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# Support Preserving Measure Algebras and Spectral Synthesis

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In this paper we consider various subspaces of first order distributions (and, in particular, pseudo-measures) as algebras with a support preserving multiplication; that is, if supp $S$, supp $T \subseteq E$ then supp $ST \subseteq E$; $E \subseteq \mathbb{R}/2\pi\mathbb{Z} \equiv \Gamma$ will always be a perfect set with Lebesgue measure $m(E)=0$. The main result says essentially that if the pseudo-measures supported by $E$ form a Banach algebra then the pseudo-measures are not only bounded but are quite close to being measures.

In Sec. 1 we define our various notation, operations, and algebras, and consider first order distributions as finitely additive set functions. Then in Sec. 2 we see that $M(E)$, the space of Radon measures with support in $E$, is a Banach algebra, and we calculate its maximal ideal space and see that $M(E)$ is symmetric (our multiplication is obviously not convolution). Sec. 3 is devoted to describing associated algebras that seem interesting in themselves and which are used to pinpoint the pseudo-measures on $E$ in Sec. 4.

## 1. Background, Notation, and Definition of Spaces

$A(\Gamma)$ is the space of absolutely convergent Fourier series $\phi \sim \sum a_n e^{in\gamma}$ with norm $\|\phi\|_A \equiv \sum |a_n|$; $A'(\Gamma)$, the *space of pseudo-measures*, is the dual of $A(\Gamma)$ with canonical norm $\|\ \|_{A'}$; and $A'(E) \equiv \{T \in A'(\Gamma)\colon \text{supp}\, T \subseteq E\}$. We designate the total variation norm on $M(E)$ by $\|\ \|_1$ and it is clear that $M(E) \subseteq A'(E)$. Notationally, we set $\mathscr{C}E \equiv \bigcup_0^\infty I_j$ where $I_j \equiv (\lambda_j, \gamma_j)$, $\varepsilon_j \equiv \gamma_j - \lambda_j$; and we refer to [4; 5] for preliminaries in pseudo-measures and Fourier analysis. Using the Hausdorff-Young theorem it is easy to see that if $T \in A'(E)$, $\hat{T}(0)=0$, then $T = f_T'$, distributionally, where

$$f_T = \sum_1 k_j \chi_{I_j} \quad \text{a.e.} \tag{1.1}$$

and $f_T \in L^p(\Gamma)$ for each $p \geqq 1$. As such we let $D_\omega(E)$ be the space of first order distributions $T$ where $T = f_T'$, $f_T \in L^p(\Gamma)$ for all $p \geqq 1$, and $f_T$ is given by (1.1). Without loss of generality we assume that if $T \in A'(E)$ (resp., $M(E)$) then $\hat{T}(0)=0$; hence, $M(E) \subseteq A'(E) \subseteq D_\omega(E)$ and $M(E)$, $A'(E)$ remain Banach spaces. Now, given $S$, $T \in D_\omega(E)$ with corresponding $f_T = \sum_1 k_j \chi_{I_j}$, $f_S = \sum_1 h_j \chi_{I_j}$, we define

$$ST \equiv (f_S f_T)',$$

noting that

$$f_S f_T = \sum_1 k_j h_j \chi_{I_j} \quad \text{a.e.} \tag{1.2}$$

Thus $U = f'_U$,

$$f_U \equiv \sum_1 \chi_{I_j} \quad \text{a.e.},$$

is a multiplicative identity in $D_\omega(E)$.

If $A'_S(E)$ consists of the elements $T$ in $A'(E)$ for which $\langle T, \phi \rangle = 0$ if $\phi = 0$ on $E$, $\phi \in A(\Gamma)$, then $E$ is a *spectral synthesis set* if $A'(E) = A'_S(E)$, and $E$ is a *Helson set* if $M(E) = A'_S(E)$. $E$ is a *set without true pseudo-measure* (or *strong spectral resolution set*) if $M(E) = A'(E)$. It is not known if every Helson set is a spectral synthesis set and so it is important to characterize sets of strong spectral resolution.

