Θ_{10}

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1 Introduction

The symplectic group $Sp(4, F)$ has a particular interesting unitary representation for F a finite or local field. When F is finite this representation was denoted Θ_{10} [30], and somehow the name has stuck. We discuss this representation in some detail in the case of $\mathbb{R}$.

This material is mostly well known, at least to the experts, and this paper is intended as a reference for non–specialists. It also serves as an introduction to some of the machinery of the Arthur conjectures as discussed in [5]. It has existed since 1990 as an informal set of notes.

It is a pleasure to dedicate this paper to Joe Shalika, my freshman number theory professor.

2 Background

Here are some references for the basic material under discussion: admissible representations of real reductive groups, Harish–Chandra modules, dual reductive pairs, the Langlands classification, and L and Arthur packets. The references are chosen for their accessibility rather than being the primary sources.

A good introduction to representation theory of real groups is A. Knapp's *Representation Theory of Semisimple Groups, An Overview Based on Examples* [21]. A quicker guide to the subject is a set of lecture notes by A. Knapp and P. Trapa from the Park City Conference 1998 [24]. The proceedings of the Edinburgh Conference 1996 [8] include a number of expository articles, including one on the Langlands Program by A. Knapp [22].

We make repeated use of the Langlands classification (sometimes referred to as the Langlands–Knapp–Zuckerman classification), and its equivalent form the Vogan classification. A summary of the statements, with references to more details, may be found in Sections 3 and 4 of D. Vogan's *The unitary dual of* G_2 [34]. Some of the methods used here are discussed in *The Kazhdan-Lusztig conjecture for real reductive groups* by D. Vogan.

For more details on some of the technology used the basic references are *Representations of Real Reductive Groups* by D. Vogan [32] and *Cohomological Induction and Unitary Representations* by A. Knapp and D. Vogan [25]. The introduction to [25] has an overview of the cohomological construction.

For the Langlands program, in addition to [22] cited above, see the article by A. Borel [9, Volume 2] in the proceedings of the Corvallis conference 1977 [10]. The real cases is explained in Section 11, pages 46–48. For Arthur's conjectures see *The Langlands Classification and Irreducible Characters for Real Reductive Groups* by J. Adams, D. Barbasch and D. Vogan [5], especially the Introduction.

For basics of dual reductive pairs see θ-series and Invariant Theory by R. Howe [15] and *Examples of dual reductive pairs* S. Gelbart [13]. For the relationship with θ_{10} see *A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups* by R. Howe and I. Piatetski–Shapiro [16]. Another good reference is *A brief survey on the theta correspondence* by D. Prasad [28]. For the connection between dual pairs and L and Arthur packets see *L-Functoriality for Dual Pairs* by J. Adams [1].

The representation θ_{10} over a finite field is discussed in [30]. It is one of the first examples of a cuspidal unipotent representation [27]. By a standard construction this also gives θ_{10} over a p–adic field and a corresponding cuspidal automorphic representation. Alternatively θ_{10} over any local field may be constructed via the dual pair correspondence with an anisotropic orthogonal group of rank 2. All of this is discussed in [16, §1].

3 Notation

In this section we establish some notation and conventions to be used throughout. Much of this is standard and the reader may want to skip ahead to Section 4 and refer back to this when necessary.

Let V be a four-dimensional real vector space with a non–degenerate symplectic form $< \, , \, >$. Let $G = Sp(V) = Sp(V, < \, , \, >)$ be the isometry group of $<\ ,\ >$. Choose a basis of V such that $<\ ,\ >$ is given by $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Then $Sp(V)$ is isomorphic to $G = Sp(4,\mathbb{R})$, the group of matrices g satisfying $gJ\,{}^tg = J$. Thus G consists of the matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where

$$A\,{}^tB = {}^t(A\,{}^tB) \quad C\,{}^tD = {}^t(C\,{}^tD) \quad A\,{}^tD - B\,{}^tC = I_2.$$

Let $\mathfrak{g}_0$ be the Lie algebra of G. Thus $\mathfrak{g}_0$ consists of matrices X satisfying $XJ + J\,{}^tX = 0$, i.e., $X = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$ where B and C are symmetric.

We will make repeated use of the Cartan subgroups of G. There are four conjugacy classes of Cartan subgroups, isomorphic to $\mathbb{R}^* \times \mathbb{R}^*, \mathbb{R}^* \times S^1, \mathbb{C}^*$ and $S^1 \times S^1$ respectively. We choose explicit representatives H^s, H^ℓ, H^{sh} and H^c as follows (ℓ and sh stand for long and short Cayley transforms respectively [21, page 417]).

Let $H^s(x,y) = diag(x, y, \frac{1}{x}, \frac{1}{y})$ $(x, y \in \mathbb{R}^*)$. This gives the Cartan subgroup H^s.

For 2×2 matrices A, B and $\theta \in \mathbb{R}$ let

$$M(A,B) = \begin{pmatrix} A_{1,1} & & A_{1,2} & \\ & B_{1,1} & & B_{1,2} \\ A_{2,1} & & A_{2,2} & \\ & B_{2,1} & & B_{2,2} \end{pmatrix} \tag{3.1}$$

$$t(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{3.2}$$

For $x \in \mathbb{R}^*, \theta \in \mathbb{R}$ let $h^\ell(x,\theta) = M(diag(x, \frac{1}{x}), t(\theta))$ and $h^{sh}(x,\theta) = diag(xt(\theta), \frac{1}{x}t(\theta))$. This gives $H^\ell \simeq \mathbb{R}^* \times S^1$ and $H^{sh} \simeq \mathbb{C}^*$ respectively.

We also need the Lie algebras. Let $X^s(x,y) = diag(x, y, -x-y)$; this gives the Lie algebra $\mathfrak{h}_0^s$ of H^s.

Let $T(y) = \begin{pmatrix} & y \\ -y & \end{pmatrix}$. Let $X^\ell(x,y) = diag(xI_2 + T(y), -xI_2 + T(y))$ and $X^{sh}(x,y) = M(diag(x,-x), T(y))$. Finally let $X^c(x,y) = M(T(x), T(y))$. This gives the Lie algebras $\mathfrak{h}_0^\ell, \mathfrak{h}_0^{sh}$ and $\mathfrak{h}_0^c$ respectively.

Write (x,y) for the element of $\mathrm{Hom}(\mathfrak{h}_0^c, \mathbb{C})$ taking $h^c(x',y')$ to $i(xx'+yy')$. Then $\alpha = (0,2), \beta = (1,-1), \gamma = (1,1)$ and $\mu = (2,0)$ are a set $\Delta^+(\mathfrak{g}, \mathfrak{h}^c)$

of positive roots of $\mathfrak{h}^c$ in $\mathfrak{g}$, and ρ (one–half the sum of the positive roots) is $(2,1)$.

