Discrete Series and Characters of the Component Group

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April 9, 2007

Suppose $\phi : W_{\mathbb{R}} \to {}^LG$ is an L-homomorphism. There is a close relationship between the L-packet associated to ϕ and characters the component group of the centralizer of ϕ. This is reinterpreted in [2], see also [1], in part to make it more canonical and a bijection. This involves a number of changes, including using the notion of strong real form, several strong real forms at once, and taking a cover of the component group. This cover is not necessarily a two-group, so the values of the character may not be just signs.

Over the years a number of people have asked me how to relate an L-packet of discrete series to characters of S_ϕ in this language. While this is a special case of [2] and [1], it isn't so easy to extract it. In these notes I work out this case, and give some details in the case of $SU(p,q)$ and $U(p,q)$. There are no proofs; see the references for more details.

Besides the basic references cited above, I make some use of [**?**].

If you want to cut to the chase the main result is Proposition 4.4. Also see Propositions 3.3, 7.3 and 8.2.

1 Basic Setup

We fix a pair (G,γ) consisting of a reductive, algebraic (or complex) group and an outer automorphism of G. This corresponds to an inner class of real forms of G. We are interested in discrete series representations of real forms of G. A real form in this inner class has discrete series representations if and only if $\gamma = 1$, so we assume this.

The extended group is a semidirect product $G^\Gamma = G \rtimes \Gamma$ where $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Since $\gamma = 1$ it is a direct product, and we may safely drop Γ from the notation. The constructions of [2],[1],[?] involving the "twist" simplify in this setting.

We fix Cartan and Borel subgroups H and B of G. Let $G^\vee$ be the dual group, and fix Cartan and Borel subgroups $H^\vee$ and $B^\vee$ of $G^\vee$. By construction we have canonical identifications

$$X^*(H) = X_*(H^\vee), \quad X_*(H) = X^*(H^\vee) \tag{1.1}$$

where X^* and X_* denote the character and co-character lattices, respectively. Let $R(G,H), P(G,H), R^\vee(G,H)$ and $P^\vee(G,H)$ be the root, weight, coroot, and coweight lattices of G with respect to H. Recall

$$\begin{aligned} P &= \{\lambda \in X^*(H)\otimes\mathbb{Q} \mid \langle\lambda,\alpha^\vee\rangle \in \mathbb{Z} \quad \text{for all } \alpha^\vee \in R^\vee\} \\ P^\vee &= \{\gamma^\vee \in X_*(H)\otimes\mathbb{Q} \mid \langle\alpha,\gamma^\vee\rangle \in \mathbb{Z} \quad \text{for all } \alpha \in R\} \end{aligned} \tag{1.2}$$

and these are lattices if and only if G is semisimple. We abbreviate these $R, P, R^\vee$ and $P^\vee$. Define $R(G^\vee, H^\vee)$ etc. similarly. Then, for example $R(G,H) = R^\vee(G^\vee, H^\vee)$. Let $W = W(G,H)$ be the Weyl group.

Write $G^{\vee\Gamma}$, instead of ${}^LG$, for the L-group of G. This is a semidirect product $G^\vee \rtimes \Gamma$.

2 Strong real forms

A strong real form of G is an element $x \in G$ satisfying $x^2 \in Z$ (Z is the center of G) (recall we are assuming $\gamma = 1$), and two such elements are said to be equivalent if they are conjugate by G. A real form of G is an involution $\theta \in \mathrm{Int}(G)$ (this is the Cartan involution), and equivalence is conjugation by G. The map $x \to \mathrm{int}(x)$ defines a surjection

$$\{\text{strong real forms of } G\}/\sim \longrightarrow \{\text{real forms of } G\}/\sim . \tag{2.1}$$

If G is adjoint this is a bijection.

Every strong real form x is conjugate to an element of H. Let

$$\mathcal{X}_1 = \{h \in H \mid h^2 \in Z(G)\}. \tag{2.2}$$

If G is semisimple this is a finite set. The Weyl group W acts on $\mathcal{X}_1$, and there is a bijection

$$\{\text{strong real forms of } G\}/\sim \xleftrightarrow{1-1} \mathcal{X}_1/W. \tag{2.3}$$

Let $\mathcal{X}_1'$ be a set of representatives of $\mathcal{X}_1/W$.

