Simple Quadrature Rules

Single Panel Midpoint Rule
\[
\int_a^b f(x)dx = (b - a)f(\frac{a + b}{2}) + E_M(f)
\]
where
\[
E_M(f) = \frac{(b - a)^3}{24}f''(\xi)
\]
for some \(\xi\) in the interval \([a, b]\).

Composite Midpoint Rule. Let \(h = (b - a)/n\), and \(x_j = a + j \cdot h, \ j = 0, \ldots, n\) and let \(s_j = (1/2)(x_{j-1} + x_j), j = 1, \ldots, n\) be the midpoint of each subinterval. Then
\[
\int_a^b f(x)dx = h[f(s_1) + f(s_2) + \cdots + f(s_n)] + E_M(f, h)
\]
where
\[
E_M(f, h) = \frac{h^2(b - a)}{24}f''(\eta)
\]
for some \(\eta\) in the interval \([a, b]\).

Single Panel Trapezoid Rule
\[
\int_a^b f(x)dx = \frac{h}{2}(f(a) + f(b)) + E_T(f)
\]
where
\[
E_T(f) = -\frac{(b - a)^3}{12}f''(\xi)
\]
for some \(\xi\) in the interval \([a, b]\).

Composite Trapezoid Rule
\[
\int_a^b f(x)dx = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] + E_T(f, h)
\]
where
\[
E_T(f, h) = -\frac{h^2(b - a)}{12}f''(\eta)
\]
for some \(\eta\) in the interval \([a, b]\).

Single Panel Simpson Rule
\[
\int_a^b f(x)dx = \frac{h}{6}[f(a) + 4f(\frac{a + b}{2}) + f(b)] + E_S(f)
\]
where
\[
E_S(f) = -\frac{(b - a)^5}{2880}f^{(4)}(\xi)
\]
Composite Simpson Rule Assume $n$ is even. Then

$$
\int_a^b f(x) \, dx = \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n)] + E_S(f, h)
$$

where

$$
E_S(f, h) = -\frac{(b-a)h^4}{180} f^{(4)}(\eta)
$$

for some $\eta$ in $[a, b]$.

Simpson’s rule is related to the trapezoid rule and the midpoint rule by the equation

$$
S(f) = \frac{2}{3} M(f) + \frac{1}{3} T(f).
$$

The midpoint rule and the trapezoid rule are both exact on polynomials of degree $\leq 1$, but not exact on $x^2$. Hence the midpoint rule and trapezoid rule and both of order 2. Simpson’s rule is exact is exact on polynomials of degree $\leq 3$, but not on $x^4$. Hence Simpson’s rule is of order 4.

Estimates of the error. Use $T_n(f)$ to denote $T(f, h)$ where $h = (b-a)/n$. From the form of the error for $T_n$, assuming $f''$ changes slowly, we can deduce that

$$
|E_T(f, n)| \approx \frac{4}{5} |T_n(f) - T_{2n}(f)|.
$$

Note that we can write

$$
T_{2n}(f) = \frac{1}{2} T_n((f) + \frac{h}{2} [f(s_1) + \cdots + f(s_n)]
$$

where $s_j$ is the midpoint of the $n^{th}$ subinterval.

For Simpson’s rule we can deduce that

$$
|E_S(f, n)| \approx \frac{16}{15} |S_n(f) - S_{2n}(f)|.
$$

We can get a new rule of higher order by forming an average of $S(f, n)$ and $S(f, 2n)$. In fact,

$$
Q(f) = \frac{16 S(f, 2n) - S(f, n)}{15}
$$

has order 6. It is a five point Newton Cotes rule.