Let $\theta(g) = {}^tg^{-1}$; this is a Cartan involution of G, and let $K = G^\theta \simeq U(2)$ be a maximal compact subgroup of G. Let $\theta(X) = -\,{}^tX$ be the corresponding Cartan involution of $\mathfrak{g}_0$, and let $\mathfrak{k}_0 = \mathfrak{g}_0^\theta$, $\mathfrak{p}_0 = \mathfrak{g}_0^{-\theta}$. For $\star = c, \ell, sh$ or s write $H^\star = T^\star A^\star$ as usual, with $T^\star = H^\star \cap K$. Let $\Delta^+(\mathfrak{k},\mathfrak{h}^c) = \Delta^+(\mathfrak{g},\mathfrak{h}^c) \cap \Delta(\mathfrak{k},\mathfrak{h}^c) = \{\beta\}$, $\Delta^+(\mathfrak{p},\mathfrak{h}^c) = \Delta^+(\mathfrak{g},\mathfrak{h}^c) \cap \Delta(\mathfrak{p},\mathfrak{h}^c) = \{\alpha,\gamma,\delta\}$. Let $\mathfrak{p}^\pm$ be the abelian subalgebra of $\mathfrak{p}$ corresponding to $\pm\Delta^+(\mathfrak{p},\mathfrak{h}^c)$,

The irreducible representations of K are parametrized by highest weights with respect to $\Delta^+(\mathfrak{k},\mathfrak{h}^c)$. For $x \geq y$, $x, y \in \mathbb{Z}$ let $\mu(x,y)$ be the irreducible finite dimensional representation with highest weight (x,y). By a highest weight module for $\mathfrak{g}$ we mean a module with a vector annihilated by $\mathfrak{p}^-$. For more on highest weight modules in general see [31].

We parametrize infinitesimal characters for $\mathfrak{g}$ by elements of $\mathrm{Hom}(\mathfrak{h}^c,\mathbb{C})$ via the Harish-Chandra homomorphism. See [12] or [24, Lecture 5] for the definitions of infinitesimal character and Harish–Chandra homomorphism. Write $\chi(x,y)$ for the infinitesimal character corresponding to (x,y) $(x,y \in \mathbb{C})$.

Let $\lambda_0 = (1,0)$, and let $\mathfrak{q} = \mathfrak{q}(\lambda_0)$ be the associated θ–stable parabolic [32, Definition 5.2.1]. Thus $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ with $\mathfrak{l} \simeq \mathbb{C} \oplus \mathfrak{sl}(2,\mathbb{C})$. With the usual notation we have $\Delta(\mathfrak{l},h^c) = \{\pm\alpha\}$ and $\Delta(\mathfrak{u},h^c) = \{\beta,\gamma,\mu\}$, $\rho_\mathfrak{l} = (0,1)$ and $\rho_\mathfrak{u} = (2,0)$.

Let L be the normalizer of $\mathfrak{q}$ as usual; $L \simeq U(1) \times SL(2,\mathbb{R})$. Note that $L \supset H^\ell$; in fact L is the centralizer of T^ℓ. Suppose $\lambda \in \mathrm{Hom}(\mathfrak{t}^\ell,\mathbb{C})$ is the differential of a character of T^ℓ; equivalently λ is the restriction of $(k,0)$ for some $k \in \mathbb{Z}$. We normalize the derived functor construction as in [32, 6.3.1] and [25, 5.3b]; let $A_\mathfrak{q}(\lambda))$ be the $(\mathfrak{g},K)$-module $R^1_\mathfrak{q}(\lambda - \rho(\mathfrak{u}))$. As usual we view $\lambda - \rho(\mathfrak{u})$ as an $(\mathfrak{l}, L\cap K)$-module. This has infinitesimal character $\chi(\lambda + \rho_\mathfrak{l}) = (k,1)$.

This is an irreducible non-tempered unitary representation if $k \geq 1$, i.e. the matrix coefficents are in $L^{2+\epsilon}(G)$ [21, page 198] or [24, Lecture 8]. It has lowest K–type $\nu(k+1,1)$ and has non–zero $(\mathfrak{g},K)$-cohomology if $k \geq 2$. See [32, Definition 1.2.10, page 52] for the notion of lowest K–type. See [37] for a discussion of representations with $(\mathfrak{g},K)$–cohomology, and in particular Theorem 4.6, p. 232 for the classification of these representations.

Let $\mathcal{O}$ be the nilpotent orbit in $\mathfrak{g}_0$ through the element $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ with $B = diag(1,0)$. This is a 4-dimensional orbit. Its closure consists of itself and the 0-orbit.

Remark 2.1 We take this oppportunity to clarify some notation. The algebraic group $Sp(2n)$ is simply connected, and is a two–fold cover of its adjoint group $PSp(2n)$. The group $PSp(2n,\mathbb{R})$ usually refers to the real points of the algebraic group $PSp(2n)$, or equivalently the real points of the complex group $PSp(2n,\mathbb{C}) = Sp(2n,\mathbb{C})/\pm I$. This group is disconnected as a real Lie group, and has identity component $Sp(2n,\mathbb{R})/\pm I$. Occasionally, for example in ([36], pp. 251–254) $PSp(2n,\mathbb{R})$ refers to $Sp(2n,\mathbb{R})/\pm I$.

Also consider the symplectic similitude group $GSp(2n)$. This has center $Z \simeq G_m$ and the adjoint group $GSp(2n)/Z$ is isomorphic to $PSp(2n)$. Furthermore the set of real points of the adjoint group is isomorphic to $GSp(2n,\mathbb{R})/Z(\mathbb{R})$. So the notation $PGSp(2n,\mathbb{R})$ is unambiguous, and perhaps for this reason some authors prefer to use this group.

Simliar remarks hold over any local field.

4 Θ_{10}

We give a number of descriptions of Θ_{10}. For unexplained notation see Section 3.

