Meshless and Generalized Finite Element Methods: A Survey of Some Major Results

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Abstract. In this lecture, we discuss Meshless and Generalized Finite Element Methods. We survey major results in this area with a unified approach.

1 Introduction

For concreteness and simplicity we will address the weak solution of the model problem

$$-\Delta u + u = f(x), \text{ on } \Omega \subset R^n, \tag{1}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \tag{2}$$

or

$$u = 0 \text{ on } \partial\Omega, \tag{3}$$

for $f \in L_2(\Omega)$ given. We will assume that Ω is a Lipschitz domain. Additional assumptions on $\partial\Omega$ will given as needed. The weak solution $u_0 \in H^1(\Omega)$ ($H^1_0(\Omega)$, respectively) satisfies

$$B(u_0, v) = F(v), \text{ for all } v \in H^1(\Omega)\,(v \in H^1_0(\Omega), \text{respectively}), \tag{4}$$

where

$$B(u,v) \equiv \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx \text{ and } F(v) \equiv \int_\Omega fv\,dx. \tag{5}$$

The energy norm of u_0 is defined by

$$\|u_0\|_E \equiv B(u_0,u_0)^{1/2} = \|u_0\|_{H^1(\Omega)}. \tag{6}$$

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We will write H instead of $H^1(\Omega)$ or $H_0^1(\Omega)$ if no misunderstanding can occur.

Let $S \subset H$ be a finite dimensional subspace, called the approximation space. Then the Galerkin approximation, $u_S \in S$, to u_0 is determined by

$$B(u_S, v) = F(v), \text{ for all } v \in S. \tag{7}$$

It is immediate that

$$\|u_0 - u_S\|_{H^1(\Omega)} = \inf_{\chi \in S} \|u_0 - \chi\|_{H^1(\Omega)}. \tag{8}$$

Hence, the main problem is the approximation of u_0 by functions in S.

Remark 1. The Finite Element Method (FEM) is the Galerkin Method where S is the span of functions with small supports. For the history of the FEM, see [5] and the reference therein.

Remark 2. The classical Ritz method uses spaces of polynomials on Ω for the approximation spaces; see, *e.g.,* [31].

As mentioned above, the Finite Element Method uses basis functions with small supports, *e.g.*, "hill" functions. The theory of approximation with general hill functions with translation invariant supports was developed in 1970 in [1] using the Fourier Transform. The results in [1] were applied to the numercial solution of PDE in [2]. A very similar theory, also based on the Fourier Transform, was later developed in [35], [36]; see also [26]. Later, hill functions were called *Particle Functions* (see [22]). In the 1990s, hill functions began to be used in the framework of *meshless methods.* For a broad survey of meshless methods see [27]. A survey of the approximation properties of radial hill functions is given in [14].

In this paper we will survey basic meshless approximation results and their use in the framework of Galerkin Methods.

2 Approximation by Particle Shape Functions Associated with Uniformly Distributed Particles in R^n. The h-version

Let

$$Z^n \equiv \{j = (j_1, j_2, \ldots, j_n) : j_1, \ldots, j_n \text{ integers}\}$$

be the integer lattice, and let

$$x_j^h \equiv \{(j_1 h, \ldots, j_n h) = hj, \text{ where } j = (j_1, \ldots, j_n) \in Z^n\}.$$

The x_j^h's are called *(uniformly distributed) particles.* When considering such a family of particles, we often construct associated shape functions as follows:

Let $\phi(x) \in H^q(R^n)$, for some $0 \leq q$, be a function with compact support. Let $\eta \equiv \text{supp } \phi$ and, for the sake of simplicity, assume that $0 \in \mathring{\eta}$ (interior of η). ϕ is called the *basic shape function*. Then, for $0 < h$ and $j \in Z^n$, define

$$\phi_j^h(x) = \phi(\frac{x}{h} - j). \tag{9}$$

We will be interested in the approximation properties of the space

$$V^h(\Phi) = \left\{ v = \sum_{j \in Z^n} w_j^h \phi_j^h(x) : w_j^h \in R \right\}, \text{ as } h \to 0. \tag{10}$$

Here Φ is the family $\{\phi_j^h\}$; ϕ_j^h's are called *particle shape functions* associated with particles x_j^h's, and w_j^h's are called *weights*. More specifically, for $1 \leq q$, given $u \in H^k(R^n)$, we will be interested in estimating

$$\inf_{v \in V^h(\Phi)} \|u - v\|_{H^1(R^n)},$$

where S in (8) is replaced by $V^h(\Phi)$. This problem was discussed in [1] and later in [36]. Essentially the same results were proved in these two papers. Since ϕ has compact support, its Fourier Transform, $\hat{\phi}(\xi)$, has derivatives of all orders with respect to ξ, (in fact, it is an entire function). We now cite the main theorem from [36].

Theorem 1 *Suppose $\phi \in H^q(R^n)$ has compact support. Then the following three conditions are are equivalent:*

1. $$\hat{\phi}(0) \neq 0 \tag{11}$$

   and

   $$D^\alpha \hat{\phi}(2\pi j) = 0, \text{ for } 0 \neq j \in Z^n \text{ and } |\alpha| \leq p. \tag{12}$$

   Here we use the usual multi-index notation for partial derivatives ($\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \geq 0$, is a multi-index, $|\alpha| \equiv \alpha_1 + \ldots + \alpha_n$, and

   $$D^\alpha \hat{\phi} = \frac{\partial^{|\alpha|} \hat{\phi}}{\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n}}).$$

2. *For $|\alpha| \leq p$,*

   $$\sum_{j \in Z^n} j^\alpha \phi(x - j) = d_\alpha x^\alpha + q_{|\alpha|-1}(x), \quad \text{where } d_\alpha \neq 0, \tag{13}$$

   and $q_{|\alpha|-1}(x)$ is a polynomial of degree less than $|\alpha|$. The equality in (13) is equality in $L_2(R^n)$, i.e., equality for almost all $x \in R^n$. The function of the right-hand side of (13) is, of course, continuous. If the function on the left-hand side is continuous, which will be the case if $q > n/2$, then (13) will hold for all $x \in R^n$.