Since $m(E) = 0$, $E$ is totally disconnected and we let $\mathscr{F}$ be the family of compact open sets in the topological space $E$. $\mathscr{F}$ is a basis for the topology on $E$ and an algebra of sets; and any distribution $T$ with support in $E$ is a finitely additive set function on $\mathscr{F}$ where

$$T(F) \equiv \langle T, \psi_F \rangle,$$

$F \in \mathscr{F}$ and $\psi_F \in C^\infty(\Gamma)$ with $\psi_F = 1$ on a neighborhood of $F$ in $\Gamma$ and $\psi_F = 0$ on a neighborhood of $E - F$ in $\Gamma$. As such, we define

$$\|T\|_v \equiv \sup_{F \in \mathscr{F}} |T(F)|,$$

and $A'(E) = M(E)$ if and only if $\|T\|_v < \infty$ for each $T \in A'(E)$ (e.g., [2; 3] for related issues). We let $\mathscr{F}\mathscr{F}$ be the elements in $\mathscr{F}$ such that real-valued $\psi_F$ can be found with the further properties that $0 \leqq \psi_F \leqq 1$ and $0 < \psi_F < 1$ on only finitely many $I_j$. Then

**Proposition 1.1.** $\mathscr{F} = \mathscr{F}\mathscr{F}$.

*Proof.* Let $F \in \mathscr{F}$ and take $\psi \in C^\infty(\Gamma)$ such that $0 \leqq \psi \leqq 1$, $\psi = 1$ on a neighborhood of $F$ in $\Gamma$, and $\psi = 0$ on a neighborhood of $E - F$ in $\Gamma$.

Let $I_j$ have the property that $0 < \psi < 1$ for some points of $I_j$; and adjust $\psi$ on $I_j$ so that $\psi = 0$ on an open interval of $I_j$ but so that it retains all its other properties. Do this for each $j$ and hence

$$\psi = \sum \psi_{F_j},$$

where $0 < \psi_{F_j} < 1$ on only two $I_k$, $\psi_{F_j} = 1$ on a neighborhood of $F_j \in \mathscr{F}$ in $\Gamma$, and $\psi_{F_j} = 0$ on a neighborhood of $E - F_j$ in $\Gamma$.

Thus $\{F_j\}$ is an open cover of $F$ so that $F$ compact implies we can cover $F$ by $F_{n_1}, \ldots, F_{n_k}$; consequently, set $\psi_F \equiv \sum_1^k \psi_{F_{n_i}}$. Q.E.D.

We say that $I_n \leqq I_m$ if $\lambda_n < \gamma_m$ and if we consider $E \subseteq [0, 2\pi)$; also $I_{n_1} \leqq \cdots \leqq I_{n_k}$ is a *partition* $P$ of $E$.

**Proposition 1.2.** *The following are equivalent for* $T\in D_\omega(E)$, $f_T=\sum_1 k_j\chi_{I_j}$:

(a) $T\in M(E)$.

(b) $f_T(\pm\gamma)$ *is defined on all of* $\Gamma$ *(by taking limits) and* $f_T$ *is of bounded variation.*

(c) *There is* $M>0$ *for which*

$$\sup_P \sum_1^{k-1} |k_{n_{j+1}}-k_{n_j}|<M. \tag{1.3}$$

*Proof.* (b) is equivalent to (a) by the Riesz representation theorem, and (b) implies (c) since $f_T$ is of bounded variation. Assume (c) and let $f_T$ be real-valued.

Set

$$(Vf_T)(\gamma)\equiv\sup_P\left\{\sum_1^{k-1}|k_{n_{j+1}}-k_{n_j}|:\ \lambda_{n_k}<\gamma,\ \gamma\in I_j \text{ for some } j\right\}.$$

For $\gamma\in\bigcup_0 I_j$ define

$$f_1(\gamma)\equiv\tfrac{1}{2}((Vf_T)(\gamma)+f_T(\gamma)),$$

$$f_2(\gamma)\equiv\tfrac{1}{2}((Vf_T)(\gamma)-f_T(\gamma)).$$

Clearly $f_T=f_1+f_2$ and in the usual way we have that $f_1$ and $f_2$ are increasing functions on $\bigcup_0 I_j$ considered as a subset of $[0,2\pi)$.

Finally, for any $\gamma\notin\bigcup_0 I_j$ set $f_1(\gamma-)\equiv\sup\{f_1(\lambda):\ \lambda\in\bigcup_0 I_j,\ \lambda<\gamma\}$, $f_1(\gamma+)\equiv\inf\{f_1(\lambda):\ \lambda\in\bigcup_0 I_j,\ \lambda>\gamma\}$, and similarly for $f_2$; because we are dealing with monotone functions these inf and sup exist and (b) follows. Q.E.D.

We set $D_b(E)$ to be the space of those elements $T$ in $D_\omega(E)$ for which $f_T\in L^\infty(\Gamma)$. Motivated by Proposition 1.2 and the properties of bounded variation functions we define the space $\mathscr{G}(E)$ of *generalized measures* to be those elements $T$ of $D_b(E)$ such that the corresponding $f_T$ has the properties that $f_T(\gamma\pm)$ exist for all $\gamma\in\Gamma$ and $f_T$ has at most countably many jump discontinuities. Also let $A_b'(E)\equiv A'(E)\cap D_b(E)$; this is the space of *bounded pseudo-measures.*

Note that the mapping $T\rightsquigarrow f_T$ for all our subspaces of $D_\omega(E)$ is bijective.

