# Comparison with Kubota's splitting of the cocycle

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Suppose $\mathbb{F}$ is non-archimedean with odd residual characteristic. Let $\mathbb{E} = \mathbb{F}(\delta)$ be a quadratic extension of $\mathbb{F}$ and embed $\mathbb{E}$ in $SL(2, \mathbb{F})$ by $\iota(a + b\delta) = \begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix}$ ($\Delta = \delta^2$). Then Kubota [2] gives a splitting of the cocycle on $SL(2, \mathcal{O})$, which contains $\mathbb{E}$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathcal{O}$ then

$$s(g) = \begin{cases} (c, d) & c \neq 0, |c| \neq 1 \\ 1 & \text{otherwise} \end{cases}$$

Since $p$ is odd $(u, v)_{\mathbb{F}} = 1$ for all units $u, v$, and in particular $(-1, -1)_{\mathbb{F}} = 1$. Also $1 + pa \in \mathbb{F}^{*2}$ for all $a \in \mathcal{O}$. Consider $s$ restricted to $\mathbb{E}^1$. I compare it to the cocycle $\zeta$ defined in [1].

**Comparison on $T^2$**

The splitting $\zeta$ has a simple formula on $T^2$:

$$\zeta(z) = (-1, \mathrm{Tr}_\delta(z))_{\mathbb{F}}$$

where

$$\mathrm{Tr}_\delta(z) = \begin{cases} \mathrm{trace}(z) & \mathrm{trace}(z) \neq 0 \\ \delta z & \mathrm{trace}(z) = 0 \end{cases}$$

Since $s$ and $\zeta$ differ by a quadratic character of $T$ they must be equal on $T^2$. This doesn't look obvious, or even true. It can be shown directly that it holds:

**Lemma 0.1**

$$s(z^2) = \zeta(z^2) \quad (z \in \mathbb{E}^1)$$

**Proof.** Note that if $\zeta = a + b\delta \in \mathbb{E}^1$ then

$$\zeta(z) = \begin{cases} (-1, a) & a \neq 0 \\ (-1, \Delta) & a = 0 \end{cases}$$

Note that if $z \in \mathbb{E}^1$ then $a = 0$ implies $N(\delta) = -\Delta = 1$ so $\Delta \equiv -1 \mod (\mathbb{F}^{*2})$.

If $a = 0$ then $-b^2\Delta = 1$, so without loss of generality $\Delta = -1$ and $b = \pm\delta$. Then $z^2 = -1$ and $s(1) = \zeta(1) = 1$. If $b = 0$ then $z = a = \pm 1$, $z^2 = 1$ and $s(1) = \zeta(1) = 1$. So from now on assume $ab \neq 0$. We have

$$s(z^2) = \begin{cases} (2ab, a^2 + b^2\Delta) & |ab| \neq 1 \\ 1 & \text{otherwise} \end{cases}$$

Now $Nz = a^2 - b^2\Delta = 1$, so $|a^2 - b^2\Delta| = 1$. By the triangle inequality this implies one of the following cases hold:

- $|a^2| = |b^2\Delta| = 1$
- $|b^2\Delta| < |a^2| = 1$
- $|a^2| < |b^2\Delta| = 1$
- $1 < |a^2|, |b^2\Delta|$

The last case cannot hold: this would give $u^2 - v^2\Delta \equiv 0(p)$ for some units $u, v$, which implies $\Delta \equiv w^2$ from some unit $w$, which implies $\Delta \in \mathbb{F}^{*2}$ by Hensel's lemma, a contradiction.

Case 1: $|\Delta| \neq 1$: Without loss of generality $|\Delta| = p$. Then $|b^2\Delta| \neq 1$, and this implies $|a| = 1$. So in this case $a$ is always a unit. Therefore $\zeta(z) = 1$ for all $z \in \mathbb{E}^1$. On the other hand $(2ab, a^2 + b^2\Delta) = (2ab, a^2 - b^2\Delta + 2b^2\Delta) = (2ab, 1 + 2b^2\Delta)$. Since $p|\Delta$, $1 + 2b^2\Delta$ is a square and $(2ab, 1 + 2b^2\Delta) = 1$.

Case 2: $|\Delta| = 1$.

The conditions on $a$ and $b$ become:

- $|a| = |b| = 1$

- $|b| < |a| = 1$
- $|a| < |b| = 1$

(a) If $|a| = |b| = 1$ then $s(z) = 1$ and $\zeta(z) = (-1, a)_{\mathbb{F}} = 1$.
(b) If $|b| < 1$ then $|a| = 1$ and $\zeta(z) = 1$. On the other hand, as in the case $|\Delta| = p$, $s(z) = (2ab, 1 + 2b^2\Delta)$ and since $p|b$, $1 + 2b^2\Delta$ is a square in $\mathbb{F}^*$.
(c) $|a| < |b| = 1$. Then $a^2 + b^2\Delta = b^2\Delta(1 + \frac{a^2}{b^2\Delta})$, and since $p|a$ (and $b, \Delta$ are units) the term in parentheses is in $\mathbb{F}^{*2}$, so modulo squares we have $a^2 + b^2\Delta = \Delta$. This gives $s(z) = (2ab, \Delta) = (\Delta, a)$.

On the other hand $\zeta(z) = (-1, a)$. Thus $s = \zeta$ if $\Delta = -1$.

Assume $\Delta \not\equiv -1 \mod (\mathbb{F}^{*2}$. Then since $|a| < 1$ we have $1 = a^2 - b^2\Delta \equiv -b^2\Delta \mod (p)$. This says $-b^2\Delta \equiv 1 \mod (p)$, which by Hensel's' lemma says $-\Delta \in \mathbb{F}^{*2}$. This contradicts the fact that $\Delta \not\equiv -1$.

This completes the proof.

■

**Lemma 0.2** *Choose a representative* $z_0$ *of* $\mathbb{E}^1/{\mathbb{E}^1}^2$*, and choose* $\zeta$ *so that* $\zeta(z_0) = s(z_0)$*. Then* $\zeta(z) = s(z)$ *for all* $z \in \mathbb{E}^1$*.*

**Proof.** We have $s(z^2) = \zeta(z^2)$, $s(z_0) = \zeta(z_0)$ and $s(z_0z^2) = s(z_0)s(z^2)c(z_0, z^2) = \zeta(z_0)\zeta(z^2)c(z_0, z^2) = \zeta(z_0z^2)$. ■

For example if $\Delta \neq -1$ we may choose $z_0 = -1$. Then $s(z_0) = 1$, so choose $\zeta(z_0) = 1$ and we have

$$s(\epsilon z^2) = (-1, \mathrm{Tr}_\delta(z))c(\epsilon, z^2)$$

# References

[1] Jeffrey Adams. Extensions of tori in sl(2). preprint.

[2] T. Kubota. Automorphic forms and the reciprocity law in a number field. 1969.