3. For each $u \in H^{p+1}(R^n)$, there are weights $w_j^h \in R$, for $j \in Z^n$ and $0 < h$, such that

$$\|u - \sum_{j \in Z^n} w_j^h \phi_j^h\|_{H^s(R^n)} \leq Ch^{p+1-s}\|u\|_{H^{p+1}(R^n)}, \text{ for } 0 \leq s \leq \min\{q, p+1\}, \tag{14}$$

and

$$h^n \sum_{j \in Z^n} (w_j^h)^2 \leq K^2 \|u\|^2_{H^0(R^n)}. \tag{15}$$

Here C and K may depend on q, p, and s, but are independent of u and h. The exponent $p+1-s$ is the best possible if p is the largest integer for which (13) holds.

If (2.5) holds, the basic shape function ϕ is said to quasi-reproduce polynomials of degree p — briefly, is *Quasi-Reproducing of Order* p. If (2.5) holds with $d_\alpha = 1$ and $q_{|\alpha|-1}(x) = 0$, ϕ is called *Reproducing of Order* p.

If ϕ is quasi-reproducing of order p (respectively, reproducing of order p), then the corresponding particle shape functions ϕ_i^h are also called quasi-reproducing of order p (respectively, reproducing of order p).

Uniformly distributed particles and associated particle shape functions are *translation invariant* in the sense that

$$x_{j+l}^h = x_j^h + x_l^h \text{ and } \phi_{j+l}^h(x) = \phi_j^h(x - x_l^h),$$

and will sometimes be referred to as *translation invariant.* They are a special case of general (nonuniformly distributed) particles, which will be addressed in the next section. We have, however, more detailed results for translation invariant particles.

In one dimension we can prove more.

Theorem 2 *[36] Let ϕ satisfy the conditions of Theorem 1 (with $n = 1$). Then*

$$\hat{\phi}(\xi) = Z(\xi) \left(\frac{\sin(\xi/2)}{\xi/2} \right)^{p+1},$$

where $Z(\xi)$ is an entire function.

This result means the ϕ is a convolution of B-splines with functions of compact support.

Theorem 3 *[1] The interval $(-\frac{p+1}{2} + \epsilon, \frac{p+1}{2} - \epsilon)$, for any $\epsilon > 0$, cannot be the support of ϕ, where ϕ satisfies the conditions of Theorem 1.*

This result means the support of the basic shape function ϕ cannot be too small.

3 Approximation by Particle Shape Functions Associated with Arbitrary (Non-Uniformly Distributed) Particles in $\boldsymbol{R^n}$. The $\boldsymbol{h}$-version

In this section we will generalize the major part of Theorem 1.

Suppose $\{X^\nu\}_{\nu \in N}$ is a family of countable subsets of points in R^n; the family is indexed by the parameter ν, which varies over the index set N. The points in X^ν are called *particles*, and will be denoted by $\underline{x}$, to distinguish them from general points in R^n. If it is necessary to underline that $\underline{x} \in X^\nu$, we will write $\underline{x} = \underline{x}^\nu$. To each $\underline{x}^\nu \in X^\nu$ we associate

- $h^\nu_{\underline{x}^\nu} = h^\nu_{\underline{x}}$ = a positive number;
- $\omega^\nu_{\underline{x}^\nu} = \omega^\nu_{\underline{x}}$ = a bounded domain in R^n;
- $\phi^\nu_{\underline{x}^\nu} = \phi^\nu_{\underline{x}}$ = a function in $H^q(R^n)$, with $\eta^\nu_{\underline{x}^\nu} = \eta^\nu_{\underline{x}} \equiv \operatorname{supp} \phi^\nu_{\underline{x}^\nu}$ assumed compact.

Regarding $\{X^\nu\}, h^\nu_{\underline{x}^\nu}, \omega^\nu_{\underline{x}^\nu}$, and $\phi^\nu_{\underline{x}^\nu}$, we make several assumptions:

1. For each ν,
$$\bigcup_{\underline{x} \in X^\nu} \omega^\nu_{\underline{x}} = R^n,$$
i.e., for each ν, $\{\omega^\nu_{\underline{x}}\}_{\underline{x} \in X^\nu}$ is an open cover of R^n.
2. For $\underline{x} \in X^\nu$, let
$$S^\nu_{\underline{x}} \equiv \{\underline{y} \in X^\nu : \omega^\nu_{\underline{x}} \cap \omega^\nu_{\underline{y}} \neq \emptyset\}.$$
There is a constant $\kappa < \infty$ depending on $\{X^\nu\}_{\nu \in N}$, but neither on ν, nor on $\underline{x} \in X^\nu$, such that
$$\text{card } S^\nu_{\underline{x}} \leq \kappa, \text{ for all } \underline{x} \in X^\nu \text{ and all } \nu \in N.$$
3. Let
$$B^\rho_{\underline{x}} = \{x \in R^n : \|x - \underline{x}\| \leq \rho\}$$
denote the ball of radius ρ centered at $\underline{x}$. There is a $0 < \gamma < 1$ such that
$$B^{\gamma h^\nu_{\underline{x}}}_{\underline{x}} \subset \omega^\nu_{\underline{x}} \subset B^{h^\nu_{\underline{x}}}_{\underline{x}}, \text{ for all } \underline{x} \in X^\nu, \text{ for all } \nu \in N.$$
4. For all $\underline{x} \in X^\nu$, for all $\nu \in N$, $\underline{x} \in \mathring{\eta}^\nu_{\underline{x}}$ (interior of $\eta^\nu_{\underline{x}}$), and $\eta^\nu_{\underline{x}} \subset \omega^\nu_{\underline{x}}$.
5. For $\underline{x} \in X^\nu$, let
$$Q^\nu_{\underline{x}} \equiv \{\underline{y} \in X^\nu : \omega^\nu_{\underline{x}} \cap \eta^\nu_{\underline{y}} \neq \emptyset\}.$$
It follows from Items 2 and 4 that
$$\text{card } Q^\nu_{\underline{x}} \leq \kappa, \text{ for all } \underline{x} \in X^\nu, \text{ for all } \nu \in N.$$

Let

$$\Omega_{\underline{x}}^{\nu} = \bigcup_{\underline{y} \in Q_{\underline{x}}^{\nu}} \omega_{\underline{y}}^{\nu}.$$