## 2. The Support Preserving Banach Algebra $M(E)$

For each $T\in M(E)$ define

$$\|T\|_{1,\infty}\equiv\|T\|_1+\|f_T\|_\infty \tag{2.1}$$

where $T=f_T'$, $f_T=\sum_1 k_j\chi_{I_j}$, and $\|f_T\|_\infty\equiv\sup_j|k_j|$. Clearly

$$\|T\|_1\leqq\|T\|_{1,\infty}\leqq 2\,\|T\|_1.$$

Generally, when dealing with Banach spaces which have a separately continuous multiplication and multiplicative unit $U$, we employ the usual trick and identify the space with an algebra of operators that has a norm $\| \ \|$ which satisfies $\|ST\| \leqq \|S\| \, \|T\|$ and $\|U\|=1$.

**Proposition 2.1.** *$M(E)$, with multiplication defined by* (1.2) *and with norm given by* (2.1), *is a commutative Banach algebra with identity $U$.*

*Proof.* Given $S$, $T \in M(E)$ with $f_S = \sum_1 k_j \chi_{I_j}$, $f_T = \sum_1 h_j \chi_{I_j}$. Letting $I_{n_1} \leqq \cdots \leqq I_{n_m}$ we have

$$\sum_1^{m-1} |k_{n_{j+1}} h_{n_{j+1}} - k_{n_j} h_{n_j}| \leqq \sum_1^{m-1} |k_{n_{j+1}} (h_{n_{j+1}} - h_{n_j})| + \sum_1^{m-1} |h_{n_j} (k_{n_{j+1}} - k_{n_j})|.$$

Thus

$$\|ST\|_1 \leqq \|f_T\|_\infty \, \|S\|_1 + \|f_S\|_\infty \, \|T\|_1 \leqq \|T\|_{1,\infty} \, \|S\|_{1,\infty}. \quad \text{Q.E.D.}$$

We designate the Gelfand transform of $T \in M(E)$ by $\tilde{T}$.

**Proposition 2.2.** *Let $T \in D_\omega(E)$, $f_T = \sum_1 k_j \chi_{I_j}$. The following are equivalent:*

(a) *$T \in M(E)$.*

(b) $\|T\|_v < \infty$.

(c) *There is $M > 0$ such that if $I_{n_1} \leqq \cdots \leqq I_{n_{2m}}$ then*

$$\left| \sum_{j=1}^m (k_{n_{2j-1}} - k_{n_{2j}}) \right| < M. \tag{2.2}$$

*Proof.* The equivalence of (a) and (b) is given in [3] and the sum in (c) is $\langle T, \psi_F \rangle$ for some $\psi_F$. Q.E.D.

We state Proposition 2.2 to observe the equivalence of (2.2) and (1.3).

Let's now describe the obvious elements of $\mathcal{M}(M(E))$, the maximal ideal space of $M(E)$:

$$X_a \equiv \{F_n \in \mathcal{M}(M(E)) \colon \ F_n(T) \equiv k_n, \ f_T = \sum_1 k_j \chi_{I_j}, \ n \geqq 1\}$$

$$X_i^+ \equiv \{F_\gamma \in \mathcal{M}(M(E)) \colon \ F_\gamma(T) \equiv f_T(\gamma+), \ \gamma \in E \text{ inaccessible}\}$$

$$X_i^- \equiv \{F_\gamma \in \mathcal{M}(M(E)) \colon \ F_\gamma(T) \equiv f_T(\gamma-), \ \gamma \in E \text{ inaccessible}\}$$

$$X_a^\lambda \equiv \{F_{\lambda_n} \in \mathcal{M}(M(E)) \colon \ F_{\lambda_n}(T) \equiv f_T(\lambda_n -), \text{ some } n\}$$

$$X_a^\gamma \equiv \{F_{\gamma_n} \in \mathcal{M}(M(E)) \colon \ F_{\gamma_n}(T) \equiv f_T(\gamma_n +), \text{ some } n\}.$$

Thus for $X \equiv X_a \cup X_i^+ \cup X_i^- \cup X_a^\lambda \cup X_a^\gamma$ we have $X \subseteq \mathcal{M}(M(E))$.

