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(LF) Spaces and Distributions on Compact Groups and Spectral Synthesis on $R/2\pi Z$

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Let $E \subsetneqq R/2\pi Z$ be closed with Lebesgue measure 0. We imbed the pseudo-measures T supported by E into a space of distributions on a specific compact connected group. The reason for this approach is to make use of the more tractable differentiable absolutely convergent Fourier series for the general problem to determine when T is a measure. The specific results are outlined in the Introduction. Applications of the techniques presented here are used to obtain new criteria that a Helson set be a set of spectral synthesis in the author's forthcoming work (viz., "A Support Preserving Hahn-Banach Property and the Helson-S Set Problem" and "The Helson-S Set Problem and Discontinuous Homomorphisms on Metric Algebras").

Let $E \subsetneqq R/2\pi Z \equiv \mathring{\Gamma}$ be closed with Haar measure $m(E)=0$. We imbed $\mathring{\Gamma}$ into a canonical compact connected group $\Gamma = \hat{G}$ (§ 1) and note that if T is a pseudo-measure with support in E, i.e., $T \in A'(E)$, then there is a corresponding well-defined linear functional $t \sim T$ on the space of C^∞ functions on Γ. We prove that T is a measure, i.e., $T \in M(E)$, if and only if t is a distribution on Γ (Theorem 6.1), i.e., $t \in \mathscr{D}'_X$. We then give conditions that $t \sim T \in A'(E)$ be a distribution (Theorem 7.2); and prove that t corresponding to any $T \in A'(E)$ is always at least a slightly more general distribution, i.e., $t \in \mathscr{D}'_Z$ or $t \in \mathscr{D}'_Y$ (§ 9), depending on whether T is a bounded pseudo-measure or not (Theorem 9.1).

We were led to this approach because of the general open problem to determine if a Helson set E is a spectral synthesis set, i.e., if $A'(E) = M(E)$. The analytic techniques available to study this question are valid, generally speaking, on $C^\infty(\mathring{\Gamma})$, whereas definitive answers depend on behavior at every $\varphi \in A(\mathring{\Gamma})$, the space of absolutely convergent Fourier series [2, Chapter 2]. Thus, by determining $C^\infty(\mathring{\Gamma})$ subgroups G of $A(\mathring{\Gamma})$ we are able to use these techniques, while changing the setting to distributions on Γ. A natural and particular G, obviously depending on E, is defined in § 8.

Although distribution theory on (compact) groups has been exposited by several authors it seems that specific problems demand different base spaces, etc. (e.g., in at least one theory, the Schwartz space of test functions is not contained in the space of absolutely convergent Fourier series!). In fact our results mentioned above only become useful if $\mathscr{D}'_X$ is quite large and if the set of elements φ for which $\hat{t}(\varphi) \neq 0$ (for a given $t \in \mathscr{D}'_X$) are somehow bounded. As such we choose $\mathscr{D}_X$, $\mathscr{D}_Y$, and $\mathscr{D}_Z$ to be definite (LF)-spaces (§ 4 and § 9).

We refer to Dieudonné and Schwartz's paper [3] for the theory of (LF)-spaces, and to [5] and [4] for the other basic functional analysis and harmonic analysis facts that we use.

0. Notation for $R/2\pi Z$

We consider $E \subsetneq [0, 2\pi)$ and write $\mathscr{C}E = \bigcup_0 I_m$ where $I_m \equiv (\lambda_m, \gamma_m)$ and $\gamma_m - \lambda_m = \varepsilon_m$. $\mathscr{F}$ is the set of open and closed subsets of E with the induced topology from $R/2\pi Z$.

$A'(\mathring{\Gamma})$ is the Banach space dual of $A(\mathring{\Gamma})$ where $\varphi(\gamma) \equiv \Sigma\, a_n\, e^{in\gamma} \in A(\mathring{\Gamma})$ is normed by $\|\varphi\|_A \equiv \Sigma\, |a_n|$. We then let

$$A'(E) \equiv \{T \in A'(\mathring{\Gamma}) : \operatorname{supp} T \subsetneq E\}\,.$$

It is easy to check that if $T \in A'(E)$, $\hat{T}(0) = 0$, then $T = f'$ distributionally, where $f = \sum_1 k_m \chi_{I^m} \in \bigcap L^p(\mathring{\Gamma})$. Thus we define the space of bounded pseudo measures to be

$$A'_b(E) \equiv \{T \in A'(E) : f \in L^\infty(\mathring{\Gamma})\}\,.$$

1. The Imbedding Group

Let $G \subsetneq A(\mathring{\Gamma})$ be *a non-trivial additive subgroup of* $A(\mathring{\Gamma})$ consisting *of real-valued elements all of which vanish at* some λ'. We take G with the discrete topology and let Γ be its compact dual.