There is a $0 < \bar{\kappa} < \infty$, which may depend on $\{X^\nu\}$, but is independent of $\underline{x}$ and ν, so that

$$\Omega_{\underline{x}}^{\nu} \subset B_{\underline{x}}^{\bar{\kappa} h_{\underline{x}}^{\nu}}, \text{ for all } \underline{x} \in X^\nu, \text{ for all } \nu \in N.$$

6. There is a set $Q_{\underline{x}}^{\nu *} \subset Q_{\underline{x}}^{\nu}$ such that given data $v(\underline{y})$, for $\underline{y} \in Q_{\underline{x}}^{\nu *}$, there is a unique polynomial $P_{\underline{x},p}(x)$ of deg $\leq p$ that minimizes

$$\sum_{\underline{y} \in Q_{\underline{x}}^{\nu *}} |v(\underline{y}) - P_{\underline{x},p}(\underline{y})|^2.$$

$P_{\underline{x},p}(x)$ satisfies

$$\|P_{\underline{x},p}\|_{L_2(\Omega_{\underline{x}}^{\nu})} \leq C_1 \max_{\underline{y} \in Q_{\underline{x}}^{\nu *}} |v(\underline{y})|,$$

where C_1 is independent of $\underline{x}$ and ν.

7. There is a one-to-one mapping $\mathcal{A}_{\underline{x}}^{\nu} : \mathcal{P}_p \to \mathcal{P}_p$ such that

$$\sum_{\underline{y} \in Q_{\underline{x}}^{\nu}} \left(\left(\mathcal{A}_{\underline{x}}^{\nu} \right)^{-1} P_p \right) (\underline{y}) \phi_{\underline{y}}^{\nu}(x) = P_p(x), \text{ for } x \in R^n, \text{ and any } P_p \in \mathcal{P}_p, \tag{16}$$

and

$$\left\| \left(\mathcal{A}_{\underline{x}}^{\nu} \right)^{-1} \right\|_{L_2(\Omega_{\underline{x}}^{\nu}) \to L_2(\Omega_{\underline{x}}^{\nu})} \leq C_2. \tag{17}$$

C_2 may depend on $\{X^\nu\}_{\nu \in N}$, but neither on $\underline{x} \in X^\nu$, nor on ν.

Remark 3. It is necessary for Item 7 of these assumptions to be valid, that for each ν,

$$\bigcup_{\underline{x} \in X^\nu} \mathring{\eta}_{\underline{x}}^{\nu} = R^n,$$

which in turn implies that *card* $Q_{\underline{x}}^{\nu}$, *card* $S_{\underline{x}}^{\nu} \geq 1$, in Items 5 and 2 respectively.

Also, Item 7, specifically (16), plays the role of (13), and characterizes the quasi-reproducing property of order p of the shape functions $\phi_{\underline{x}}^{\nu}$ in the non-uniform case. The shape functions $\phi_{\underline{x}}^{\nu}$ are said to be quasi-reproducing of order p if, for $|\alpha| \leq p$,

$$\sum_{\underline{x} \in X^\nu} \underline{x}^\alpha \phi_{\underline{x}}^{\nu}(x) = d_\alpha x^\alpha + q_{|\alpha|-1}(x),$$

where $d_\alpha \neq 0$ and degree $q_{|\alpha|-1}(x) < |\alpha|$.

Now we get,

Theorem 4 *Suppose* $\{X^\nu\}, h^\nu_{\underline{x}^\nu}, \omega^\nu_{\underline{x}^\nu}$, *and* $\phi^\nu_{\underline{x}^\nu}$ *satisfy Assumptions 1–7. Suppose*

$$\sum_{\underline{x}\in X^\nu} \|u\|^2_{H^{r^\nu_{\underline{x}}+1}(B^{\overline{\rho}h^\nu_{\underline{x}}}_{\underline{x}})} < \infty, \text{ where } r^\nu_{\underline{x}} \le p, \text{ for all } \underline{x} \in X^\nu, \text{ for all } \nu \in N,$$

with $\overline{\rho} > 1$ *independent of* $\underline{x}$ *and* ν. *Then there are weights* $w^\nu_{\underline{x}} \in R$, *for* $\underline{x} \in X^\nu$, *for all* $\nu \in N$, *such that*

$$\begin{aligned}\|u - \sum_{\underline{x}\in X^\nu} w^\nu_{\underline{x}} \phi^\nu_{\underline{x}}\|_{H^l(R^n)} \\ \le C \left(\sum_{\underline{x}\in X^\nu} (h^\nu_{\underline{x}})^{(r^\nu_{\underline{x}}+1-l)2} \|u\|^2_{H^{r^\nu_{\underline{x}}+1}(B^{\overline{\rho}h^\nu_{\underline{x}}}_{\underline{x}})} \right)^{1/2},\end{aligned} \tag{18}$$

for $0 \le l \le l^\nu \equiv \min\{q, r^\nu_{\underline{x}} + 1 : \underline{x} \in X^\nu\}$ *for all* $\nu \in N$. C *here depends on the constants in assumptions 1–7, but neither on* u, *nor on* ν.

The proof of this theorem is given in [12].

Remark 4. For uniformly distributed particles (and associated particle shape functions) Assumption 1–7 are satisfied with $h^\nu_{\underline{x}^\nu} = h$ and $\underline{x}^\nu = x^h_j$.

Remark 5. In [12], we will address some ways to construct particle shape functions that satisfy Assumption 1–7. In general, the verification of these assumption is not easy for higher p; for $p = 0$ the construction and verification is usually quite easy.

Remark 6. Theorem 4 is a generalization of the h-version of the FEM. It permits approximation of functions that are characterized by weighted Sobolev spaces.

So far we have addressed the case when one particle function is associated with each particle. Let us now consider the case when more shape functions are used.