**Proposition 2.3.** (a) *The elements of $X$ are identified with monotone convergent sequences $\{\lambda_{m_n}\}$, $\lambda_{m_n}$ accessible in $E$.*

(b) *$F \in X$ if and only if there is a subsequence $\{I_{m_n}\}$ such that for all $T \in M(E)$, $f_T = \sum_1 k_j \chi_{I_j}$,*

$$F(T) = \lim_n k_{m_n}. \tag{2.3}$$

*Proof.* (a) Given $\{\lambda_{m_n}\}$ a monotone (decreasing, say) convergent sequence in $[0, 2\pi)$ and let $\lambda_{m_n} \to \gamma$; $\gamma \in E$ since $E$ is closed.

For $T \in M(E)$, $f_T = \sum_1 k_j \chi_{I_j}$, $\lim_n k_{m_n}$ exists.

We set $F_\gamma(T) \equiv \lim_n k_{m_n}$. $F_\gamma$ is clearly a homomorphism, and, by monotonicity, $F_\gamma(T) = f_T(\gamma+)$.

Conversely, if $F_\gamma \in X$ then without loss of generality we have $F_\gamma(T) = f_T(\gamma+)$ for all $T \in M(E)$.

Since the accessible points are dense in $E$ we choose $\lambda_{m_n} \to \gamma$, and, since in this case we are dealing with right hand limit points, we take $\lambda_{m_n}$ monotone decreasing to $\gamma$.

(b) For $F_\gamma \in X$ we take a monotone sequence as in (a) and the corresponding intervals $\{I_{m_n}\}$.

For any $T \in M(E)$, since $F_\gamma(T) = f_T(\gamma+)$, say, and

$$\lim_n k_{m_n} = f_T(\gamma+), \qquad f_T = \sum_1 k_j \chi_{I_j},$$

we have (2.3).

Assume (2.3); that is, let $\{\lambda_{m_n}\}$ have the property that for all $T \in M(E)$, $f_T = \sum_1 k_j \chi_{I_j}$, $\lim_n k_{m_n}$ exists — we designate this limit by $F(T)$.

It is easy to see that we can choose a monotone subsequence of $\{\lambda_{m_n}\}$, call it $\{\lambda_{m_n}\}$ again, and hence apply (a). Q.E.D.

Obviously in the correspondence of Proposition 2.3 there are many monotone subsequences for any $F \in X$. Also, in the second part of the argument of Proposition 2.3b the existence of $\lim_n k_{m_n}$ for all $T$ implies the existence of a limit $\gamma$ of $\{\lambda_{m_n}\}$ with the property that $\lambda_{m_n} \geqq \gamma$ (or $\gamma \geqq \lambda_{m_n}$) for all but a finite number of the $\lambda_{m_n}$.

**Theorem 2.1.** (a) $X = \mathcal{M}(M(E))$.

(b) *$M(E)$ is a symmetric, semi-simple algebra with $\overline{X}_a = \mathcal{M}(M(E))$.*

*Proof.* (a) Clearly $X \subseteq \mathcal{M}(M(E))$.

For $T \in M(E)$, $f_T = \sum_1 k_j \chi_{I_j}$, we define $f_{\bar{T}} \equiv \sum_1 \bar{k}_j \chi_{I_j}$ and note that for any partition $I_{n_1} \leqq \cdots \leqq I_{n_m}$,

$$\sum_{j=1}^{m-1} |\bar{k}_{n_{j+1}} - \bar{k}_{n_j}| = \sum_{j=1}^{m-1} |k_{n_{j+1}} - k_{n_j}|;$$

thus $f'_{\bar{T}} \equiv \overline{T} \in M(E)$.

Also, if $F_\gamma \in X$ we define

$$M_{F_\gamma} \equiv \{T \in M(E) \colon F_\gamma(T) = 0\};$$

since $F_\gamma(T) = f_T(\gamma+)$, say, it is easy to see that $M_{F_\gamma}$ is a maximal (and hence closed) ideal in $M(E)$.

Taking any proper ideal $I \subseteq M(E)$ we show $I \subseteq M_{F_\gamma}$ for some $F_\gamma \in X$, and this proves that $X$ consists of all maximal ideals.

If $I \nsubseteq M_{F_\gamma}$ for some $F_\gamma \in X$ then for all $F \in X$ there is $T_F \in I$ such that $F(T_F) \neq 0$; we get a contradiction to this assumption.

Define $S_{T_F} \equiv g'_{T_F}$ where

$$g_{T_F} \equiv f_{T_F} f_{\bar{T}_F}, \tag{2.4}$$

so that since $I$ is an ideal we have $g_{T_F} \in I$ and for all $G \in X$ and all $T_F$

$$G(S_{T_F}) \geqq 0; \tag{2.5}$$

(2.5) follows by definition of $X$, from (2.4), and because $G \in X$.