For $\gamma^\vee \in P^\vee$ let $x(\gamma) = \exp(\pi i \gamma^\vee) \in \mathcal{X}_1$ and $z(\gamma^\vee) = \exp(2\pi i \gamma^\vee) \in Z$. This gives an isomorphism $\mathcal{X}_1 \simeq P^\vee/2X_*$.

The element $x(\rho^\vee) = \exp(\pi i \rho^\vee)$ plays a special role ($\rho^\vee$ is one-half the sum of the positive co-roots). The corresponding real form is quasisplit. The element $z(\rho^\vee) = x(\rho^\vee)^2$ is independent of the choice of positive roots.

It is often convenient consider only those x with x^2 fixed. So fix $z \in Z$ and let

$$\mathcal{X}_1[z] = \{x \in H \mid x^2 = z\}, \quad \mathcal{X}_1'[z] = \{x \in \mathcal{X}_1' \mid x^2 = z\}. \tag{2.4}$$

3 L-parameters

Suppose ϕ is an admissible homomorphism of the Weil group of $\mathbb{R}$ into $G^{\vee\Gamma}$. Associated to $x \in \mathcal{X}_1$ and ϕ is a finite set $\Pi(x,\phi)$, the L-packet for the strong real form x corresponding to ϕ. This may be empty. We have $\Pi(x,\phi) = \Pi(x',\phi)$ if and only if x is equivalent to x', i.e. $x' = wx$ for some $w \in W$. Define

$$\Pi(\phi) = \coprod_{x \in \mathcal{X}_1'} \Pi(x,\phi), \tag{3.1}$$

the L-packet associated to ϕ.

Unless G is adjoint these sets are often larger than necessary. For one thing if G is not semisimple they are infinite. Secondly, multiple strong real forms corresponding to the same real form can occur. See the examples.

For these reasons it is sometimes helpful to fix an element $z \in Z$, and define

$$\Pi(z,\phi) = \coprod_{x \in \mathcal{X}_1'[x]} \Pi(x,\phi). \tag{3.2}$$

(If we think of $z \in Z$ as an element of Z or $\mathcal{X}_1$ then $\Pi(z,\phi)$ has two possible meanings; hopefully this will not cause any confusion.)

These sets are finite, and the number of strong real forms mapping to single real form is small, and often 1. On the other hand for a fixed z some real forms may fail to occur. See the examples.

Let λ be the infinitesimal character for G determined by ϕ. For the purposes of this discussion this infinitesimal character is not important, and we can take it to be ρ, the infinitesimal character of the trivial representation.

Proposition 3.3 *Suppose ϕ is a discrete series L-parameter for G. There are natural bijections*

$$\mathcal{X}_1 \overset{1-1}{\longleftrightarrow} \Pi(\phi) = \coprod_{x \in \mathcal{X}_1'} \Pi(x,\phi) \tag{3.4}$$

and

$$\mathcal{X}_1[z] \overset{1-1}{\longleftrightarrow} \Pi(z,\phi) = \coprod_{x \in \mathcal{X}_1'[z]} \Pi(x,\phi). \tag{3.5}$$

This proposition practically proves itself. Write the infinitesimal character corresponding to ϕ as $\lambda \in \mathfrak{h}^*$, dominant with respect to the fixed choice of Borel subgroup B. Associated to $x \in \mathcal{X}_1$ is a real form with Cartan involution $\theta_x = \mathrm{int}(x)$ and (complexified) maximal compact subgroup $K_x = \mathrm{Cent}_G(x)$. Associated to x is a discrete series representation, which we denote $\pi_x(\lambda)$ of this real form, with Harish-Chandra parameter λ. The key point is that if $x' = wx$ with $w \in W$, there is a natural way to identify the representation $\pi_{wx}(\lambda)$ with $\pi_x(w^{-1}\lambda)$. See [**?**].

4 Characters of S_ϕ

Fix a discrete series L-homomorphism ϕ and let $S_\phi = \mathrm{Cent}_{G^\vee}(\phi)$. This is a two-group. The theory of Langlands and Shelstad relates the elements of $\Pi(\phi)$ to characters of S_ϕ.