Theorem 5 *Suppose* $\{X^\nu\}, h^\nu_{\underline{x}^\nu}, \omega^\nu_{\underline{x}^\nu}$, *and* $\phi^\nu_{\underline{x}^\nu}$ *satisfy Assumptions 1–7. In terms of* $\phi^\nu_{\underline{x}}$, $\underline{x} \in X^\nu$, *for all* $\nu \in N$, *define the shape functions*

$$\psi_{\underline{x},\beta^\nu_{\underline{x}}} = \phi^\nu_{\underline{x}}(x)(x - \underline{x})^{\beta^\nu_{\underline{x}}},$$

where $|\beta^\nu_{\underline{x}}| \le t^\nu_{\underline{x}}$. *Suppose*

$$\sum_{\underline{x}\in X^\nu} \|u\|^2_{H^{r^\nu_{\underline{x}}+1}(B^{\overline{\rho}h^\nu_{\underline{x}}}_{\underline{x}})} < \infty, \text{ where } r^\nu_{\underline{x}} \le p + t^\nu_{\underline{x}},$$

for all $\underline{x} \in X^{\nu}$, *for all* $\nu \in N$. *Then there are weights* $w^{\nu}_{\underline{x},\beta^{\nu}_{\underline{x}}} \in R$ *such that*

$$\begin{aligned} &\|u - \sum_{\underline{x}\in X^{\nu}} \sum_{|\beta^{\nu}_{\underline{x}}|\leq t^{\nu}_{\underline{x}}} w^{\nu}_{\underline{x},\beta^{\nu}_{\underline{x}}} \psi_{\underline{x},\beta^{\nu}_{\underline{x}}} \|_{H^{l}(R^{n})} \\ &\quad \leq C \left(\sum_{\underline{x}\in X^{\nu}} (h^{\nu}_{\underline{x}})^{(r^{\nu}_{\underline{x}}+1-l)2} \|u\|^{2}_{H^{r^{\nu}_{\underline{x}}+1}(B^{\overline{\rho} h^{\nu}_{\underline{x}}}_{\underline{x}})} \right)^{1/2}, \end{aligned} \tag{19}$$

for $0 \leq l \leq l^{\nu} \equiv \min\{q, r^{\nu}_{\underline{x}} + t^{\nu}_{\underline{x}} : \underline{x} \in X^{\nu})\}$, *for all* ν. *C here depends on the constants in assumptions 1–7, but neither on* u, *nor on* ν.

The proof of this result is given in [12].

Remark 7. Theorem 5 avoids the difficulties connected with the construction of particle shape function for higher p (see Remark 5) because we construct the functions $\phi^{\nu}_{\underline{x}^{\nu}}$ only for small values of p.

Remark 8. Although Theorem 5 is formulated in the framework of the h version of the FEM, it is essentially a generalization of the $h-p$ version. See Section 4.

Remark 9. It is also possible to prove an inverse theorem that characterizes the smoothness of the approximated function in terms of the convergence rate. This analysis is similar to that in [8].

In Section 2 we defined the space V^{h} for the case of uniformly distributed particle and associated shape functions. We will also use the same symbol, V^{h} (instead of V^{ν}), for the general case of non-uniformly distributed particles, which was addressed in Theorems 4 and 5.

4 The Generalized Finite Element Method

In the previous section we addressed the approximation by particle functions of the form $\phi^{\nu}_{\underline{x}^{\nu}}(x)(x - \underline{x}^{\nu})^{\beta^{\nu}_{\underline{x}^{\nu}}}$. We now further generalize the character of these shape functions. We let $\{\phi^{\nu}_{\underline{x}}\}_{\underline{x}\in X^{\nu}}$, for $\nu \in N$, be shape functions with compact supports $\eta^{\nu}_{\underline{x}}$, and assume they are a *partition of unity, i.e.,*

$$\sum_{\underline{x}\in X^{\nu}} \phi^{\nu}_{\underline{x}} = 1, \text{ for all } \nu \in N. \tag{20}$$

We note that we do not assume Items 1–7, given in Section 3, for the particles considered in this section. However, (20) implies that $\bigcup_{\underline{x}\in X^{\nu}} \mathring{\eta}^{\nu}_{\underline{x}} = R^{n}$, for each ν. We also assume that the pointwise overlap

$$M \equiv \sup_{y\in R^{n},\, \nu\in N} \text{card}\, \{\underline{x} \in X^{\nu} : y \in \eta^{\nu}_{\underline{x}}\} < \infty.$$

We further assume that

$$\|\phi^\nu_{\underline{x}}\|_{L^\infty(R^n)} \le C_1(\underline{x}), \qquad \text{for all } \underline{x} \in X^\nu, \text{ for all } \nu, \tag{21}$$

$$\|\nabla\phi^\nu_{\underline{x}}\|_{L^\infty(R^n)} \le \frac{C_2(\underline{x})}{\text{diam } \eta^\nu_{\underline{x}}}, \qquad \text{for all } \underline{x} \in X^\nu, \text{ for all } \nu, \tag{22}$$

and

$$\Psi^\nu_{\underline{x}} \subset H^1(\eta^\nu_{\underline{x}}) \text{ is an } N^\nu(\underline{x}) \text{ dimensional space defined on } \eta^\nu_{\underline{x}}, \tag{23}$$

for $\underline{x} \in X^\nu$, for all ν. Then we have

Theorem 6 *Let $u \in H^1(R^n)$ and suppose for every $\underline{x} \in X^\nu$, for all ν, there exists $v^\nu_{\underline{x}} \in \Psi^\nu_{\underline{x}}$ such that*

$$\|u - v^\nu_{\underline{x}}\|_{L^2(\eta^\nu_{\underline{x}})} \le \epsilon_1(\underline{x}), \qquad \textit{for all } \underline{x} \in X^\nu, \textit{ for all } \nu \tag{24}$$

and

$$\|\nabla u - \nabla v^\nu_{\underline{x}}\|_{L^2(\eta^\nu_{\underline{x}})} \le \epsilon_2(\underline{x}), \qquad \textit{for all } \underline{x} \in X^\nu, \textit{ for all } \nu. \tag{25}$$

Then

$$\phi^\nu_{\underline{x}} v^\nu_{\underline{x}} \in H^1(R^n), \textit{ with compact support}, \textit{ for all } \underline{x} \in X^\nu, \textit{ for all } \nu \tag{26}$$

and

$$\|u - \sum_{\underline{x}\in X^\nu} \phi^\nu_{\underline{x}} v^\nu_{\underline{x}}\|_{H^1(R^n)} \tag{27}$$

$$\le \left(2M\left(\sum_{\underline{x}\in X^\nu} \epsilon_1(\underline{x})^2 \left(C_1(\underline{x})^2 + \frac{C_2(\underline{x})^2}{(\textit{ diam } \eta^\nu_{\underline{x}})^2}\right) + \sum_{\underline{x}\in X^\nu} \epsilon_2(\underline{x})^2 C_1(\underline{x})^2\right)\right)^{1/2}$$

for all ν.