We now show that $X$ is closed in $\mathscr{M}(M(E))$, where, of course, we have the weak $*$ topology from $M(E)$.

First let $F \in X$ and say that for all $T \in M(E)$, $F(T) = f_T(\gamma+)$, some $\gamma \in E$. A subbasic neighborhood of $F$ is

$$N \equiv \{H \in \mathscr{M}(M(E)) \colon |H(T) - F(T)| < \varepsilon\},$$

and if $f_T = \sum_1 k_j \chi_{I_j}$ we have $k_{n_j} \to F(T)$ where $\lambda_{n_j}$ is monotone convergent (in $[0, 2\pi)$); thus if $F_{n_j} \in X_a$ corresponds to $I_{n_j}$ we have $F_{n_j} \to F$.

Therefore $\bar{X}_a = X$ where $X$ has the induced weak $*$ topology.

Thus $\bar{X}_a = \bar{X}$ where $\bar{X}$ is the weak $*$ closure in $\mathscr{M}(M(E))$ of $X$.

Hence, for $F \in \bar{X}$ there is $\{F_{n_j}\} \subseteq X_a$ such that $F_{n_j} \to F$ – that is, if $f_T = \sum_1 k_j \chi_{I_j}$, $k_{n_j} \to F(T)$; consequently, we apply Proposition 2.3 and so $F \in X$ and $X$ is closed in $\mathscr{M}(M(E))$.

Because $X$ is weak $*$ closed in $\mathscr{M}(M(E))$ and $\mathscr{M}(M(E))$ is weak $*$ compact we have $X$ weak $*$ compact in $\mathscr{M}(M(E))$.

Without loss of generality take $F(T_F) = 1$ and hence $F(S_{T_F}) = 1$.

Now, for all $F \in X$ let $N_F \subseteq \mathscr{M}(M(E))$ be a weak $*$ neighborhood of $F$ such that $|\tilde{S}_{T_F}| > \frac{1}{2}$ on $N_F$; there is no problem about doing this since $\tilde{S}_{T_F}$ is continuous and $\tilde{S}_{T_F}(F) \equiv F(S_{T_F}) = 1$.

Further, $\tilde{S}_{T_F} > \frac{1}{2}$ on $X \cap N_F$, and since $X$ is weak $*$ compact, $X \subseteq N_{F^1}, \ldots, N_{F^k}$, $F^j \in X$, and

$$\tilde{S} \equiv \sum_{j=1}^{k} \tilde{S}_{T_{Fj}} > \tfrac{1}{2} \quad \text{on } X.$$

Therefore, $I$ an ideal implies $S \in I$. Thus, if $f_S = \sum_1 h_j \chi_{I_j}$ there is $f_{S^{-1}} \equiv \sum_1 \frac{1}{h_j} \chi_{I_j}$ with $S^{-1} \in M(E)$ because $X_a \subseteq X$ and $\tilde{S} > \frac{1}{2}$ on $X$.

Consequently, $U = SS^{-1} \in I$ and hence $I = M(E)$, a contradiction.

(b) We showed $\bar{X}_a = X$ in (a) so that since $X = \mathscr{M}(M(E))$ we have $\bar{X}_a = \mathscr{M}(M(E))$.

For the symmetry, recall from (a) that if $T \in M(E)$ then

$$\bar{T} \equiv f'_{\bar{T}} \in M(E).$$

Hence, $\tilde{\bar{T}} = \bar{\tilde{T}}$ on $X_a$ which does it.

For the semi-simplicity let $\tilde{T} \equiv 0$ on $X$, $T \in M(E)$. Then if $f_T = \sum_1 k_j \chi_{I_j}$ we have each $k_j = 0$ since $\tilde{T}(F_j) = 0$ and $\tilde{T}(F_j) = F_j(T) = k_j$. Q.E.D.

## 3. The Algebras $D_\omega(E)$ and $\mathcal{G}(E)$

For $D_\omega(E)$ we define the natural metric topology given by the countable family of norms

$$\|T\|_p \equiv \|f_T\|_p \equiv \left(\frac{1}{2\pi}\int_0^{2\pi} |f_T(\gamma)|^p\, d\gamma\right)^{1/p}, \qquad T\in D_\omega(E),\ p=1,2,\ldots$$

**Proposition 3.1.** *$D_\omega(E)$ is a Fréchet space and a continuous topological algebra with unit.*

*Proof.* Note that the metric space $D_\omega(E)$ is complete; in fact, if $\{T_n\}\subseteq D_\omega(E)$ is Cauchy we have $f_T\in L^p(\Gamma)$ such that $\|f_T-f_{T_n}\|_p \xrightarrow{n} 0$ for all $p\geqq 1$. In particular $f_{T_n}\to f$ in measure and so there is a subsequence (call it $\{f_{T_n}\}$ again) which converges to $f$ a.e.

