The Splitting of Murase and Sugano

Jeffrey Adams

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We consider the cocycle on $\mathbb{E}^1$ of [1]. That is $\mathbb{E}$ is a quadratic extension of $\mathbb{F}$, ω is the quadratic character of $\mathbb{F}$ associated to $\mathbb{E}$, ψ is an additive character of $\mathbb{F}$, and $\lambda_{\mathbb{E}}(\psi)$ is the Weil index of $z \rightarrow \psi(z\overline{z})$ $(z \in \mathbb{E}^*)$. Write $(x,y)_{\mathbb{F}}$ for the Hilbert symbol of $\mathbb{F}$. Then (cf. [1], Section 3.2):

$$\lambda_{\mathbb{E}}(\psi)^2 = \omega(-1) = (-1,\Delta)_{\mathbb{F}}$$

This also follows from [5], Appendix.

Choose $\delta \in \mathbb{E}$ so that $\mathbb{E} = \mathbb{F}(\delta)$ with $\Delta = \delta^2 \in \mathbb{F}$. For $z, w \in \mathbb{E}^1$ define

$$c_0(z,w) = \begin{cases} 1 & z=1, w=1 \text{ or} zw=1 \\ \lambda_{\mathbb{E}}(\psi)\omega(\delta\frac{1-zw}{(1-z)(1-w)}) & \end{cases}$$

Note that $c_0(z,w)^2 = (-1,\Delta)_{\mathbb{F}}$, so

$$c_0(z,w) \in \begin{cases} \mu_2 & (-1,\Delta)_{\mathbb{F}} = 1 \\ \mu_4 & (-1,\Delta)_{\mathbb{F}} = -1 \end{cases}$$

We replace c_0 by a cocycle $c \in H^2(\mathbb{E}^1, \mu_2)$ which is in the same cohomology class. Let

$$\alpha(z) = \begin{cases} \lambda_{\mathbb{E}}(\psi)^{-1} & z \neq 1 \\ 1 & z = 1 \end{cases}$$

and let

$$c(z,w) = c_0(z,w)\alpha(z)\alpha(w)\alpha(zw)^{-1}$$

Then c_0, c represent the same cohomology class, and a short calculation gives

$$c(z,w) = \begin{cases} \omega(\delta\frac{1-zw}{(1-z)(1-w)}) & z \neq 1, w \neq 1, zw \neq 1 \\ 1 & z = 1 \text{ or } w = 1 \\ (-1,\Delta)_{\mathbb{F}} & z = w = -1 \end{cases}$$

In particular $c(z,w) \in \mu_2$. Note that if $(-1,\Delta)=1$ then by changing ψ if necessary we can assume $\lambda_{\mathbb{E}}(\psi)=1$, and $c(z,w)=c_0(z,w)$.

Finally a calculation as in [1] shows that if we choose a character χ of $\mathbb{E}^*$ satisfying

$$\chi|_{\mathbb{F}^*} = \omega$$

and let

$$\zeta(z) = \begin{cases} \chi(\frac{1-z}{\delta}) & z \neq 1 \\ 1 & z = 1 \end{cases}$$

then ζ is a splitting of c:

$$c(z,w) = \zeta(z)\zeta(w)\zeta(zw)^{-1}$$

Going back to c we have that

$$\zeta_0(z) = \begin{cases} \chi(\frac{1-z}{\delta})\lambda_{\mathbb{E}}(\psi) & z \neq 1 \\ 1 & z = 1 \end{cases}$$

is a splitting of $c_0(z,w)$.

A straightforward calculation gives:

$$\zeta(\frac{z}{\bar{z}})^2 = \chi(z^2)$$

Therefore $\zeta \in \mu_2$ if $\chi^2 = 1$. Such a choice of χ exists if and only if $(-1,\Delta)_{\mathbb{F}} = 1$.

Suppose $(-1,\Delta)=1$.

Claim: let $n=2^k$ be minimal such that $\delta \notin \mathbb{E}^n$. Then we can take $\chi^{2n}=1$ and therefore $\zeta^{2n} = \zeta^{2^{k+1}} = 1$.

For example we can choose $\chi^4 = \zeta^4 = 1$ unless $(-1,\Delta)_{\mathbb{F}} = -1$ and $\delta \in \mathbb{E}^{*2}$.

Remark

Write $c^*(z,w) \in \mu_2$ for the cocycle on $\mathbb{E}^1$ obtained by restricting the usual cocycle on $SL(2)$ to $\mathbb{E}^1$ ([4], cf. [3], pg. 41 and [2], Section 2). Then $c^*(z,w)$ has a μ_n splitting for $n=2,4$ or 8 in the p–adic case [2].

The minimal splitting group for a cocycle $c(z,w)$ on $\mathbb{E}^1$ with values in A is defined to be the smallest subgroup A_{min} of $\mathbb{C}^*$ for which $c \equiv 0$ in $H^2(\mathbb{E}^1, A_{min})$, or equivalently it is the smallest group A_{min} for which there is a splitting ζ of c with values in A_{min}. The minimal splitting group is well–defined on $H^2(\mathbb{E}^1, A)$, i.e. if two cocycles are equivalent in $H^2(\mathbb{E}^1, A)$ they have the same minimal splitting group.

Note, however, that the minimal splitting group is not well–defined on $H^2(\mathbb{E}^1, \mathbb{C}^*)$. That is if two cocycles represent the same cohomology class in $H^2(\mathbb{E}^1, \mathbb{C}^*)$ they do *not* necessarily have the same splitting group. This is obvious: to say a cocycle admits a splitting is to say it is trivial in $H^2(\mathbb{E}^1, \mathbb{C}^*)$, so any two cocycles which admit splittings are equivalent in $H^2(\mathbb{E}^1, \mathbb{C}^*)$. (Note that a cocycle admits a $\mathbb{C}^*$ splitting if and only if it is symmetric; equivalently $Ext(\mathbb{E}^1, \mathbb{C}^*) = 1$.)

Now our cocycles $c(z, w)$ and $c^*(z, w)$ are not necessarily equivalent in $H^2(\mathbb{E}^1, \mu_2)$. This is not surprising, since $c(z, w)$ is defined using the lattice model, and $c^*(z, w)$ is defined using the Schroedinger model. So there is no reason for them to have the same minimal splitting group.

In fact:

$$c(\ ,\ ) = c^*(\ ,\ ) \text{ in } H^2(\mathbb{E}^1, \mu_2) \Leftrightarrow (-1, \Delta)_{\mathbb{F}} = (-1, -1)_{\mathbb{F}}$$

This follows from ([2], Lemma 1.1).

Finally, note that all we are given is the projective Weil representation of $SL(2)$, restricted to $\mathbb{E}^1$. The only natural cohomology object associated to this is an element of $H^2(\mathbb{E}^1, \mathbb{C}^*)$, by the usual construction. That is, for each $z \in \mathbb{E}^1$ choose $\pi(z)$; this gives a cocycle, whose class in $H^2(\mathbb{E}^1, \mathbb{C}^*)$ is independent of the choices. In fact this cohomology class is trivial (equivalently the cocycle is symmetric).

Now there are many ways to choose the operators $\pi(z)$ so that the cocycle has values in μ_2. There is no reason for two such choices to be equivalent in $H^2(\mathbb{E}^1, \mathbb{C}^*)$. For example you can choose the operators so the cocycle is trivial, or so that you get the Kubota cocycle. These are not equivalent in $H^2(\mathbb{E}^1, \mu_2)$ if $(-1, -1)_{\mathbb{F}} = -1$.

References

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