1. (25) For each of the following matrices:
   + Find its eigenvalues and an eigenvector for each eigenvalue.
   + If possible, find a (possibly complex) matrix $P$ and a diagonal matrix $D$ so that the 
given matrix equals $PDP^{-1}$.
   + If possible, find a real matrix $Q$ so that the given matrix is $QCQ^{-1}$ where $C$ is of the 
form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

   a) $\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}$
   b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
   c) $\begin{bmatrix} -6 & -15 \\ 3 & 6 \end{bmatrix}$

Answer: The characteristic polynomial of a) is $(4 - \lambda)(-2 - \lambda) - 4(-2) = \lambda^2 - 2\lambda$ 
so the eigenvalues are 0 and 2. For $\lambda = 2$ the eigenvectors are nonzero vectors in the 
null space of $\begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}$ so an eigenvector is $(2, -1)^T$. For $\lambda = 0$ the eigenvectors are 
nonzero vectors in the null space of $\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}$ so an eigenvector is $(1, -1)^T$. So the 
matrix is diagonalizable and equals $PDP^{-1}$ where $P = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

It cannot be $QCQ^{-1}$ since its eigenvalues would then be the eigenvalues of $C$ which are 
a $\pm b\sqrt{-1}$. The characteristic polynomial of b) is $(1 - \lambda)(1 - \lambda) - 2(0) = (\lambda - 1)^2$ so the 
only eigenvalue is 1. The eigenvectors are nonzero vectors in the null space of $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ so 
an eigenvector is $(1, 0)^T$. It is not diagonalizable since the eigenvalue 1 has multiplicity 2 
but its eigenspace is only one dimensional. It is not $QCQ^{-1}$ since the eigenvalues of $C$ are 
a $\pm b\sqrt{-1}$ which would mean $a = 1$ and $b = 0$ so $C$ would be diagonal, but the matrix 
is not diagonalizable. The characteristic polynomial of c) is $(-6 - \lambda)(6 - \lambda) - 3(-15) = \lambda^2 + 9$ so the eigenvalues are $\pm 3\sqrt{-1}$. For $\lambda = 3\sqrt{-1}$ the eigenvectors are nonzero vectors 
in the null space of $\begin{bmatrix} -6 - 3\sqrt{-1} & -15 \\ 3 & 6 - 3\sqrt{-1} \end{bmatrix}$ so an eigenvector is $(6 - 3\sqrt{-1}, -3)^T$ or 
more simply $(-2 + \sqrt{-1}, 1)^T$. For $\lambda = -3\sqrt{-1}$ an eigenvector is the complex conjugate, 
$(-2 - \sqrt{-1}, 1)^T$. So the matrix is $PDP^{-1}$ with $P = \begin{bmatrix} -2 + \sqrt{-1} & -2 - \sqrt{-1} \\ 1 & 1 \end{bmatrix}$ and 
$D = \begin{bmatrix} 3\sqrt{-1} & 0 \\ 0 & -3\sqrt{-1} \end{bmatrix}$. It is also $QCQ^{-1}$ where $Q = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$ has columns which are 
the real and imaginary parts of the eigenvectors.

2. (20) A 5 × 5 matrix $A$ has three eigenvalues 1, 3, and 6. The eigenspace of $A$ 
corresponding to $\lambda = 1$ is three dimensional.
   a) What is the characteristic polynomial of $A$?

Answer: We know that $\lambda = 1$ has multiplicity $\geq 3$. So since it has degree 5 the only 
possibility is $-(\lambda - 1)^3(\lambda - 3)(\lambda - 6)$. (The minus sign comes because the coefficient
of $\lambda^n$ in Lay’s version of the characteristic polynomial is $(-1)^n$ for an $n \times n$ matrix. I did not take off points if you got the sign wrong.)

b) Must $A$ be diagonalizable? Why or why not?
   
   **Answer:** Since the eigenspace dimensions all equal the multiplicities we know $A$ must be diagonalizable.

c) Must $A$ be invertible? Why or why not?
   
   **Answer:** Since 0 is not an eigenvalue, the null space of $A$ is trivial, so $A$ is invertible.

d) Find the ranks of the matrices $A$, $A - I_5$ and $A - 3I_5$.
   
   **Answer:** Since $A$ is invertible, $\text{rank}(A) = 5$. The eigenspace for $\lambda = 1$ is the null space of $A - I_5$ so $\text{rank}(A - I_5) = 5 - \text{dim}(\text{Null}(A - I_5)) = 2$. Likewise, $\text{rank}(A - 3I_5) = 5 - \text{dim}(\text{Null}(A - 3I_5)) = 4$ since the eigenspace for $\lambda = 3$ has dimension at most the multiplicity 1.

3. (15) Suppose $A$ is a $3 \times 3$ matrix so that

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
2 & 2 \\
0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix},
\]

and so that

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

is in the null space of $A$.

a) What are the eigenvalues of $A$?
   
   **Answer:** Since $A \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
0 & 0
\end{bmatrix}$ we know 2 is an eigenvalue, since $A \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}$ we know $-1$ is an eigenvalue, and since $A \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}$ we know 0 is an eigenvalue.

   Since $A$ is $3 \times 3$ it has at most 3 eigenvalues so the eigenvalues are 2, $-1$, and 0.

b) Determine $A$. (You may leave your answer as a product of matrices and their inverses.)
   
   **Answer:** We know from part a) that $A = PD P^{-1}$ where $P = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}$ and

\[
D = \begin{bmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

c) Determine $A^k$ for any integer $k > 0$. (You may leave your answer as a product of matrices and their inverses.)
   
   **Answer:** We know $A^k = PD^k P^{-1}$ where $P$ and $D$ are as in part b), so

\[
A^k = P \begin{bmatrix}
2^k & 0 & 0 \\
0 & (-1)^k & 0 \\
0 & 0 & 0
\end{bmatrix} P^{-1}
\]
4. (10) Let \( T : \mathbb{P}_2 \to \mathbb{P}_2 \) be the linear transformation which takes a polynomial \( p(t) \) to \( tp'(t) \). Find the matrix \([T]_B\) of \( T \) with respect to the basis \( B = \{1, t, t^2\} \).

Answer: \( T(1) = t \cdot 1' = 0, T(t) = t \cdot t' = t, \) and \( T(t^2) = t \cdot (t^2)' = 2t^2. \) So \([T]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \).

5. (15) Suppose \( B = P^{-1}AP \).

a) Then by definition, \( A \) is ________ to \( B \).

Answer: similar

b) Show that if \( x \) is an eigenvector of \( A \) with eigenvalue \( \lambda \), then \( P^{-1}x \) is an eigenvector of \( B \) with eigenvalue \( \lambda \).

Answer: We are given that \( x \neq 0 \) and \( Ax = \lambda x \). If \( P^{-1}x = 0 \) then \( x = PP^{-1}x = P(0) = 0 \), a contradiction, so \( P^{-1}x \neq 0 \). Now \( BP^{-1}x = P^{-1}APP^{-1}x = P^{-1}Ax = P^{-1}(\lambda x) = \lambda P^{-1}x \), so \( P^{-1}x \) is an eigenvector of \( B \) with eigenvalue \( \lambda \).

c) Show that \( \det A = \det B \).

Answer:

\[
\det(B) = \det(P^{-1}AP) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)(1/\det(P)) = \det(A)
\]

6. (15) Same as a problem on Test 2.