Theorem 4.1 *There is a unique irreducible representation* Θ_{10} *of* $Sp(4,\mathbb{R})$ *satisfying the following equivalent conditions:*

1. Θ_{10} *is a highest weight module for* $\Delta^+(\mathfrak{k},\mathfrak{h}^c) \cup \Delta(\mathfrak{p}_0^-,\mathfrak{h}^c)$*, with highest weight* $(2,2)$*;*

2. Θ_{10} *is the endpoint of the continuous spectrum in the parametrization [38] of unitary highest weight modules of* G *with one–dimensional lowest* K*-types;*

3. *In the Langlands-Knapp-Zuckerman classification,* Θ_{10} *is a limit of holomorphic discrete series [21, Chapter XII,§7, page 460], with infinitesimal character* $\chi(1,0)$*. Thus* $\Theta_{10} = \Theta^G(\lambda, C, \chi)$ *in the notation of [23, Theorem 1.1], with* $\lambda = (1,0)$*,* C *the holomorphic discrete series chamber (given by simple roots* α,β*), and* χ *the trivial character of the center of* G*;*

4. $\Theta_{10} = \psi_\alpha(\pi')$ *where* π' *is the holomorphic discrete series with infinitesimal character* $\chi(\rho)$*, and* ψ_α *is the translation functor to the* α*-wall (cf. [32, 8.2.6]);*

5. Θ_{10} *has infinitesimal character* $\chi(1,0)$ *and contains the K-type* $\mu(2,2)$*;*

6. *In the Vogan-Zuckerman classification,* $\Theta_{10} = \overline{X}(\mathfrak{q}, \delta \otimes \nu)(\mu(2,2))$*, where* $\delta \otimes \nu \in H^{\ell^*}$ *is given by* $\delta \otimes \nu(h^\ell(\theta, x)) = e^{-i\theta} sgn(x)$ *([32], Definition 6.5.11);*

7. Θ_{10} *is the direct summand of* $Ind_{MAN}^G(\pi_2 \otimes 1 \otimes 1)$ *containing* $\mu(2,2)$*, where* $MA \simeq SL(2,\mathbb{R}) \times \mathbb{R}^*$ *is the centralizer of* A^ℓ *and* π_2 *is the lowest holomorphic discrete series representation of* $SL(2,\mathbb{R})$*;*

8. Θ_{10} *has K-spectrum* $\sum_{1 \leq m \leq n} n\, \mu(2m, 2n)$*;*

9. Θ_{10} *corresponds to the sgn representation of* $O(2)$ *in the dual pair correspondence for* $(O(2), Sp(4,\mathbb{R}))$*;*

10. Θ_{10} *is the direct summand of* $A_{\mathfrak{q}}(0)$ *containing* $\mu(2,2)$*; equivalently it is the tempered summand of this representation;*

11. Θ_{10} *is tempered, has infinitesimal character* $\chi(1,0)$ *and wave–front set* $\overline{\mathcal{O}}$ *[17].*

Θ_{10} *is unitary, tempered, and has Gelfand-Kirillov dimension* 2 *(cf. [35]).*

Remark 4.2 *There is an outer automorphism of* $Sp(4,\mathbb{R})$ *which takes* Θ_{10} *to its contragredient* Θ_{10}^**. There is no intrinsic way to choose one of these two representations. The explict choices of the previous section enable us to make such a choice and denote it* Θ_{10}*.*

If we replace $Sp(4,\mathbb{R})$ *with* $G = PSp(4,\mathbb{R})$ *(cf. Remark 2.1) there is a unique representation, whose restriction to the identity component* G^0 *of* G *is* $\Theta_{10} \oplus \Theta_{10}^*$ *(pushed down to* $G^0 = Sp(4,\mathbb{R})/\pm I$*).*

Proof. Most of these facts may be found in the literature, generally as special cases. We sketch the arguments.

The equivalence of (1) and (2) is a matter of reading the definitions of [38]. Thus the endpoint of the continuous spectrum in (2) is a limit of discrete series with highest weight $(2,2)$. The relation between the infinitesimal character $\chi(x,y)$ and highest weight $\tau = (x', y')$ is given by $(x', y') = (x+1, y+2)$, so $\chi = \chi(1,0)$. Note for later use that this implies the only highest weight modules with this infinitesimal character have highest weight $(1,1)$ or $(2,2)$. Also note that the series [38] of unitary highest weight modules for the universal cover of G has four isolated points, two of which correspond to representations which factor to G.

The data (λ, C, χ) [23] giving the limit of discrete series described by (2) are the infinitesimal character, the "type" of discrete series (holomorphic) and the central character ($\mu(2,2)$ is trivial on the center of G). Thus (2) and (3) are equivalent. Item (4) is simply the definition of limit of discrete series in terms of translation functors [23, Section 1].

Let π be any representation with lowest K-type $\mu(2,2)$ or $\mu(2,0)$ and infinitesimal character $\chi(1,0)$. The θ-stable data attached to this data [32, Definition 5.4.8 and Corollary 5.4.9], are computed as follows. The element λ of [32, Definition 5.3.22] is $(1,0)$, so the θ-stable parabolic is $\mathfrak{q}$ (cf. Section 3). The $L \cap K$-type π^0 has highest weight $(-1,1)$, and so δ is this weight restriced to H^c is as stated. Then ν is determined in this case by the infinitesimal character, and is trivial.

Thus π is a subquotient of the standard module $X = X(\mathfrak{q}, \delta \otimes \nu)$ [32, Definition 6.5.2, page 392]. Now X is computed as follows [32, Definition 6.5.1]. We have $L \simeq U(1) \times SL(2,\mathbb{R})$ and X_L is a principal series of L with odd K-types (because $\delta_{T^\ell \cap A^\ell}$ is the sgn character) and infinitesimal character 0 (because $d\nu = 0$) on $SL(2,\mathbb{R})$. Therefore X_L is the sum of the limits of discrete series of $SL(2,\mathbb{R})$ tensored with $\delta|_{U(1)}$ which is $e^{-i\theta}$ on $U(1)$. Thus by [32, 6.5.10] $X(\mathfrak{q}, \delta \otimes \nu)$ is the direct sum of two pieces, each with a unique irreducible summand. By [32, , 6.5.10] again this standard module has two lowest K-types $\mu(2,2)$ and $\mu(2,0)$ each of which is the unique lowest K-type of a summand. Thus Θ_{10} is the summand of X containing $\mu(2,2)$, i.e. $\overline{X}(\mathfrak{q}, \delta \otimes \nu)(\mu(2,2))$.