Thus if $\gamma, \lambda\in I_j$ and $f_{T_n}(\gamma), f_{T_n}(\lambda)$ converge to $f(\gamma), f(\lambda)$, respectively, we have $f(\gamma)=f(\lambda)$ since $f_{T_n}(\gamma)=f_{T_n}(\lambda)$. Thus $f_T'\in D_\omega(E)$.

Now, for $S, T\in D_\omega(E)$ and $q\geqq 2$ we note that $(f_S f_T)^q\in L^1(\Gamma)$.

In fact, if $s\geqq 1$ $f_S^q, f_T^q\in L^s(\Gamma)$; and so if $\dfrac{1}{p}+\dfrac{1}{p'}=1$ we have $f_S^q\in L^p(\Gamma), f_T^q\in L^{p'}(\Gamma)$ so that $(f_S f_T)^q\in L^1(\Gamma)$ by Hölder's inequality.

Hence $ST\in D_\omega(E)$, and, again by Hölder, $(ST)\rightsquigarrow ST$ is continuous. Q.E.D.

Notationally we set $\mathcal{M}(D_\omega(E))\equiv\{F\in(D_\omega(E))':\ F\not\equiv 0,\ F(ST)=F(S)\,F(T)\}$. Also if $M(E)\subseteq B\subseteq D_\omega(E)$ is any Banach algebra define $\mathcal{M}(B)\equiv\{F\in B':\ F\not\equiv 0,$ $F(ST)=F(S)\,F(T)\}$; and for each $T\in B$, $\tilde{T}$ is the Gelfand transform of $T$. For example, $D_b(E)$, when normed by $\|T\|_b\equiv\|f_T\|_\infty$, $T\in D_b(E)$, is a Banach algebra.

**Proposition 3.2.** (a) $\overline{M(E)}=D_\omega(E)$.

(b) *$X_a=\mathcal{M}(D_\omega(E))$ and so $\mathcal{M}(D_\omega(E))$ is dense in $\mathcal{M}(M(E))$.*

*Proof.* (a) Let $T\in D_\omega(E), f_T=\sum_1 k_j\chi_{I_j}$, and set $f_{T_n}\equiv\sum_1^n k_j\chi_{I_j}$.

Letting $p\geqq 1$

$$2\pi\|T-T_n\|_p^p=\int_0^{2\pi}\left|\sum_{n+1}^{\infty} k_j\chi_{I_j}(\gamma)\right|^p d\gamma=\sum_{n+1}^{\infty}|k_j|^p\,\varepsilon_j;$$

but

$$2\pi\|T\|_p^p=\sum_1^{\infty}|k_j|^p\,\varepsilon_j$$

and so

$$\lim_n\|T-T_n\|_p=0.$$

(b) Let $F_n\in X_a$ and let $T_m\to 0$ in $D_\omega(E)$, $T_m\in D_\omega(E)$. If $f_{T_m}\equiv\sum_{j=1} k_{m,j}\chi_{I_j}$ and $p\geqq 1$ we have $F_m(T_m)=k_{m,n}$ and

$$\frac{1}{2\pi}\varepsilon_n|k_{m,n}|^p\leqq\frac{1}{2\pi}\sum_{j=1}^{\infty}|k_{m,j}|^p\,\varepsilon_j=\|T_m\|_p^p.$$

Thus, with $n$ and $p$ fixed, $k_{m,n}\to 0$ as $m\to\infty$ since $\|T_m\|_p\to 0$; consequently, $F_n\in\mathcal{M}(D_\omega(E))$.

Now if $F\in\mathcal{M}(D_\omega(E))$ let $T\in D_\omega(E)$ be such that $F(T)\neq 0$.

Setting $f_T \equiv \sum k_j \chi_{I_j}$ let $f_{T_N} \equiv \sum_1^N k_j \chi_{I_j}$ have the property that $F(T_N) \neq 0$ by (a). Therefore, if $n > N$ and $S \equiv \chi'_{I_n}$

$$F(S)\,F(T_N) = F(S T_N) = 0$$

so that $F(S) = 0$.

By linearity of $F$ there is $1 \leq n \leq N$ such that $F(P) \neq 0$, $P = \chi'_{I_n}$; if $R = \chi'_{I_k}$, $k \neq n$ and $1 \leq k \leq N$, then

$$F(R)\,F(P) = F(RP) = 0$$

so that $F(R) = 0$.

Also $F(P) = 1$ since $F(P) = F(PP)$.