In [3] S_ϕ is replaced by a certain cover of it, denoted $\widetilde{S_\phi}$, and defines a canonical bijection between $\Pi(\phi)$ and $\widetilde{S_\phi}\widehat{\ }$. Unlike S_ϕ, however, $\widetilde{S_\phi}$ is not a two-group, and if G is not semisimple is not even finite. This version of the theory has the advantage that it is canonical, but it is also somewhat unsatisfying, and harder to relate to the classical theory.

For this reason we fix an element $z \in Z$, and work with $\mathcal{X}_1[z] \simeq \Pi(z,\phi)$. Fix $x_0 \in \mathcal{X}_1[z]$. The map $x \to x x_0^{-1}$ defines an isomorphism

$$\mathcal{X}_1[z] \simeq H_2 \simeq X_*/2X_*, \tag{4.1}$$

the elements of order 2 in H.

The lack of a canonical choice of $x_0 \in \mathcal{X}_1[z]$ is problematic. The case of $z = z(\rho^\vee)$ is an important one, for which there is a canonical such choice. In this case we can take $x_0 = x(\rho^\vee) = \exp(\pi i \rho^\vee)$. The corresponding (strong) real form is quasisplit, and x_0 in the bijection of Proposition 3.3 x_0 will corresponds to a generic discrete series representation of the quasisplit form.

The strong real forms $x \in \mathcal{X}_1[z(\rho^\vee)]$ are called *pure* [4]. (In [4] these are given by $x^2 = 1$; the difference is due to the fact that we defined G^Γ with respect to the maximally compact real form, instead of the quasisplit one.) Not every real form is represented by a pure strong real form. For such real forms a choice is necessary. See the examples.

Let $\mathcal{X}_{1,pure} = \mathcal{X}_1[z(\rho^\vee)]$ and $\mathcal{X}'_{1,pure} = \mathcal{X}'_1[z(\rho^\vee)]$.

Now consider S_ϕ. Without loss of generality we may assume $\phi(\mathbb{C}^\times) \subset H^\vee$. Then $\phi(j)$ normalizes $H^\vee$ and (since this is a discrete series parameter) acts on $H^\vee$ by $h \to h^{-1}$. Then it is easy to see

$$S_\phi = H_2^\vee \simeq X_*(H^\vee)/2X_*(H^\vee) \simeq X^*/2X^*. \tag{4.2}$$

Write $s(\mu)$ for the element of S_ϕ corresponding to $\mu \in X^*$.

For $x \in \mathcal{X}_1$ let χ_x be the corresponding character of S_ϕ, i.e.

$$\chi_x(s(\mu)) = \mu(x) \quad (\mu \in X^*). \tag{4.3}$$

The natural pairing between these $X_*/2X*$ and $X^*/2X^*$ gives:

Proposition 4.4 *Suppose ϕ is a discrete series L-parameter. For $x \in \mathcal{X}_{1,pure}$ let*

$$\tau_x = \chi_{xx(\rho^\vee)^{-1}} = \chi_x \chi_{x(\rho^\vee)}^{-1}. \tag{4.5}$$

Then the map $x \to \tau_x$ induces a canonical bijection

$$\coprod_{\{x \in \mathcal{X}'_{1,pure}\}} \Pi(x,\phi) \overset{1-1}{\longleftrightarrow} \widehat{S_\phi}. \tag{4.6}$$

In this bijection the trivial character of S_ϕ corresonds to a generic discrete series representation of a quasisplit (strong) real form.

More generally fix $z \in Z$ and $x_0 \in \mathcal{X}_1[z]$. The map

$$x \to \tau_x = \chi_x \chi_{x_0}^{-1} = \chi_{xx_0^{-1}} \tag{4.7}$$

induces a bijection

$$\coprod_{\{x \in \mathcal{X}'_1[z]\}} \Pi(x,\phi) \overset{1-1}{\longleftrightarrow} \widehat{S_\phi} \tag{4.8}$$

depending on the choice of x_0.

The left hand side of (4.6) is the disjoint union, over various strong real forms of G, of a classical L-packet for the corresponding real form. For any real form of G in this inner class there may be a pure strong real form mapping to it, and if so there may be more than one. The pure strong real form $x(\rho)$ is quasisplit.

Every real form occurs in (4.8) for some z.