Thus there are unique irreducible representations with infinitesimal character $\chi(1,0)$ and lowest K-types $\mu(2,2)$ and $\mu(2,0)$ respectively, and they are distinct. Now let π be any irreducible representation with this infinitesimal character and containing the K-type $\mu(2,2)$ (not necessarily lowest). By definition of the ordering of K-types and elementary K-type considerations, the only possible lower K-types are $\mu(2,0)$, which is ruled out by the above discussion, or trivial. The spherical principal series with infinitesimal character ρ contains the holomorphic discrete series as a constituent (see the Appendix). Translated to infinitesimal character $\chi(1,0)$ we obtain the standard module X, so by (3) X contains Θ_{10}. Since X contains $\mu(2,2)$ with multiplicity one we see the irreducible spherical constituent does not contain $\mu(2,2)$. This proves the representation defined by (5) is unique and equal to that defined by (1-4), and (6).

Item (7) is equivalent to (6) by [32, 6.6.2, 6.6.12-14]. The representation defined by (3) has the K-spectrum indicated in (8) by the Blattner formula [14]. Conversely if an irreducible representation π has this K-spectrum it is clearly a highest weight module (since operators from $\mathfrak{p}_0^-$ lower weights), so this representation is Θ_{10} by (1).

For (9), the *sgn* representation of $O(2)$ corresponds to an irreducible representation of $Sp(4,\mathbb{R})$ with highest weight $(2,2)$ [20], which is isomorphic to Θ_{10} by (1). See Section 5.

For (10) we compute the K-spectrum of $A_{\mathfrak{q}}(0)$ by the Blattner formula. We see it has K-types $\mu(1,1), \mu(2,2), \mu(3,1), \ldots$, and as in (8) any constituent of this representation has a highest weight. Considering infinitesimal characters as in (2) we immediately see $A_{\mathfrak{q}}(0)$ has two constituents, with highest weights $(1,1)$ and $(2,2)$. By (2) the term with highest weight $(1,1)$ is not tempered. On the other hand $A_{\mathfrak{q}}(0)$ is completely reducible. This proves (10) is equivalent to (1).

For (11) Θ_{10} has the indicated wave–front set, and any representation with this wave–front set is a highest weight module. Using the infinitesimal character and the discussion in (2) we see the only other possible highest weight is $(1,1)$. This representation has the same wave–front set, but is not tempered. This proves (11).

This completes the proof of the Theorem. ■

5 Reductive Dual Pairs

We now list some representations of $Sp(4,\mathbb{R})$ coming from dual pairs. Fix a non-trivial unitary character of $\mathbb{R}$, and let ω_n be the corresponding oscillator representation of $Sp(2n,\mathbb{R})$. Fix n, and let $< \, , \, >_i$ be a set of representatives of the isomorphism classes of symmetric bilinear forms of dimension $2n$. There are $2n+1$ such forms, one each of signature (p,q) $(p+q=2n)$. Let G_i be isometry group of $< \, , \, >_i$. We write $G_i = G_{p,q}$ if $< \, , \, >_i$ has signature (p,q). Then for all p,q $(G_{p,q}, Sp(4,\mathbb{R}))$ is a reductive dual pair in $Sp(8n,\mathbb{R})$. Even though $G_{p,q} \simeq G_{q,p}$ this notation keeps track not just of G but of the form and the embedding (which it is necessary to choose consistently). See [3] for details. Let $\iota : G_{p,q} \times Sp(4,\mathbb{R}) \to Sp(8n,\mathbb{R})$ be the corresponding map.

Let $\widetilde{Sp}(2n,\mathbb{R})$ be the non–trivial two–fold cover of $Sp(2n,\mathbb{R})$. If H is a subgroup of $Sp(2n,\mathbb{R})$, let $\tilde{H}$ denote its inverse image in $\widetilde{Sp}(2n,\mathbb{R})$. In particular $\tilde{U}(n)$ is isomorphic to the $det^{\frac{1}{2}}$ cover of $U(n)$:

$$\tilde{U}(n) \simeq \{(g,z)|\, g \in U(n), z \in \mathbb{C}^*, det(g) = z^2\}.$$

Consequently if $(O(p,q), Sp(2n,\mathbb{R}))$ is a dual pair in $Sp(2n(p+q),\mathbb{R})$, then $\tilde{O}(p,q) \simeq \{(g,z)|\, det(g)^n = z^2\}$. If n is even this covering splits over $O(p,q)$, by the map $g \to (g, det(g)^{n/2})$.

The covering of $Sp(2n(p+q),\mathbb{R})$ splits over $Sp(2n,\mathbb{R})$ if and only if $p+q$ is even, in which case the splitting is unique. Thus if $p+q$ and n are even we obtain a map $\gamma : G_{p,q} \times Sp(2n\,R) \to \widetilde{Sp}(2n(p+q),\mathbb{R})$.

The oscillator representation ω of $Sp(8n,\mathbb{R})$ now yields a bijection between subsets of the admissible duals of $\tilde{G}_{p,q}$ $(p+q=2)$ and $\tilde{S}p(4,\mathbb{R})$. We are in the setting of the previous paragraph and via γ we obtain a representation correspondence between representations of $G_{p,q}$ and $Sp(4,\mathbb{R})$.

Let $\pi^+[p,q]$ be the irreducible representation of $Sp(4,\mathbb{R})$ corresponding (via the embedding coming from the form of signature p,q) to the trivial representation of $G_{p,q}$. Similary $\pi^-[p,q]$ corresponds to the sgn representation. The following identifications are not difficult to deduce from the literature. We use the notation of the Appendix. All representations have infinitesimal character $(1,0)$ and are unitary.

1. $\pi^+[2,0]$: highest weight module, with lowest K–type $\mu(1,1)$; non–tempered; $\overline{C}_\alpha$

2. $\pi^+[0,2] = \pi^+[2,0]^*$; LKT $\mu(-1,-1)$; $\overline{D}\alpha$,

3. $\pi^-[2,0] = \Theta_{10}$; LKT $\mu(2,2)$; $\overline{I}_\alpha$,

4. $\pi^-[0,2] = \Theta_{10}^*$; LKT $\mu(-2,-2)$; $\overline{L}_\alpha$,

5. $\pi^+[1,1]$: spherical; $\overline{B}_\alpha$,

6. $\pi^-[1,1]$: LKT $(1,-1)$; $\overline{H}_\alpha$,

7. $\pi^+[4,0] = \Theta_{10}$,

8. $\pi^+[0,4] = \Theta_{10}^*$,

9. $\pi^+[2,2] = \pi^+[1,1]$

Here is a brief justification for this table. In each example the lowest K–type of the representation of $Sp(4,\mathbb{R})$ is known by Howe's theory of joint harmonics [19, Proposition 3.4]; see [4, Proposition 1.4, page 7]. Also the infinitesimal character is determined; see [29] or [1, page 107].