Theorem 7 *Suppose $1 \in \Psi^\nu_{\underline{x}^\nu}$ and that*

$$\inf_{\lambda\in R} \|u - \lambda\|_{L^2(\eta^\nu_{\underline{x}})} \le C_P\,(\textit{diam } \eta^\nu_{\underline{x}})\,|u|_{H^1(\eta^\nu_{\underline{x}})}, \textit{ for all } \underline{x} \in X^\nu, \textit{ for all } \nu, \tag{28}$$

with $C_P < \infty$ independent of $\underline{x}$ and ν, and where $|\cdot|_{H^1(\eta^\nu_{\underline{x}})}$ is the semi-norm. Also assume that C_1, C_2 in (21) and (22), respectively, are independent of $\underline{x}$ and ν. Then there exists $v^\nu_{\underline{x}} \in \Psi^\nu_{\underline{x}}$, satisfying (25), such that

$$\|u - \sum_{\underline{x}\in X^\nu} \phi^\nu_{\underline{x}} v^\nu_{\underline{x}}\|_{H^1(R^n)} \le C\left(\sum_{\underline{x}\in X^\nu} \epsilon_2(\underline{x})^2\right)^{1/2}, \qquad \textit{for all } \nu, \tag{29}$$

where C depends on C_1, C_2, M, and C_P, but is independent of u and ϵ_2.

Remark 10. Inequality (28) is the Poincaré inequality. It holds if there are balls $B^{\nu}_{\underline{x}^{\nu}}$ and $B^{\nu *}_{\underline{x}^{\nu}} \supset B^{\nu}_{\underline{x}^{\nu}}$, with diameters $\rho_{\underline{x}^{\nu}}$ and $\rho^{*}_{\underline{x}^{\nu}}$, respectively, such that $\rho^{*}_{\underline{x}^{\nu}} \leq \kappa \rho_{\underline{x}^{\nu}}$, with κ independent of $\underline{x}^{\nu}$ and ν, and $B^{\nu}_{\underline{x}^{\nu}} \subset \eta^{\nu}_{\underline{x}^{\nu}} \subset B^{\nu *}_{\underline{x}^{\nu}}$.

Remark 11. If $\Psi^{\nu}_{\underline{x}^{\nu}}$ is a space of polynomials, then Theorem 6 leads to Theorem 5. Moreover, if the assumptions of Theorem 7 hold, then Theorem 7 also leads to Theorem 5.

Remark 12. Theorem 6 and Theorem 7 are generalizations of the p-version of the FEM.

Remark 13. Theorem 6 allows the use of additional information about the approximated function. For example, if u is harmonic, then $\Psi^{\nu}_{\underline{x}^{\nu}}$ can be the space of harmonic polynomials; if u has various singularities, as in the neighborhood of the corners or has boundary layer behavior, $\Psi^{\nu}_{\underline{x}^{\nu}}$ can be chosen accordingly. For proofs of Theorem 6 and Theorem 7, we refer to [7], [9], [12], [30].

Remark 14. If $u = 0$ on part of $\eta^{\nu}_{\underline{x}^{\nu}}$, we do not include the constant function in the space $\Psi^{\nu}_{\underline{x}^{\nu}}$.

Remark 15. The GFEM is obviously a generalization of the meshless method.

5 Approximation in Bounded Domains

In the previous sections we have addressed the approximation of functions in R^n by linear combinations of particle shape functions. These results lead immediately to approximation results in Lipschitz domains. We need only to use an extension theorem.

Theorem 8 *[33] Suppose Ω is a Lipschitz domain and suppose $k \geq 0$. Then there is a bounded extension operator $\mathcal{E} : H^k(\Omega) \to H^k(R^n)$, i.e., an operator $\mathcal{E}$ such that for all $u \in H^k(\Omega)$, $v = \mathcal{E}(u)$ has compact support in R^n,*

$$\|v\|_{H^k(R^n)} \leq C\|u\|_{H^k(\Omega)}, \text{ for all } u \in H^k(\Omega),$$

and

$$v(x) = u(x), \text{ for all } x \in \Omega.$$

Here C is independent of u (it does depend on Ω and k).

Theorem 8 yields the desired result when usual Sobolev spaces are used. Theorems 4 and 5 address the approximation by particle shape functions with refinements. As stated in Remark 6, we can approximate functions that lie in the weighted Sobolev spaces. Here we need extension of weighted Sobolev spaces when the weight is a power of the distance from the vertex of the domain where the solution of a PDE has singular behavior. Such theorems for $\Omega \subset R^n$ with piecewise smooth boundary were proved in [8].

Remark 16. We addressed only Lipschitz domains, which exclude slit domains. These domains require special treatment, which, because of the length restriction of this paper, we will not discuss.

Let us now introduce the notion of a (t,k)-regular system of functions. For $0 \leq k \leq t$, supposes $\{\gamma_h^{t,k}(\Omega)\}_{0<h\leq 1}$ is a-one parameter family of linear spaces. We say $\{\gamma_h^{t,k}(\Omega)\}_{0<h\leq 1}$ is a (t,k)-*regular system* if

1. $\gamma_h^{t,k}(\Omega) \subset H^k(\Omega)$, for $0 < h \leq 1$, and
2. For every $w \in H^l(\Omega)$, with $0 \leq l$, there is a function $g \in \gamma_h^{t,k}$ such that

$$\|w - g\|_{H^s(\Omega)} \leq Ch^{\mu}\|w\|_{H^l(\Omega)}, \text{ for all } 0 \leq s \leq \min\{l,k\}, \tag{30}$$

where $\mu = \min\{t-s, l-s\}$. The constant C is independent of g and h.

Remark 17. In the definition of a (t,k)–regular system, we can have $\Omega = R^n$.

We now introduce two additional notions. A (t,k)-regular system $\{\gamma_h^{t,k}(\Omega)\}$ is said to satisfy a *local assumption* if for $w \in H^l(\Omega)$ with compact support ω, the function $g \in \gamma_h^{t,k}(\Omega)$ can be chosen so that the support ω_h of g has the property that

$$\omega_h \subset \omega^{\lambda h} \equiv \{x \in \Omega : d(x,\omega) \leq \lambda h\},$$

where $d(x,\omega)$ is the distance form x to ω, and λ is independent of ω, s, k. We say that $\{\gamma_h^{t,k}(\Omega)\}$ satisfies an *inverse assumption* if

$$\|g\|_{H^k(\Omega)} \leq Ch^{-(k-s)}\|g\|_{H^s(\Omega)}, \text{ for all } 0 \leq s \leq k \text{ and all } g \in \gamma_h^{t,k}(\Omega).$$

In [6] many important properties of (t,k)–regular systems are proved. Some of these properties will be used in Section 7.