We are able to write down explicitly a large number of elements of Γ. In fact, we say $f \in X$, where f corresponds to a sequence $\{\lambda_j\} \subsetneq \mathring{\Gamma}$ and $\{r_j\} \subsetneq R$ with $\Sigma\, |r_j| < \infty$, and define

$$\forall\, \varphi \in G,\ (f, \varphi) \equiv \Pi \exp\{i r_j\, \varphi(\lambda_j)\}\,;$$

we frequently write $f \sim r_j, \lambda_j$.

Proposition 1.1. a) *Γ is a compact connected group.*

b) *$X \subsetneq \Gamma$ is a dense subgroup.*

c) *b is independent of the subgroup $G \subsetneq A(\mathring{\Gamma})$ that we choose with the above properties (i.e., given G and X we form Γ, depending only on G, and b holds).*

Proof. a) Γ is connected since G contains no compact subgroup different from $\{0\}$.

b) X is a subgroup because $\Sigma\, |r_j| < \infty$ for $f \sim r_j, \lambda_j$. Obviously we identify the zero element of Γ with any $f \sim r_j, \lambda_j$ for which $r_j = 0$ for all j, or $\varphi(\lambda_j) = 0$ for all j and all $\varphi \in G$, etc. We choose $0 \equiv f \sim \lambda'$.

If $\bar{X} \neq \Gamma$ we proceed in the standard way taking $f + \bar{X} \neq \bar{X}$, some $f \in \Gamma$, and defining $F \in \widehat{\Gamma/\bar{X}}$ such that

$$(F, f + \bar{X}) \neq 1\,.$$

From this we define the continuous homomorphism $\varphi \in G$ with the property $g \mapsto (F, g + \bar{X})$; consequently, $\varphi \neq 0$.
Now, take any $\psi \in G$ such that $\forall\, f \sim \lambda,\ \lambda \in \mathring{\Gamma}$,

$$(\psi, f) = e^{i\psi(\lambda)} = 1\,.$$

Thus, $\psi(\lambda) \equiv 2\pi\, k$, some $k \in Z$ and all $\lambda \in \mathring{\Gamma}$ (since ψ is continuous).

Clearly $k=0$ since $\psi(\lambda')=0$; and so $\psi\equiv 0$. Therefore $(\psi,\Gamma)=1$ and in particular $\varphi=0$, a contradiction.

c) Clear. q.e.d.

Thus there is canonical map

$$j:\mathring{\Gamma}\to\Gamma$$
$$\lambda\to f\sim\lambda\,,$$

that is, $(j\lambda,\varphi)=e^{i\varphi(\lambda)}$ for all $\varphi\in G$. In order that j be injective we assume that $\forall\lambda,\gamma\in\mathring{\Gamma}$, $\lambda\neq\gamma$, $\exists\varphi\in G$ *such that* $\forall k\in Z$

$$\varphi(\lambda)-\varphi(\gamma)\neq 2\pi\,k\,.$$

Proposition 1.2. *j iska continuous injection.*

Proof. If $\lambda_\alpha\to\lambda$ in $\mathring{\Gamma}$ then $\varphi(\lambda_\alpha)\to\varphi(\lambda)$ for all $\varphi\in G$; and so, $\forall\varphi\in G$, $(f_\alpha,\varphi)\to(f,\varphi)$ in Γ, where $f_\alpha\sim\lambda_\alpha$, $f\sim\lambda$, by the definition of the dual topology on Γ. q.e.d.

2. Associated Canonical Maps

For every $\Psi\in C(\Gamma)$ we associate the function

$$j'\Psi:\mathring{\Gamma}\to C(\Gamma)$$
$$\lambda\to\Psi(j\lambda)\,,$$

i.e., $\langle j'\Psi,\lambda\rangle\equiv\langle\Psi,j\lambda\rangle$, the canonical transpose.

Proposition 2.1. a) $j':C(\Gamma)\to C(\mathring{\Gamma})$.

b) *$j':C(\Gamma)\to C(\mathring{\Gamma})$ is a continuous homomorphism (when both range and domain have sup norm).*

Proof. a) Clear since j is continuous.

b) The homomorphism part is obvious, and so the continuity follows because we are dealing with Banach algebra and $C(\mathring{\Gamma})$ is semi-simple. q.e.d.