In (1-4) the orthogonal groups $O(2,0)$ and $O(0,2)$ are compact, and by [18] the corresponding representations of $Sp(4,\mathbb{R})$ are highest weight modules. These are determined by their lowest K–type and infinitesimal character. Similar comments apply to cases (7) and (8). In cases (5) and (9) the representation of $Sp(4,\mathbb{R})$ is spherical by the preceding paragraph, and is determined by its infinitesimal character, which determines the representation. Finally the representation in (6) is determined by its lowest K–type and infinitesimal character.

6 Packets

We now describe some L and Arthur packets for G. For generalities on L–packets

Let ${}^LG$ be the L–group for G [26]. The identity component ${}^\vee G$, sometimes denoted ${}^LG^0$, is a complex connected group with root system of type $B_2 \simeq C_2$, and is adjoint since G is simply connected. Therefore ${}^\vee G$ is isomorphic to the special orthogonal group on a complex 5–dimensional space. We define ${}^\vee G = SO(5,\mathbb{C})$ with respect to the standard inner product. Then ${}^LG$ is a trivial extension of ${}^\vee G$ by $\Gamma = Gal(\mathbb{C}/\mathbb{R})$, i.e. ${}^LG \simeq {}^\vee G \times \Gamma$. As usual we may drop the extension since it is trivial, and write ${}^LG = {}^\vee G$.

Let $W_\mathbb{R}$ be the Weil group of $\mathbb{R}$ [9]. This is the unique non–split central extension of $Gal(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^*$. It is given by generators and relations as $W_\mathbb{R} =< \mathbb{C}^*, j >$ where $j^2 = -1$ and $jzj^{-1} = \overline{z}$.

6.1 L–packets

We now define the L-homomorphism $\phi : W_\mathbb{R} \to {}^LG$ whose corresponding L-packet Π_ϕ contains Θ_{10}. We begin by describing the more general L-homomorphism giving a general L-packet of discrete series. For $k, \ell \in \mathbb{Z}$ let $\phi(re^{i\theta}) = diag(t(k\theta), t(\ell\theta), 1)$ (cf. 3.2). To extend this to a homomorphism of $W_\mathbb{R}$ we need to choose $x = \phi(j)$ such that

$$x^2 = diag((-1)^k, (-1)^k, (-1)^\ell, (-1)^\ell, 1)$$

and such that conjugation by x acts by inverse on $\phi(\mathbb{C}^*)$. This forces $x = diag(1,-1,1,-1,1)t$ with $t \in \phi(\mathbb{C}^*)$, which in turn implies k,ℓ are even. Then ϕ is an admissible homomorphism [26].

Definition 6.1 *For $k,\ell \in \mathbb{Z}$ let $\phi_{k,\ell}$ be the admissible homomorphism from $W_{\mathbb{R}}$ to ${}^LG$ defined by:*

$$\phi_{k,\ell}(re^{i\theta}) = diag(t(2k\theta), t(2\ell\theta), 1), \quad \phi_{k,\ell}(j) = J.$$

For $k \neq \ell$ both non-zero the image of $\phi_{k,\ell}$ is contained in no proper Levi subgroup of ${}^LG$, and the corresponding L-packet consists of four discrete series with infinitesimal character $\chi(k,\ell)$. Up to conjugation we may assume $k > \ell > 0$.

The L–homomorphism giving Θ_{10} is $\phi_{1,0}$:

Definition 6.2

$$\phi(re^{i\theta}) = diag(t(2\theta), 1, 1, 1), \quad \phi(j) = diag(1,-1,1,-1,1)$$

The next proposition follows immediately.

Proposition 6.3 *The L-packet Π_ϕ contains Θ_{10}, and consists of four limits of discrete series representations.*

We describe this L–packet in more detail. The L-packet $\Pi_{\phi_{2,1}}$ contains four discrete representations, with Harish–Chandra parameters $(2,\pm 1), (\pm 1, -2)$. Each of these representations is non-zero when we translate to the α–wall, with infinitesimal character $\chi(1,0)$, and these are the four limits of discrete series in Π_ϕ. They have lowest K–types $\mu(2,2), \mu(2,0), \mu(0,-2)$ and $\mu(-2,-2)$ respectively. We can think of them as having Harish–Chandra parameters $(1,0^+), (1,0^-), (0^+,-1)$ and $(0^-,-1)$ respectively. Here $0^\pm$ indicates the limit; for example $(1,0^+)$ is the translation to the wall of the discrete series representation with Harish–Chandra parameter $(2,1)$.

Write γ for the limit of discrete series representation with lowest K–type $\mu(2,0)$. Then

$$\Pi_\phi = \{\Theta_{10}, \gamma, \gamma^*, \Theta_{10}^*\}.$$

The centralizer S_ϕ of ϕ in ${}^\vee G$ is computed as follows. The centralizer of $\phi(\mathbb{C}^*)$ is isomorphic to $\mathbb{C}^* \times SO(3,\mathbb{C})$, and $\phi(j)$ acts on the centralizer as an involution. The fixed point set of this action is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times O(2)$.

The component group $\mathbf{S}_\phi$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The four characters of this group correspond to the four representations in the L-packet Π_ϕ, with the trivial representation corresponding to γ. Note that in [2] and [5] we discussed the larger super-packet containing Θ_{10} which contains representations of the groups $Sp(p,q)$. In this case this is not necessary: we have obtained a bijection $\Pi_\phi \to \widetilde{S}_\phi\hat{}$.

6.2 Arthur Packets

We describe some Arthur packets of unipotent representations of $Sp(4,\mathbb{R})$ [5]. There is some overlap with [7, Examples 1.4.2-3] and [5, Example 27.14].

We describe unipotent orbits by their Jordan form [11]: $SO(5,\mathbb{C})$ has four unipotent orbits: $\mathcal{O}(5), \mathcal{O}(1,1,1,1,1), \mathcal{O}(3,1,1)$ and $\mathcal{O}(2,2,1)$. That is if λ is a partition of 5 then λ determines a nilpotent orbit in $GL(5,\mathbb{C})$. If the multiplicity of every even entry of λ is even this orbit interesects $SO(5,\mathbb{C})$ and determines a unipotent orbit of $SO(5,\mathbb{C})$. We consider parameters ψ corresponding to the first three cases in turn.

(1) $\mathcal{O}(5)$.

This is the principal orbit for the dual group, and it follows that $\psi(j) = I_5$. The corresponding representations have infinitesimal character ρ and trivial associated variety: $\Pi_\psi = \{trivial\}$. The centralizer of the image of ψ is trivial, so there is no endoscopy for this packet. (There would be if we consider packets for all real forms of $Sp(2n,\mathbb{R})$ as in [5].) See [5, Theorem 27.18].

