1 = cT(x_1) = T(cx_1)$ which is in $T(U)$ since $cx_1$ is in $U$. Finally $0 = T(0)$ so $0$ is in $T(U)$. So $T(U)$ is a subspace of $W$ since it satisfies the three subspace conditions.