Hence, by applying (a) again, we have $F(T) = k_n$ and so $F \equiv F_n \in X_a$. Q.E.D.

*Remark.* 1. Since $D_\omega(E)$ is not locally $m$-convex, a fact which is clear by the properties of $L^p$-spaces, we expect [7, p. 355] that there is a non-invertible $T \in D_\omega(E)$ such that for all $F \in \mathcal{M}(D_\omega(E))$, $F(T) \neq 0$; and this is obviously the case.

2. It is also easy to see that $\mathcal{M}(D_\omega(E))$ is not weak $*$ compact; for if it were, $X_a$ would be weak $*$ compact in $\mathcal{M}(M(E))$, by the continuity of the natural injection (by Proposition 3.2a) of $\mathcal{M}(D_\omega(E))$ into $\mathcal{M}(M(E))$, and this contradicts Theorem 2.1b.

The following is easy to prove from the properties of $D_\omega(E)$, and we refer to [1; 7] for general and related results.

**Proposition 3.3.** *$\mathcal{M}(D_\omega(E))$ is the space of closed maximal ideals in $D_\omega(E)$.*

It is also clear (e.g., Theorem 4.1) that –

**Proposition 3.4.**

(a) *$\mathcal{G}(E)$ is a closed subalgebra of $D_b(E)$.*

(b) *$\mathcal{M}(\mathcal{G}(E)) = \mathcal{M}(M(E))$.*

(c) *The space $C(\mathcal{M}(\mathcal{G}(E)))$ of continuous functions on $\mathcal{M}(\mathcal{G}(E))$ is precisely $\{\tilde{T}: T \in \mathcal{G}(E)\}$.*

## 4. Subalgebras of $\mathcal{G}(E)$

**Theorem 4.1.** *Let $M(E) \subseteq B \subseteq D_\omega(E)$, $B$ a Banach algebra with*

$$\mathcal{M}(B) \subseteq \mathcal{M}(M(E)). \tag{4.1}$$

*Then*

(a) *$\mathcal{M}(B) = \mathcal{M}(M(E))$, as sets and topologically.*

(b) *$B \subseteq \mathcal{G}(E)$.*

*Proof.* Since $\overline{M(E)} = D_\omega(E)$ we have $\bar{B} = D_\omega(E)$ and hence the canonical adjoint $D'_\omega(E) \to B'$ is injective; thus

$$\mathcal{M}(D_\omega(E)) \subseteq \mathcal{M}(B).$$

From Theorem 2.16, Proposition 3.26, and (4.1) we have

$$\overline{\mathcal{M}(B)} = \mathcal{M}(M(E)). \tag{4.2}$$

It is easy to check that the natural injection $\mathscr{M}(B)\to\mathscr{M}(M(E))$ is continuous, where both domain and range have their respective weak $*$ topologies.

By this continuity and (4.2) we have $\mathscr{M}(B)=\mathscr{M}(M(E))$ as sets and (a) follows by properties of compact spaces.

Let $T\in B$; then there is $\{T_n\}\subseteq M(E)$ such that $\tilde{T}_n\to\tilde{T}$ in the sup norm topology of $C(\mathscr{M}(M(E)))$ since $M(E)$ is symmetric.

Because $X_a\subseteq\mathscr{M}(B)$ and $\tilde{T}_n(F_j)=k_{n,j}$, for $f_{T_n}=\sum_{j=1}k_{n,j}\chi_{I_j}$ and $F_j\in X_a$, we have $f_{T_n}\to f_T$ uniformly on $\cup I_j$.

Similarly, if $F_\gamma\in X-X_a$ assume, without loss of generality, that $\tilde{S}(F_\gamma)=F_\gamma(S)=f_S(\gamma+)$, $\gamma$ an inaccessible point of $E$ and $S\in M(E)$.

Let $\{\lambda_{m_j}\}$ be monotone decreasing (as a subset of $[0,2\pi)$) and with the property that for all $S\in M(E)$, $f_S(\gamma+)=\lim_j h_{m_j}$ where $f_S=\sum_1 h_j\chi_{I_j}$; we can do this from the results of Sec. 2.