A linear system of particle shape functions is a (t,k)–regular system; more precisely we have,

Theorem 9 *Let* $\{\underline{x}^{\nu}\}$ *and* $\{\phi_{\underline{x}}^{\nu}\}$*, for* $\underline{x}^{\nu} \in X^{\nu}$ *and all* $\nu \in N$*, be a system of particles and associated particle shape functions that are quasi-reproducing of order* p*. Suppose that the particles are quasi-uniform, i.e.,*

$$\kappa_1 \leq \frac{h_{\underline{x}}^{\nu}}{h_{\underline{y}}^{\nu}} \leq \kappa_2, \text{ for all } \underline{x}, \underline{y} \in X^{\nu}, \text{ for all } \nu \in N,$$

where $0 < \kappa_1 < \kappa_2 < \infty$ *are independent of* $\underline{x}, \underline{y}$*, and* ν*. Then the associated linear system of particle shapes functions is a* $(p+1,q)$*-regular system.*

The proof of this theorem is given in [12].

Remark 18. With some mild assumptions on the distribution of particle near the boundary of Ω, the particle shape functions are a $(p+1,q)$-regular system that satisfies an inverse assumption [12].

6 Construction of Particle Shape Functions and Some of Their Properties

Various particle shape functions are used today in meshless methods. Some major representatives are

1. Reproducing Kernel Particle (RKP) shape functions. These are conceptually based on smooth particle hydrodynamics (SPH) approximations [22]. For RKP, see *e.g.,* [12], [28].
2. Moving Least Squares (MLS) particle shape functions [25]. This idea was further generalized into Moving Least Squares Kernel (MLSK). See *e.g.,* [27], [29].
3. In the GFEM often the classical finite element shape functions are used as partition of unity functions [9], [11], [30].

One of the major problems in this area is the assessment of the performance of the shape functions, respectively, their selection. The shape functions, constructed in Items 1 and 2 above, are reproducing of order p. Error estimates for the meshless approximation, using such shape functions, are obtained via an interpolation error estimate. For translation invariant shape functions (addressed in Theorem 1) associated with uniformly distributed particles, the error can be analyzed in detail. Let u be sufficiently smooth, and consider the "interpolant" $I^h u$ of u defined as

$$I^h u(x) = \sum_j u(x_j^h)\phi_j^h(x), \quad x_j^h = hj,$$

where $\{\phi_j^h\}$ are reproducing of order p. The accuracy of Galerkin approximation (8) of a smooth function u (using u instead of u_0), by these shape functions, is indicated by the estimate

$$\inf_{v \in S} \|u - v\|_{H^1(R^n)} \le \|u - I^h u\|_{H^1(R^n)} \le Ch^p, \tag{31}$$

where $S = \text{span}\ \{\phi_j^h\}$. The following result is proved in [13] for $n = 1$.

Theorem 10 *Suppose u is a sufficiently smooth function and consider the shape functions that are reproducing of order p. Then*

$$\sup_{u \in H^{p+2}(R)} \lim_{h \to 0} \frac{\|u - I^h u\|^2_{H^1(R)}}{h^{2p}[|u|^2_{H^{p+1}(R)} + |u|^2_{H^{p+2}(R)}]} = \frac{|\xi_{p+1}|^2_{H^1(0,1)}}{(p+1)!^{\,2}}, \tag{32}$$

where

$$\xi_{p+1}(x) = x^{p+1} - \sum_{j \in Z} j^{p+1}\phi(x - j).$$

$\xi_{p+1}(x)$ is a periodic function with period 1, and $\phi(x)$ is the basic shape function.

A similar result also holds in R^n for bounded domains.

Theorem 11 *Suppose the shape functions are reproducing of order p. Let $u \in H^k(R)$, with k sufficiently large, and suppose*

$$\|u - I^h u\|_{L_2(R)} \leq C h^{p+1+\epsilon},$$

where C is independent of h. Then $u = 0$. (Note that $u(x) = 0$ is the only polynomial contained in $H^k(R)$.)

Remark 19. In [13], we computed $|\xi_{p+1}|_{H^1(0,1)}$ for several particle shape functions as a comparison of the performance of various particle shape functions.

Remark 20. Theorem 11 is a saturation theorem, and is proved in [13]. It can serve as a basis for the verification of the implementation code, as h approaches 0.

Remark 21. In [10], [11] we introduced the theory of the robustness of the selection of the spaces $\Psi^h_{x^h_j}$. We note that $\Psi^h_{x^h_j}$ is $\Psi^\nu_{\underline{x}^\nu}$ (defined in Section 4) for uniformly distributed particles.

We note that the use of reproducing shape functions of order p, and the associated analysis in terms of interpolation may give a pessimistic order of convergence of the usual Sobolev norm error in the approximation of smooth functions. Thus the use of quasi-reproducing shape functions of order p in the approximation analysis is appropriate, while the use of reproducing shape functions of order p is not.

Let us consider the situation where the particles are uniformly distributed. From the definition of reproducing and quasi-reproducing shape functions, it is clear that if a basic shape function $\phi(x)$ is reproducing of order p, then it is also quasi-reproducing of order p. But a basic shape function may be reproducing of order p, and quasi-reproducing of higher order $p+k,\ k \geq 1$, in which case the approximation error $\inf_{v \in S} \|u - v\|_{H^1(R)} \approx O(h^{p+k})$ for smooth u by Theorem 1. But note that (31), which is based on interpolation, will only yield $O(h^p)$. The basic shape function $\phi(x)$, with Fourier Transform $\hat{\phi}(\xi) = (\sin(\xi/2)/(\xi/2))^4$ (B-spline of order 4), is reproducing of order 1, but quasi-reproducing of order 3.