(2) $\mathcal{O}(1,1,1,1,1)$.

This is the 0–orbit for the dual group, with dual orbit the principal orbit of $Sp(4,\mathbb{R})$, and infinitesimal character 0. We take $\psi(z) = 1$ ($z \in \mathbb{C}^*$). Up to conjugacy there are three possibilities for $\psi(j)$. We write $\psi^\dagger$ with $\dagger = a, b, c$, where:

$$\psi^\dagger(j) = \begin{cases} I_5 & \dagger = a \\ diag(-1,-1,1,1,1) & \dagger = b \\ diag(-1,-1,-1,-1,1) & \dagger = c \end{cases}$$

Let $y = \psi(j)$. Then y acts as an involution on ${}^\vee G$, and thereby defines a real form of ${}^\vee G$. These real forms are $SO(5,0), SO(3,2)$ and $SO(4,1)$ respectively. The representations in each packet have infinitesimal character

0. There are three (minimal) principal series representations of $Sp(4,\mathbb{R})$ with infinitesimal character 0, the spherical one of which is irreducible, and the two others each have two irreducible components.

In terms of the tables in the Appendix we see this as follows. After translation we are considering representations with infinitesimal character ρ, and both simple roots not in the τ-invariant. See [32, Definition 7.3.8, p. 472] for the definition of the τ–invariant. These are dual to representations of the real forms of ${}^\vee G$ with both roots in the τ-invariant, i.e. one–dimensional representations. These are parametrized by ${}^\vee G(\mathbb{R})/{}^\vee G(\mathbb{R})^0$; there are $1, 2$ and 2 of them respectively. In the first case Π consists of the single irreducible spherical representation π_{sph} with infinitesimal character 0 (this representation is itself a block). In the second case Π consists of the two large discrete series representations $\overline{J}, \overline{K}$ translated to infinitesimal character 0. We denote these $\overline{J}_0, \overline{K}_0$. The final case consists of the two representations $\overline{Y}, \overline{Z}$ in the other block for $Sp(4,\mathbb{R})$, translated to $\overline{Y}_0, \overline{Z}_0$ at infinitesimal character 0.

It is immediate that π_{sph}, $\overline{K}_0 + \overline{L}_0$ and $\overline{Y}_0 + \overline{Z}_0$ are stable.

We turn now to endoscopy. See [5], Chapters 22 and 26 for details. Computing centralizers, we see $\mathbf{S}_\psi$ consists of $1, 2$ and 2 elements respectively. Given $s \in \mathbf{S}_\psi$ let ${}^\vee H$ be the identity component of the centralizer of s in ${}^\vee G$. Associated to ${}^\vee H$ and ψ is an endoscopic group H, stable Arthur packet of unipotent representations of H, and virtual character of $Sp(4,\mathbb{R})$ obtained by lifting. The identity element corresponds to the stable sums above.

We consider endoscopy coming from the non–trivial elements. For ψ^a we have $H = GL(1,\mathbb{R}) \times SL(2,\mathbb{R})$; $\overline{K}_0 - \overline{L}_0$ is the lift from H of sgn on $GL(1,\mathbb{R})$ and the irreducible (spherical) principal series of infinitesimal character 0 on the $SL(2,\mathbb{R})$ factor. In this case lifting is induction from a real parabolic subgroup. The corresponding construction at infinitesimal character ρ yields $E - F = \overline{E} - \overline{F} + \overline{I} + \overline{J} - \overline{K} - \overline{L}$, and translating to infinitesimal character 0 all terms vanish except $\overline{K}_0 - \overline{L}_0$.

Similarly we obtain $\overline{Y}_0 - \overline{Z}_0$ as the lift from $GL(1,\mathbb{R}) \times SL(2,\mathbb{R})$, of the trivial representation on $GL(1,\mathbb{R})$ times the reducible principal series of infinitesimal character 0 on $SL(2,\mathbb{R})$.

We summarize this as follows.

Proposition 6.4 *(1)* $\Pi_{\psi^a} = \{\pi_{sph}\}$
(2) $\Pi_{\psi^b} = \{\overline{J}_0, \overline{K}_0\}$
(2a) $\overline{J}_0 - \overline{K}_0$ *is the lift of an irreducible principal series representation of* $GL(1,\mathbb{R}) \times SL(2,\mathbb{R})$*,*

(3) $\Pi_{\psi^c} = \{\overline{Y}_0, \overline{Z}_0\}$
(3a) $\overline{Y}_0 - \overline{Z}_0$ *is the lift of a reducible principal series representation of* $GL(1,\mathbb{R}) \times SL(2,\mathbb{R})$.

(3) $\mathcal{O}(3,1,1)$

This is the most interesting case. To be concrete, let ι denote the embedding of $SO(3,\mathbb{C})$ given by $\iota(g) = diag(I_2, g)$. Let $\pi : SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$ be the covering map and let $\psi = \iota \circ \pi : SL(2,\mathbb{C}) \to SO(5,\mathbb{C})$.

We take $\psi(z) = 1$ ($z \in \mathbb{C}^*$). The centralizer of $\psi(SL(2,\mathbb{C}))$ is isomorphic to $O(2)$. Up to conjugation $O(2)$ has three elements of order two. Hence there are three Arthur parameters ψ for this orbit, written $\psi^\dagger$ with $\dagger = a, b, c$, where

$$\psi^\dagger(j) = \begin{cases} I_5 & \dagger = a \\ diag(-1,-1,1,1,1) & \dagger = b \\ diag(1,-1,-1,-1,-1) & \dagger = c \end{cases}$$

As in [5, 22.8] we obtain an element $y \in {}^\vee G$ of order 2, defining a Cartan involution θ_y. For $\dagger = a, b, c$ we have $y = diag(1,1,-1,1,-1), diag(-1-1-,1,-1)$ and $diag(1,-1,1,-1,1)$ respectively. The Cartan involutions θ_y define the real forms $SO(3,2), SO(4,1)$ and $SO(3,2)$, respectively. We obtain a block for this real form, and the representations in $\Pi_{\psi^\dagger}$ are dual in the sense of [33] to these.