We make these maps explicit. For $x \in \mathcal{X}_{1,pure}$ we have

$$\tau_x(\mu) = \mu(xx(\rho^\vee)^{-1}) \quad (\mu \in X^*/2X^*). \tag{4.9}$$

More generally given $x_0 \in \mathcal{X}_1[z]$ we have

$$\tau_x(\mu) = \mu(xx_0^{-1}) \quad (\mu \in X^*/2X^*). \tag{4.10}$$

Alternatively write $x_0 = x(\gamma_0^\vee)$; recall $\gamma_0^\vee = \rho^\vee$ in the case of pure strong real forms. Then

$$\chi_{x(\gamma^\vee)}(s(\mu)) = (-1)^{\langle \mu, \gamma^\vee - \gamma_0^\vee \rangle} \tag{4.11}$$

5 Example: Strong Real forms of $SU(p,q)$

Let $n = p + q$.

First assume n is odd. Then $z(\rho^\vee) = I$ and $x = \text{diag}(\pm 1, \ldots, \pm 1)$ with an even number of minus signs. There is a bijection between real forms and strong real forms. For example if $n = 5$ we have $SU(5,0), SU(3,2)$ and $SU(1,4)$.

Suppose $n = 2m$ is even. Then $z(\rho^\vee) = -I$, and

$$\mathcal{X}_1[z(\rho^\vee)] = \text{diag}(\epsilon_1 i, \ldots, \epsilon_n i) \tag{5.1}$$

with $\epsilon_i = \pm 1$ and $\prod \epsilon_i = (-1)^m$. In particular $x(\rho^\vee) = \text{diag}(\overbrace{i, \ldots, i}^{m}, \overbrace{-i, \ldots, -i}^{m})$ and this corresponds to the quasisplit form $SU(m,m)$. The strong real forms in $\mathcal{X}_{1,pure}$ correspond to real forms $SU(p,q)$ with $p \equiv q \pmod 2$; the non-quasisplit ones each occur twice. If m is even we can think of these as $SU(2m, 0), SU(2m-2, 2), \ldots SU(0, 2m)$. If m is odd we have $SU(2m-1, 1), SU(2m-3, 3), \ldots, SU(1, 2m-1)$.

For example if $n = 2m = 4$ we can think of the strong real forms as $SU(4,0), SU(2,2)$ and $SU(0,4)$. If $n = 2m = 6$ we get $SU(5,1), SU(3,3)$ and $SU(1,5)$.

In this case not every real form occurs. To get all real forms we must choose another element z. The most convenient such choice is $z = I$ if m is odd, or $z = iI$ if m is even.

Suppose m is odd. Let $z = I$. Then the elements $x = \mathrm{diag}(\pm 1, \ldots, \pm 1)$ with and even number of minus signs give the groups $SU(2m, 0), SU(2m - 2)$,..., $SU(0, 2m)$; each real form occurs twice. For example for $n = 2m = 6$ we have $SU(6, 0), SU(4, 2), SU(2, 4)$ and $SU(0, 6)$.

Now assume m is even. Then take $z = iI$ and let $\alpha = e^{\pi i/4}$. Then $x = \alpha(\pm 1, \ldots, \pm 1)$ with an odd number of minus signs. We get each $SU(p, q)$ with p, q odd counted twice. For example if $n = 2m = 8$ we have $SU(7, 1), SU(5, 3), SU(3, 5)$ and $SU(1, 7)$.

The following table summarizes the situation. The number of groups in the row labelled by z is the number of inequivalent strong real forms corresponding to z. The occurence of two groups $SU(p, q)$ and $SU(q, p)$ means that the map from strong real forms to this real forms is two to one. If only one of these groups occurs it is one to one.

1. $n = 2$
   (a) $z = z(\rho^\vee) = -I$: $SU(1, 1)$
   (b) $z = I$: $SU(2, 0)$, $SU(0, 2)$
2. $n = 3$
   (a) $z = z(\rho^\vee) = I$: $SU(3, 0), SU(1, 2)$
3. $n = 4$:
   (a) $z = z(\rho^\vee) = -I$: $SU(4, 0), SU(2, 2), SU(0, 4)$
   (b) $z = iI$: $SU(3, 1), SU(1, 3)$
4. $n = 5$:
   (a) $z = z(\rho^\vee) = I$: $SU(5, 0), SU(3, 2), SU(1, 4)$
5. $n = 6$:
   (a) $z = z(\rho^\vee) = -I$: $SU(5, 1), SU(3, 3), SU(1, 5)$,
   (b) $z = I$: $SU(6, 0), SU(4, 2), SU(2, 4), SU(0, 6)$

It isn't entirely necessary, but to be precise below we specify names for strong real forms of $SL(n,\mathbb{C})$. Let $\alpha = 1, i$ or $e^{\pi i/4}$, and let $SU(p,q)$ be the strong real form defined by

$$x = \operatorname{diag}(\overbrace{\alpha,\dots,\alpha}^{p},\overbrace{-\alpha,\dots,-\alpha}^{q}). \tag{5.2}$$

(assuming this has determinant 1).