We further note that a basic shape function $\phi(x)$ may be reproducing of order p, and not quasi-reproducing of any higher order. In that case, Theorem 1 and (31) yield the same order of convergence. But even in this case, $\phi(x)$ may be "almost quasi-reproducing" of order $(p+1)$, *i.e.,*

(a) $\phi(x)$ is reproducing of order p

(b) $\xi_{p+1}(x) \equiv x^{p+1} - \sum_{i \in Z} i^{p+1} \phi(x - i) \approx$ Constant.

For such basic shape functions, the approximation error (for smooth functions) may decrease at a higher rate in the *pre-asymptotic range* (*i.e.,* for larger h), than is predicted. We note, however, that the pre-asymptotic range can be so large that for practical accuracy, the asymptotic range is not visible. It can be shown, using Theorem 10, that

$$\|u - I^h u\|_{H^1(R)} \approx \left(\frac{|\xi_{p+1}|_{H^1(0,1)} h^p}{(p+1)!} + O(h^{p+\epsilon}) \right) [|u|^2_{H^{p+1}(R)} + |u|^2_{H^{p+2}(R)}]^{1/2}.$$

For an "almost quasi-reproducing" basic shape function of the order $(p+1)$ (which is also reproducing of order p), we have $|\xi_{p+1}|_{H^1(0,1)} \approx 0$, and $\|u - I^h u\|_{H^1(R)}$ will exhibit higher order of convergence than the expected order, which is $O(h^p)$, in the pre-asymptotic range. An example of a RKP-shape function, which is almost quasi-reproducing of order 2, is given in [13].

7 Solution of Elliptic Boundary Value Problems

In Section 1 we introduced the boundary value problem (1), (2) ((3), respectively), its variational formulation, and its approximate solution. Equation (8) shows that the approximation theory introduced in the previous sections, with $S = V^h$, can be used directly for the approximate solution and for the error estimation when purely natural boundary conditions (2) are prescribed. Essential (Dirichlet) boundary conditions (3) require the construction of approximation functions that vanish on $\partial\Omega$, or the approximate satisfaction of the boundary condition. The GFEM naturally allows the implementation of Dirichlet boundary conditions by the appropriate choice of the space $\Psi^{\nu}_{\underline{x}^{\nu}}$. See also Remark 14.

Let us now consider several methods for boundary value problems with Dirichlet boundary conditions that use the same approximation space V^h used for Neumann boundary conditions. We will consider three methods:

1. The Penalty Method
2. Nitsche's Method
3. The Characteristic Function Method.

The aim is to obtain as much as possible the same rate of convergence as for the Neumann boundary conditions, *i.e.,* that the Dirichlet conditions will not adversely influence the rate of convergence. For simplicity we will assume here that the boundary, $\partial\Omega$, is sufficiently smooth and that the distribution of particles is quasi-uniform. We will assume the space V^h is a (t,k)–regular family of spaces (introduced in Section 5, see Theorem 9) satisfying a local assumption.

1. The Penalty Method

Here, instead of the bilinear form (5), we will use

$$B_\sigma(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,dx + h^{-\sigma}\int_{\partial\Omega} uv\,ds, \tag{33}$$

with $\sigma > 0$, and as before

$$L(v) = \int_\Omega fv\,dx. \tag{34}$$

The penalty method approximation $w_{\sigma,h} \in V^h$ satisfies

$$B_\sigma(w_{\sigma,h}, v) = L(v), \text{ for all } v \in V^h. \tag{35}$$

Then we have

Theorem 12 *[3] [4] Let u_0 be the solution of the problem (1), (3) ($H = H_0^1(\Omega)$) with $f \in H^l(\Omega)$, $l \geq 0$. Let $w_{\sigma,h} \in V^h$ be the approximate solution, defined in (35), with $k \geq 1$, and $t \geq l+2$. Then*

$$\|u_0 - w_{\sigma,h}\|_{H^1(\Omega)} \leq C(\epsilon)h^{\mu-\epsilon}\|f\|_{H^l(\Omega)}, \tag{36}$$

for $\epsilon > 0$ arbitrary, where $C(\epsilon)$ is independent of f and h, and

$$\mu = \min(\sigma, \frac{\sigma+1}{2} + l, \frac{t-\tau}{t-1}(l+1)), \tag{37}$$

where

$$\tau = \max(1, \frac{1}{2}(\sigma+1)). \tag{38}$$

Remark 22. For the Neumann boundary condition we have

$$\|u_0 - w_{\sigma,h}\|_{H^1(\Omega)} \leq Ch^{\mu}\|f\|_{H^l(\Omega)},$$

where $\mu = \min(l+1, t-1)$, which is the maximal rate of convergence. We see a loss in the rate of convergence, but not a significant one. For example, using $l = 2, t = 4$, the maximal rate of convergnce is 3, while with a certain value of σ, which minimizes (37), we get the rate 2.74.

2. Nitsche's Method

Here we use the bilinear form $B_{N,\gamma}$ defined on $V^h \times V^h$ by

$$\begin{aligned} B_{N,\gamma}(u,v) = &\int_\Omega (\nabla u \cdot \nabla v + uv)\,dx \\ &- \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,ds - \int_{\partial\Omega} \frac{\partial v}{\partial n} u\,ds + \gamma h^{-1}\int_{\partial\Omega} uv\,ds, \end{aligned} \tag{39}$$

with $\gamma > 0$, and as before

$$L(v) = \int_\Omega fv\,dx. \tag{40}$$

The Nitsche's method approximation $u_{\gamma,h} \in V^h$ satisfies

$$B_{N,\gamma}(u_{\gamma,h}, v) = L(v), \text{ for all } v \in V^h. \tag{41}$$

Let us assume that

$$h \int_{\partial\Omega} (\frac{\partial v}{\partial n})^2 \, ds \leq C_1 \|\nabla v\|^2_{H^0(\Omega)}, \text{ for all } v \in V^h, \text{ with } C_1 < \infty. \tag{42}$$

Then we have from [32] (see also [34]),

Theorem 13 *Let u_0 be the solution of the problem (1), (3) ($H = H^1_0(\Omega)$) with $f \in H^l(\Omega)$, $l \geq 0$. Let $u_{\gamma,h} \in V^h$ be the approximate solution, defined by (41), with $t \geq 1$ and $k \geq 1.5$. Then*

$$\|u_0 - u_{\gamma,h}\|_{H^1(\Omega)} \leq C(\epsilon) h^{\mu-\epsilon} \|f\|_{H^l(\Omega)}, \ \epsilon > 0 \ \textit{arbitrary}, \tag{43}$$

where $C(\epsilon)$ is independent of f and h, and

$$\mu = \min(l + 1, t - 1). \tag{44}$$

Remark 23. Condition (42) can be guaranteed by the proper selection of the particles, *e.g.*, $|\eta^{\nu}_{\underline{x}^{\nu}} \cap \Omega| \geq \beta |\eta^{\nu}_{\underline{x}^{\nu}}|$, with $\beta > 0$ independent of $\underline{x}^{\nu}$ and ν.