Using some facts about associated varieties for representations of $SO(3,2)$ and $SO(4,1)$ we conclude (notation as in Section 5 and the Appendix):

Theorem 6.5 *The Arthur packets defined by* ψ^a, ψ^b *and* ψ^c *are:*

1. $\Pi_{\psi^a} = \{\overline{B}_\alpha, \overline{H}_\alpha\} = \{\pi^+[1,1], \pi^-[1,1]\}$.
2. $\Pi_{\psi^b} = \{\overline{W}_\alpha, \overline{X}_\alpha\}$
3. $\Pi_{\psi^c} = \{\overline{C}_\alpha, \overline{D}_\alpha, \overline{I}_\alpha, \overline{L}_\alpha\} = \{\pi^+[2,0], \pi^+[0,2], \pi^-[2,0] = \Theta_{10}, \pi^-[0,2] = \Theta^*_{10}\}$.

Now we compute the centralizer of the image of ψ. As noted the centralizer of $\psi(SL(2,\mathbb{C}))$ is isomorphic to $O(2)$. To compute the centralizer of the image of ψ we compute the fixed points of $\psi(j)$ on this group.

Lemma 6.6 *The centralizer S_ψ and its component group $\mathbf{S}_\psi$ are as follows.*
(1) If $\psi = \psi^a$ or ψ^b then $S_\psi \simeq O(2)$, and $\mathbf{S}_\psi = \mathbb{Z}/2\mathbb{Z}$, with
$s = diag(1, -1, -1, -1, -1)$.

(2) $S_{\psi^c} = \mathbf{S}_{\psi^c} = S[O(1) \times O(1) \times O(1)] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Explicitly $\mathbf{S}_{\psi^c} = \{s_1 = I_5, s_2 = diag(-1, -1, 1, 1, 1), s_3 = diag(1, -1, -1, -1, -1), s_4 = s_2 s_3\}$.

In case (1) the identity component ${}^\vee H$ of the centralizer of the non–trivial element of $\mathbf{S}$ is $SO(4, \mathbb{C})$. In case a (respectively b) θ_y gives to the real form $SO(2,2)$ (resp. $SO(3,1)$). The corresponding endoscopic groups are $SO(2,2)$ and $SO(3,1)$ repsectively.

In case (2) the identity components of the centralizers of the elements of $\mathbf{S}$ are $SO(5), SO(3) \times SO(2), SO(4)$ and $SO(4)$, respectively. The corresponding real forms defined are $SO(3,2), \mathbb{R}^* \times SO(2,1), SO(2,2)$ and $SO(3,1)$ respectively. The corresponding endoscopic groups are isomorphic to $Sp(4,\mathbb{R})$, $U(1) \times SL(2,\mathbb{R}), SO(2,2)$ and $SO(3,1)$ respectively.

Let *sgn* be the non–trivial one–dimensional representation of $SO(2,2)$ or $SO(3,1)$

Proposition 6.7 *Each Arthur packet $\Pi_{\psi^\dagger}$ is in bijection with $\mathbf{S}_{\psi^\dagger}$. The lifted characters in the Arthur packets defined by ψ^a, ψ^b and ψ^c are the following.*
(1) ψ^a:
(1a) $\overline{B}_\alpha + \overline{H}_\alpha$ is stable,
(1b) $\overline{B}_\alpha - \overline{H}_\alpha$ is the lift from $SO(2,2)$ of the sgn representation,
(2) ψ^b:
(2a) $\overline{W}_\alpha + \overline{X}_\alpha$ is stable,
(2b) $\overline{W}_\alpha - \overline{X}_\alpha$ is the lift from $SO(3,1)$ of the sgn representation,
(3) ψ^c:
(3a) $\overline{C}_\alpha + \overline{D}_\alpha + \overline{I}_\alpha + \overline{L}_\alpha$ is stable,
(3b) $\overline{C}_\alpha - \overline{D}_\alpha - \overline{I}_\alpha + \overline{L}_\alpha$ is the lift of the trivial representation from $SL(2,\mathbb{R}) \times U(1)$,
(3c) $\overline{C}_\alpha - \overline{D}_\alpha + \overline{I}_\alpha - \overline{L}_\alpha$ is the lift of the trivial representation from $SO(3,1)$.
(3d) $\overline{C}_\alpha + \overline{D}_\alpha - \overline{I}_\alpha - \overline{L}_\alpha$ is the lift of the trivial representation from $SO(2,2)$.

Remark 6.8 *There is a further choice required to define the lifting [5, Definition 26.15(iii)]. The affect of this choice is to interchange the trivial and sgn representations of the endoscopic group H. We have made a particular such choice above.*

Remark 6.9 *The packets given by ψ^b, ψ^c are those of [7, 1.4.3] and [5, 27.27(a-b)]. The case of ψ^c is [7, 1.4.2] and [5, 27.17(c)].*

Proof. The proofs of these facts are all similar, based on the character table in the Appendix. The basic technique is that standard representations of H at infinitesimal character ρ for G lift to standard representations in a simple way. To compute the lift of an irreducible representation, in particular the trivial representation, we write it as a linear combination of standard representations. We then compute the corresponding lift at ρ, and then translate to a wall (cf. [32, 8.2.6]). The information which we need is either in the Appendix or is readily obtained from smaller groups such as $SL(2,\mathbb{R})$.

We give a few examples.

ψ^a:

$$\begin{aligned}\overline{B}+\overline{H} &= (B-G-I-L)+(H-J-K)\\ &= B+H+G-(I+J+K+L)\end{aligned}$$

which is stable. Translating to the α wall we conclude that $\overline{B}_\alpha + \overline{H}_\alpha$ is stable. On the other hand the lift of the sgn representation of $SO(2,2)$ is the translation to the α wall of

$$\begin{aligned}B-G-G+(-I+J+K-L) &= (\overline{B}+\overline{E}+\overline{F}+\overline{G}+\overline{H}+\overline{I}+\overline{J}+\overline{K}+\overline{L})\\ &\quad -\overline{2}(G+\overline{E}+\overline{F}+\overline{H}+\overline{J}+\overline{K})-\overline{I}+J+K-\overline{L}\\ &= \overline{B}-\overline{E}-\overline{F}-\overline{G}-\overline{H}\end{aligned}$$

Translating to the α wall we have

$$Lift(sgn) = \overline{B}_\alpha - \overline{E}_\alpha - \overline{F}_\alpha - \overline{G}_\alpha - \overline{H}_\alpha = \overline{B}_\alpha - \overline{H}_\alpha$$

since α is in the τ–invariant of $\overline{E}, \overline{F}$ and $\overline{G}$.

The case of ψ^b is similar.

ψ^c:

$$\begin{aligned}\overline{C}+\overline{D}+\overline{I}+\overline{L} &= (C-E-H+I+J+K)+\\ &\quad (D-H-F+J+K+L)+I+L\\ &= (C+D)-(E+F)-2H+2(I+J+K+L),\end{aligned}$$

This is stable, and remains so upon passing to the α–wall.