6 Example: Strong Real Forms of $U(p,q)$

Since G is not semisimple the theory is complicated if we allow all strong real forms. On the other hand if we fix $z = z(\rho^\vee)$ the situation is quite simple.

Let $n = p + q$. We have $z(\rho^\vee) = I$ if n is odd, and $-I$ if n is even.

Suppose n is odd. Then $x = \operatorname{diag}(\pm 1,\dots,\pm 1)$ with any number of minus signs. Write $U(p,q)$ for the strong real form defined by x where 1 is an eigenvalue of multiplicity p and -1 has multiplicity q. Note that the strong real forms $U(p,q)$ and $U(q,p)$ are not equivalent; they map to the same real form.

Suppose n is even. Then $z(\rho^\vee) = -I$, and we take $x = i(\pm 1,\dots,\pm 1)$. Write $U(p,q)$ for the strong real form defined by x with i^{n-1} having multiplicity p and $-(i^{n-1})$ having multiplicity q. (The exponent isn't critical, and is included for the sake of Proposition 7.3). If $p \ne q$ the strong real forms $U(p,q)$ and $U(q,p)$ are inequivalent, and both map to the same real form, which is therefore counted twice. There is unique quasisplit strong real form denoted $U(m,m)$ mapping to the quasisplit real form.

In each case the strong real forms are labelled $U(n,0), U(n-1,1),\dots,U(0,n)$.

7 Discrete series of $U(p,q)$

Let $G = GL(n,C)$ and suppose ϕ is a discrete series L-parameter. Then $S_\phi \simeq \mathbb{Z}/2\mathbb{Z}^n$, and $\widehat{S_\phi}$ is in bijection with a set of discrete series representations of various strong real forms of G. We make this more explicit.

We fix $z = z(\rho^\vee) = (-1)^{n+1}I$. Let

$$\begin{aligned} z_0 &= z(\rho^\vee) = (-1)^{n-1}I \\ x_0 &= x(\rho^\vee) = i^{n-1}\operatorname{diag}(1,-1,\dots,(-1)^{n-1}) \end{aligned} \tag{7.1}$$

Write $S_\phi = \{\mathrm{diag}(\pm 1, \dots, \pm 1)\}$. For $\epsilon_i = \pm 1$ let $\chi(\epsilon_1, \dots, \epsilon_n)$ be the corresponding character of S_ϕ. Then $\chi(\epsilon_1, \dots, \epsilon_n) = \chi_x$ with

$$\begin{aligned} x &= \mathrm{diag}(\epsilon_1, \dots, \epsilon_n)x_0 \\ &= i^{n-1}\mathrm{diag}(\epsilon_1, -\epsilon_2, \epsilon_3, -\epsilon_4, \dots, (-1)^{n-1}\epsilon_n). \end{aligned} \tag{7.2}$$

The upshot is this.

Proposition 7.3 *Let $G = GL(n, \mathbb{C})$ and suppose ϕ is a discrete series L-parameter for G. There is a canonical bijection between $\widehat{S_\phi}$ and the discrete series representations of the strong real forms $\{U(p,q) \mid 0 \le p, q \le n, p+q = n\}$.*

Fix a character $\chi(\epsilon_1, \dots, \epsilon_n)$ and let

$$v = (\epsilon_1, -\epsilon_2, \epsilon_3, -\epsilon_4, \dots, (-1)^{n-1}\epsilon_n). \tag{7.4}$$

Then χ corresponds to a representation of the strong real form $U(p,q)$ where p is the number of $1'$s in v, and q is the number of $-1'$s.

For example the trivial character goes to the quasisplit strong real form $U(m,m)$ or $U(m+1,m)$.