By hypothesis, $\lim_n \tilde{T}_n(F_\gamma)=\tilde{T}(F_\gamma)$ exists, and we show that

$$\lim_j k_{m_j}=\tilde{T}(F_\gamma),\qquad f_T=\sum_1 k_j\chi_{I_j}. \tag{4.3}$$

Now given $\varepsilon>0$, for any $j$,

$$|\tilde{T}(F_\gamma)-k_{m_j}|\leqq|\tilde{T}(F_\gamma)-\tilde{T}_n(F_\gamma)|+|\tilde{T}_n(F_\gamma)-k_{m_j}|,$$

and

$$|\tilde{T}_n(F_\gamma)-k_{m_j}|\leqq|f_{T_n}(\gamma+)-k_{n,m_j}|+|k_{n,m_j}-k_{m_j}|.$$

There is $N$ such that for all $n\geqq N$ and for all $j$,

$$|k_{n,m_j}-k_{m_j}|<\varepsilon/4\quad\text{and}\quad|\tilde{T}(F_\gamma)-\tilde{T}_n(F_\gamma)|<\varepsilon/2.$$

For this $N$ there is $J_N$ such that for all $j\geqq J_N$, $|f_{T_N}(\gamma+)-k_{N,m_j}|<\varepsilon/4$.

Thus, for $\varepsilon>0$ we've found $J\equiv J_N$ so that if $j\geqq J$, $|\tilde{T}(F_\gamma)-k_{m_j}|<\varepsilon$ and hence (4.3) holds.

Finally, to show $B\subseteq\mathscr{G}(E)$ we must prove that $f_T(\gamma+)\neq f_T(\gamma-)$ for at most countably many $\gamma\in E$.

Given $k>0$. There is $N>0$ such that for all $n\geqq N$ and for all $\gamma\in E$

$$|f_T(\gamma\pm)-f_{T_n}(\gamma\pm)|<1/4k.$$

For any fixed $n\geqq N$ there are at most countably many $\gamma$ for which

$$|f_{T_n}(\gamma+)-f_{T_n}(\gamma-)|>1/k;$$

thus for any $\lambda$, not one of these $\gamma$,

$$\begin{aligned}&|f_T(\lambda+)-f_T(\lambda-)|\\&\quad\leqq|f_T(\lambda+)-f_{T_n}(\lambda+)|+|f_{T_n}(\lambda+)-f_{T_n}(\lambda-)|+|f_{T_n}(\lambda-)-f_T(\lambda-)|\\&\quad\leqq 1/k.\end{aligned}$$

Therefore for a given $k$ there are at most countably many $\gamma$ for which $|f_T(\gamma+)-f_T(\gamma-)|>1/k$. Q.E.D.

**Corollary 4.1.1.** *Let $M(E) \subseteq B \subseteq D_\omega(E)$, $B$ a Banach algebra, and assume $M(E)$ is weakly dense (and hence norm dense) or dense in the spectral norm in $B$. Then*

(a) $\mathscr{M}(B) = \mathscr{M}(M(E))$, *as sets and topologically.*

(b) $B \subseteq \mathscr{G}(E)$.

*Proof.* By the weak denseness or spectral denseness (4.1) holds, and we apply the theorem. Q.E.D.

*Remark.* If $E$ is Helson and a spectral synthesis set then $A'(E)$ is the Banach algebra $M(E)$. If $E$ is not Helson but $A'(E)$ is a Banach algebra (containing $M(E)$) then $A'(E) \subseteq \mathscr{G}(E)$ if $E$ satisfies either of the denseness conditions of Corollary 4.1.1, or, more generally, if (4.1) is satisfied.

**Theorem 4.2.** *Let $M(E) \subseteq B \subseteq D_\omega(E)$, $B$ a Banach algebra, and assume that the identity homomorphism $M(E) \to M(E)$ extends to a homomorphism $j: B \to M(E)$. Then $B \subseteq \mathscr{G}(E)$.*

*Proof.* Since $j$ is surjective $j(B)$ is dense in $M(E)$ with the spectral norm.

Thus $\mathscr{M}(M(E)) \hookrightarrow \mathscr{M}(B)$ homeomorphically.

For $T \in B$, $\tilde{T}_r$ is the restriction of $\tilde{T}$ to $\mathscr{M}(M(E))$.

Since $\tilde{T}_r \in C(\mathscr{M}(M(E)))$ and $M(E)$ is symmetric $\tilde{T}_r$ is the uniform limit of $\tilde{T}_n$, $T_n \in M(E)$.

By the calculation at the end of Theorem 4.1, $B \subseteq \mathscr{G}(E)$. Q.E.D.

*Remark.* Generally, when one wishes to show $M(E) = Y$, for some subspace $Y$ of $A'(E)$, it is natural to extend the identity map $M(E) \to M(E)$ to a linear transformation $j$: $Y \to M(E)$, $Y$ a Banach space [2; 6]; this process has built into it that $j$ is injective. In Theorem 4.2 we need more initial structure on the space (viz., $B$ must be a Banach algebra not just a Banach space) and on the map (viz., $j$ must be a homomorphism) but we do not make any requirements concerning injectiveness.

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