Gaussian Quadrature

We begin with a three point rule on $[-1, 1]$. Let

$$
Q(g) = A_1 g(t_1) + A_2 g(t_2) + A_3 g(t_3).
$$
We want to choose the weights \( A_1, A_2, A_3 \) and the nodes \( t_1, t_2, t_3 \) so as to maximize the order of \( Q \). To economize in the derivation, we shall assume some symmetry: We take \( t_1 = -t_3 \) and \( t_2 = 0 \), and \( A_1 = A_3 \). Thus our rule becomes

\[
Q(g) = A_1 g(-t_3) + A_2 g(0) + A_1 g(t_3).
\]

With this symmetry, we have that for any power \( t^p \) with \( p \) odd, \( Q(t^p) = 0 = \int_{-1}^{1} t^p dt \). Thus we shall determine \( A_1, A_2 \) and \( t_3 \) by requiring that \( Q \) be exact on the even powers \( g(t) \equiv 1, g(t) = t^2 \) and \( g(t) = t^4 \). This yields the equations

\[
Q(1) = A_1 + A_2 + A_1 = \int_{-1}^{1} 1 dt = 2
\]

or

\[
2A_1 + A_2 = 2. \tag{1}
\]

\[
Q(t^2) = A_1 t_3^2 + A_1 t_3^2 = \int_{-1}^{1} t^2 dt = 2/3
\]

or

\[
A_1 t_3^2 = 1/3. \tag{2}
\]

Finally we require that

\[
Q(t^4) = A_1 t_3^4 + A_1 t_3^4 = \int_{-1}^{1} t^4 dt = 2/5
\]

or

\[
A_1 t_3^4 = 1/5. \tag{3}
\]

Dividing equation (3) by equation (2), we find \( t_3^2 = 3/5 \), whence \( t_3 = \sqrt{3/5} \). Substitution of this value of \( t_3 \) into (2) yields \( A_1 = 5/9 \), and finally (1) yields \( A_2 = 8/9 \). Hence our three-point Gaussian quadrature rule is

\[
G_3(g) = \frac{5}{9} g(-\sqrt{3/5}) + \frac{8}{9} g(0) + \frac{5}{9} g(\sqrt{3/5}).
\]

It has order 6 because it integrates exactly \( g(t) = 1, t, t^2, t^3, t^4 \), but not \( g(t) = t^5 \). Use the map \( \varphi(t) = a + \frac{(b-a)}{2} (t + 1) \) to make the change of variable from \([-1, 1]\) to \([a, b]\). If \( f \) given on \([a, b]\), set \( g(t) = f(\varphi(t)) \). Then

\[
G_3(f) = G_3(g) = \frac{(b-a)}{2} \frac{5}{9} f(\varphi(-\sqrt{3/5}) + \frac{8}{9} f(\varphi(0)) + \frac{5}{9} f(\varphi(\sqrt{3/5}))
\]

and the error is

\[
E_3(f) = \left[ \frac{(b-a)}{2} \right] \frac{1}{15750} f^{(6)}(\eta).
\]

There are Gaussian quadrature rules of all orders. If there are \( n \) weights and \( n \) nodes to be chosen, there are \( 2n \) degrees of freedom, and we can choose them so that \( G_n \) integrates all polynomials of degree \( \leq 2n - 1 \), but not of degree \( 2n \). Hence \( G_n \) is order \( p = 2n \). The nodes of \( G_n \) are the zeros of the Legendre polynomial \( \theta_n \) of degree \( n \). The weights and nodes for Gaussian quadrature can easily be found on the web.
Lobatto quadrature

Lobatto quadrature using 4 points is a rule similar to Gaussian quadrature but which uses the end points of the interval. On the interval \([-1, 1]\), the rule is

\[
L_4(g) = A_1 g(-1) + A_2 g(-t_1) + A_2(t_1) + A_1 g(1).
\]

Because of symmetry, \(L_4\) is already exact on all odd powers \(t^p\). We impose the conditions that \(L_4\) be exact on the polynomials \(1, t^2\) and \(t^4\). This yields the equations

\[
A_1 + A_2 = 1
\]

\[
A_1 + A_1 t_1^2 = \frac{1}{3}
\]

and

\[
A_1 + A_2 t_1^4 = \frac{1}{5}.
\]

The solutions for \(A_1, A_2\) and \(t_1\) yield the Lobatto rule

\[
L_4(g) = \frac{1}{6} g(-1) + \frac{5}{6} g(-\sqrt{1/5}) + \frac{5}{6} g(\sqrt{1/5}) + \frac{1}{6} g(1).
\]

The Lobatto rule \(L_4\) has order \(p = 6\). There are Lobatto rules of all orders. The interior nodes \(t_i, i = 1, \ldots, n - 2\) are the zeros of the derivatives \(\theta_n'\) of the Legendre polynomials \(\theta_n\).