Note that each real form $U(p,q)$ occurs twice, as the strong real forms $U(p,q)$ and $U(q,p)$, unless $p = q$. The total number of discrete series representation of $U(n,0), U(n-1,1), \dots, U(0,n)$ is $2^n = |S_\phi|$.

8 Discrete series of $SU(p,q)$

This case is a bit more complicated than $U(p,q)$. Write $S_\phi = \{\mathrm{diag}(\pm 1, \dots, \pm 1)\}$ with and even number of minus signs, and write $\chi(\epsilon_1, \dots, \epsilon_n)$ accordingly.

If n is odd things are reasonably straightforward. Recall (Section 5) the map from strong real forms to real forms is a bijection.

In this case

$$\begin{aligned} z(\rho^\vee) &= I \\ x(\rho^\vee) &= (-1)^{\frac{n-1}{2}}\mathrm{diag}(1, -1, \dots, 1). \end{aligned} \tag{8.1}$$

Proposition 8.2 *Let $G = SL(n, \mathbb{C})$ with n odd. There is a canonical bijection between $\widehat{S_\phi}$ and the discrete series representations of the real forms, equivalently strong real forms,*

$$\{SU(p,q) \mid 0 \le p, q \le n, p+q = n, q \text{ even}\}. \tag{8.3}$$

Let

$$v = (-1)^{\frac{n-1}{2}}(\epsilon_1, -\epsilon_2, \epsilon_3, \ldots, -\epsilon_{n-1}, \epsilon_n). \tag{8.4}$$

Then χ corresponds to the real form $SU(p,q)$ where p is number of $1's$ in v, and q the number of $-1's$.

Suppose n is even. Recall not every real form occurs in $\mathcal{X}_{1,pure}$ in this case, and the map from strong real forms to real forms is not bijective.

Proposition 8.5 *Let $G = SL(n,\mathbb{C})$ with $n = 2m$. Let $z_0 = z(\rho^\vee)$. There is a canonical bijection between $\widehat{S_\phi}$ and the discrete series representation of the strong real forms*

$$\{SU(p,q) \,|\, 0 \le p, q \le n, p+q = n, p \equiv q \equiv \frac{n}{2} \pmod 2\}. \tag{8.6}$$

These are the pure strong real forms.

Let

$$v = (\epsilon_1, -\epsilon_2, \epsilon_3, \ldots, \epsilon_{n-1}, -\epsilon_n). \tag{8.7}$$

Then χ corresponds to the real form $SU(p,q)$ where p is number of $1's$ in v, and q the number of $-1's$.

If m is even let $\alpha = e^{\pi i/4}$ and $z_0' = iI$. If m is odd let $\alpha = 1$ and $z_0' = I$. Choose

$$x_0 = \alpha\, diag(\delta_1, \ldots, \delta_n) \tag{8.8}$$

with $\delta_i = \pm 1$. Then $x_0^2 = z_0$, and there is a bijection between $\widehat{S_\phi}$ and the discrete series representations of the strong real forms

$$\{SU(p,q) \,|\, 0 \le p, q \le n, p+q = n, p \equiv q \equiv \frac{n}{2} + 1 \pmod 2\}. \tag{8.9}$$

Let

$$v = (\delta_1\epsilon_1, \ldots, \delta_n\epsilon_n). \tag{8.10}$$

Then χ corresponds to the real form $SU(p,q)$ where p is number of $1's$ in v, and q the number of $-1's$.

We mention just one example of the group $\widetilde{S_\phi}$. Let $G = SL(2, \mathbb{C})$. Then $S_\phi = \mathbb{Z}/2\mathbb{Z}$ and $\widetilde{S_\phi} = \mathbb{Z}/4\mathbb{Z}$. There is a natural bijection between the discrete series representations of the strong real forms $SU(2, 0)$, $SU(0, 2)$ and $SU(1, 1)$ and characters of $\widetilde{S_\phi}$.

The two characters of $\widetilde{S_\phi}$ of order 2 factor to S_ϕ; these correspond to the two discrete series representation of the (pure) strong real form $SU(1, 1)$ as before. The trivial representations of $SU(2, 0)$ and $SU(0, 2)$ correspond to the two characters of $\widetilde{S_\phi}$ of order 4. Via the non-canonical choice of (8.10) these also correspond to the two characters of $\widetilde{S_\phi}$ of order 2, or equivalently the two characters of S_ϕ.

References

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