3. The Characteristic Function Method
Let Φ be a smooth function on Ω such that

$$\Phi > 0 \text{ on } \Omega, \tag{45}$$

$$\Phi = 0 \text{ on } \partial\Omega, \tag{46}$$

and

$$|\nabla\Phi| > 0 \text{ on } \partial\Omega. \tag{47}$$

Let $V^h_{\Phi} = \{u : u = \Phi v, v \in V^h\}$. Then obviously $V^h_{\Phi} \subset H^1_0(\Omega)$, and we have (see [12]) the following result.

Theorem 14 *Let $u_h \in V^h_{\Phi}$ be the solution of (7) with $S = V^h_{\Phi}$. Then*

$$\|u_0 - u_h\|_{H^1(\Omega)} \leq C h^{\mu} \|u\|_{H^l(\Omega)}, \tag{48}$$

where C is independent of u and h, and

$$\mu = \min(l + 1, t - 1). \tag{49}$$

Remark 24. (48), (49) gives the optimal rate of convergence.

Remark 25. We have assumed homogeneous boundary conditions (2), quasi-uniform particles, and a smooth boundary. This method can be generalized for non-smooth boundaries.

8 Certain Implementational Aspects

The implementation of meshless and GFE methods involve certain basic steps, which are similar to standard FEM. These steps are as follows:

(a) Construction of the stiffness matrix and the load vector
(b) Solving the system of linear equations, and computation of data of interest.

Let us discuss these steps separately:

(a) Computation of the elements of the stiffness matrix and the load vector for meshless methods, which must be done numerically, is much more expensive than with standard FEM, because numerical integration cannot be performed on the "master" element. If the supports of the shape functions corresponding to the particles are circles (or spheres), then the underlying code becomes complex, and the computation of these elements become expensive (see *e.g.,*[18]). Also, the quadrature error can have a severe negative influence on the accuracy of the computed solution (see *e.g.,*[15], [18]). The quadrature error analysis, in the context of FEM, has been discussed in detail in Chapter 4 of [16]. This analysis is essentially based on the fact that $D^{\alpha}\phi \approx O(1)h^{-|\alpha|}$ for $|\alpha| \leq p$, and $D^{\alpha}\phi = 0$ for $\alpha \geq p+1$, where ϕ is a "straight" element of degree p. For curvilinear elements, the additional fact that the mapping from master element to physical element converges to a linear mapping as $h \rightarrow 0$, is used in the analysis. For meshless methods, the shape functions ϕ satisfy $D^{\alpha}\phi \approx O(1)h^{-|\alpha|}$ for all $|\alpha| \leq p$, but we do not have $D^{\alpha}\phi = 0$ for $|\alpha| \geq p+1$. Moreover, to maintain the optimal order of convergence, the numerical quadrature rule must approximate the elements of the stiffness matrix with a relative accuracy of $O(h^{p+2})$. In practice, the use of adaptive integration to compute the diagonal elements of the stiffness matrix with relative error $\leq 0.01h^{p+2}$ is quite sufficient. The quadrature rule, which is used to compute the diagonal elements, is then used for the off-diagonal elements of the same row and the corresponding element of the load vector, *i.e.,* the elements are computed simultaneously. In GFEM, the use of partition of unity function ϕ, based of finite elements, simplifies numerical quadrature. Nevertheless, the adaptive quadrature is essential if special functions, *e.g.,* functions with corner singularities, are used to construct the shape functions in GFEM. In most cases, numerical integration is the most expensive part of the computation.

(b) The stiffness matrix in meshless methods may be ill-conditioned for various distribution of particle points. The GFEM may give rise to a singular stiffness matrix, because of the linear dependency of the basis functions. For example, if the space $\varPsi^{\nu}_{\underline{x}^{\nu}}$ (see Section 4) is the space of polynomials, and the partition of unity functions ϕ are piecewise polynomial finite elements functions, then the basis functions are linearly dependent. In such situations, for each row of the stiffness matrix, every element of that row and the corresponding element of the load vector have to be computed simultaneously using the same adaptive quadrature procedure. Numerical integration is performed only locally over the "elements", which have non-empty intersection with the support of ϕ. This is essential to keep the linear system consistent. The computed stiffness matrix is positive semi-definite, and elimination with pivoting, or band elimination with perturbation and iteration, can be implemented successfully. Certainly, such a linear system has many solutions, and this leads to "erratic" computed solution. Nevertheless, the FEM solution is unique. This "erratic" behaviour disappears when data of interest are computed. In fact, the effect of round-off errors and global accuracy are similar as in the classical FEM (see [37]).

We further note that, the character of the stiffness matrix (*i.e.,* the distribution of non-zero entries) obtained from GFEM is different from that obtained from classical FEM [20]. This influences the performance of sparse elimination solvers. However, one may still use solvers based on the conjugate gradient method. We note that the multigrid method is not applicable to these problems, because the eigenfunctions associated with the zero eigenvalue are oscillatory. Nevertheless, changing the partition of unity function ϕ (at the expense of decreased rate of convergence) gives rise to a non-singular stiffness matrix, and some form of multigrid method can be used to solve the linear system (see [23], [24]).

9 Applications

The meshless method is applied in various fields. For a survey, we refer to [27]. In [38], a survey of GFEM with emphasis on complex geometries is given.

10 Additional Comments

Meshless methods (also referred to as Meshfree methods) and GFEM appear in the literature under different names, although their content is essentially same. For example, the partition of unity method, the GFEM, the extented FEM, and the method of clouds are essentially identical methods. See *e.g.,* [17], [19], [21]

Meshless methods are based on meshless approximation and discretization. The discretization could be based on a variational formulation or on collocations, and can be used for linear and non-linear problems.

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