The lift of the trivial representation of $SO(2,2)$ is

$$\begin{aligned} A - H - H + (-I + J + K - L) &= (\overline{A} + \overline{C} + \overline{D} + \overline{E} + \overline{F} + \overline{G} + \\ &\quad 2\overline{H} + \overline{J} + \overline{K}) \\ &- 2(\overline{H} + \overline{J} + \overline{K}) - \overline{I} + \overline{J} + \overline{K} - \overline{L} \\ &= \overline{A} + \overline{C} + \overline{D} + \overline{E} + \overline{F} - \overline{I} - \overline{L} \end{aligned}$$

Translating to the α wall A, E, F vanish to give

$$Lift(\mathbb{C}) = \overline{C}_\alpha + \overline{D}_\alpha - \overline{I}_\alpha - \overline{L}_\alpha.$$

We leave verification of the remaining cases to the reader. ■

6.3 Relation with the theta correspondence

Note that by 6.5(3) Π_{ψ^c} consists of the representations corresponding to the trivial and sgn representations of $O(2,0)$ and $O(0,2)$. This is an example of the philosophy of [1].

This can also be described in terms of derived functors. With notation as in Section 3 $A_{\mathfrak{q}}(k,0)$ is an irreducible unitary representation if $k \geq 1$. We define $A_{\mathfrak{q}'}(0,-k)$ analogously; $A_{\mathfrak{q}'}(0,-k) \simeq A_{\mathfrak{q}}(k,0)^*$. If $k \geq 1$ then $\{A_{\mathfrak{q}}((k,0)), A_{\mathfrak{q}}((0,-k))\}$ is an Arthur packet of a particularly simple type as in [6].

Now take $k = 0$. Then $A_{\mathfrak{q}}(0)$ is unitary and reducible; in fact

$$A_{\mathfrak{q}}(0) = \Theta_{10} \oplus \pi^+[2,0], \quad A_{\mathfrak{q}'}(0) = \Theta_{10}^* \oplus \pi^+[0,2]$$

Therefore Π_{ψ^c} consists of the four consitituents of $A_{\mathfrak{q}}(0)$ and $A_{\overline{\mathfrak{q}}}(0)$, and this is an Arthur packet of the previous type at singular infinitesimal character.

7 Appendix: Character Tables

We give some information about the representations of $Sp(4,\mathbb{R})$ with infinitesimal character ρ. This information is reasonably well-known, if not necessarily readily accessible. Each table lists standard modules $A, B, \ldots$ with their irreducible quotients $\overline{A}, \overline{B}, \ldots$. The composition series of the standard modules, and the expressions of the irreducible modules (in the Grothendieck group) in terms of standard modules are given. The final column shows the τ-invariant of each irreducible representation, with α (respectively β) a long (respectively short) simple root. For general information see [32] and [33].

There are three blocks for $Sp(4, \mathbb{R})$ with infinitesimal character ρ. One of these is the singleton consisting of the irreducible principle series module, which is dual to the trivial representation of $SO(5, 0)$. The other two are the block of the trivial representation (dual to a block for $SO(3, 2)$) and a block dual to the block of the trivial representation of $SO(3, 1)$.

Table I is from [36]. The information in Table II may all be read off via duality from the corresonding dual block of the trivial representation of $SO(4, 1)$. We note that $SO(4, 1)$ has two one-dimensional representations *trivial* and χ, and two representations with $(\mathfrak{g}, K)$-cohomology. We denote the latter $A_{\mathfrak{q}}(\lambda)$ and $A(\mathfrak{q}, \lambda)' = A_{\mathfrak{q}}(\lambda) \otimes \chi$.

If α is not contained in the τ-invariant of a representation $\overline{X}$ occuring in this list, we let $\psi_\alpha(\overline{X}) \neq 0$ be the translate of $\overline{X}$ to the α wall.

Table I: Block of the Trivial Representation					
Representation	Description	Length	Composition Series	Formal Character of Standard Module	τ − Invariant
$\overline{A}$	trivial	3	$\overline{A}+\overline{C}+\overline{D}+\overline{E}+\overline{F}+\overline{G}+2\overline{H}+\overline{J}+\overline{K}$	A-C-D-G+E+F+ H-I-J-K-L	α, β
$\overline{B}$	non-spherical	3	$\overline{B}+\overline{E}+\overline{F}+\overline{G}+\overline{H}+\overline{I}+\overline{J}+\overline{K}+\overline{L}$	B-G-I-L	β
$\overline{C}$		2	$\overline{C}+\overline{E}+\overline{H}+\overline{J}$	C-E-H+I+J+K	β
$\overline{D}$		2	$\overline{D}+\overline{F}+\overline{H}+\overline{K}$	D-H-F+J+K+L	β
$\overline{E}$	$A(2,0)$ − highest weight	1	$\overline{E}+\overline{I}+\overline{J}$	E-I-J	α
$\overline{F}$	$A(0,2)$ − lowest weight	1	$\overline{F}+\overline{K}+\overline{L}$	F-K-L	α
$\overline{G}$		2	$\overline{G}+\overline{E}+\overline{F}+\overline{H}+\overline{J}+\overline{K}$	G-E-F-H+ I+J+K+L	α
$\overline{H}$	A(1/2,1/2)	1	$\overline{H}+\overline{J}+\overline{K}$	H-J-K	β
$\overline{I}$	holomorphic discrete series	3	$\overline{I}$	I	β
$\overline{J}$	discrete series	3	$\overline{J}$	J	$*$
$\overline{K}$	discrete series	3	$\overline{K}$	K	$*$
$\overline{L}$	anti-holomorphic discrete series	3	$\overline{L}$	L	β

Table II: Block of Sp(4,$\mathbb{R}$) dual to block of the trivial representation of SO(4,1)					
Representation	Description	Length	Composition Series	Formal Character of Standard Module	$\tau -$ Invariant
$\overline{V}$	dual to discrete series	3	$\overline{V}+\overline{W}+\overline{X}+\overline{Y}+\overline{Z}$	V-W-X	β
$\overline{W}$	dual to $A_{\mathfrak{q}}(\lambda)$	2	$\overline{W}+\overline{Y}$	W-Y	α
$\overline{X}$	dual to $A_{\mathfrak{q}}(\lambda)'$	2	$\overline{X}+\overline{Z}$	X-Z	α
$\overline{Y}$	dual to one-dimensional	1	$\overline{Y}$	Y	α, β
$\overline{Z}$	dual to trivial	1	$\overline{Z}$	Z	α, β

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