We next assume that G *be a real vector space.*

hom(G,R) is the space of continuous homomorphisms

$$h:G\to R$$

where R is considered additively and G *is normed by* $\|\ \|_G$, *where* $\|\ \|_G$ *is at least as strong as uniform convergence on* $\mathring{\Gamma}$. hom(G,R) is then a real vector space with canonical norm $\|\ \|_{\text{hom}}$; we let hom$(G,R)'$ be its Banach space dual with norm $\|\ \|'$.

For all $h\in\text{hom}(G,R)$ we define

$$\hbar:R\to\Gamma$$

where

$$\forall\varphi\in G\,,\qquad(\hbar(r),\varphi)\equiv e^{irh(\varphi)}\,.$$

Obviously $\hbar$ is a well-defined homomorphism, and if $h=0$ then $\hbar=0$.

Remark. There is a natural map

$$X \to \hom(G, R)$$

where if $f \sim r_j, \lambda_j$ we define the corresponding h as

$$\forall \varphi \in G, \qquad h(\varphi) \equiv \Sigma\, r_j\, \varphi(\lambda_j)\,.$$

h is certainly a homomorphism $G \to R$, and if $\|\varphi_n\|_G \to 0$ we have

$$|h(\varphi_n)| \leqq \sum_j |r_j|\, |\varphi_n(\lambda_j)| \leqq \|\varphi_n\|_G \sum_j |r_j| \to 0\,.$$

Proposition 2.2. *There is a natural continuous injective homomorphism*

$$k: G \to \hom(G, R)'$$

(the domain G considered as a discrete group) where

$$\forall h \in \hom(G, R), \qquad \langle k(\varphi), h\rangle \equiv h(\varphi)\,.$$

Proof. To show $k(\varphi) \in \hom(G, R)'$

$$\begin{aligned} k(\varphi)\,(\alpha_1 h_1 + \alpha_2 h_2) &= (\alpha_1 h_1 + \alpha_2 h_2)\,(\varphi) = \alpha_1 h_1(\varphi) + \alpha_2 h_2(\varphi) \\ &= \alpha_1 k(\varphi)\,(h_1) + \alpha_2 k(\varphi)\,(h_2)\,. \end{aligned}$$

If $h_n \to 0$ then

$$\sup\{|h_n(\psi)| : \|\psi\|_G \leqq 1\} \to 0 \quad \text{and so} \quad h_n(\varphi) \equiv \langle k(\varphi), h_n\rangle \to 0\,.$$

k is a homomorphism since

$$\langle k(\varphi_1 + \varphi_2), h\rangle = h(\varphi_1) + h(\varphi_2) = \langle k(\varphi_1), h\rangle + \langle k(\varphi_2), h\rangle = \langle k(\varphi_1) + k(\varphi_2), h\rangle\,.$$

To show k is injective take $\varphi \neq \psi$, $\varphi, \psi \in G$ and let $\varphi(\lambda) \neq \psi(\lambda)$; from the previous Remark let $h \sim \lambda \in \mathring{\Gamma}$, and hence $h \in \hom(G, R)$ has the property $h(\varphi) \neq h(\psi)$. q.e.d.

3. C^∞ Spaces

Let $\mathscr{T}$ be the vector space of trigonometric polynomials on Γ, i.e., functions of the form

$$\Psi(f) \equiv \sum_1^n a_j(f, \varphi_j) \equiv \Sigma\, a_j\, \Psi_{\varphi_j}(f)\,,$$

$\varphi_j \in G$, $\alpha_j \in C$.

A function $\Psi: \Gamma \to C$ is *differentiable in the direction* $h \in \hom(G, R)$ *at the point* $f \in \Gamma$ if

$$D_h \Psi(f) \equiv \frac{d}{du}\,(\Psi(f + \hbar(u))|_{u=0}$$

exists; in this case the *derivative of* Ψ *in direction* h *at* f is $D_h \Psi(f)$. For $h \equiv 0$ we set $D_0\, \Psi(f) \equiv \Psi(f)$.

Proposition 3.1. *$\forall \varphi \in G$ and $\forall h \in \hom(G, R)$*

$$D_h \Psi_\varphi(f) = i h(\varphi)\, \Psi_\varphi(f), \qquad \Psi_\varphi(f) \equiv (f, \varphi).$$

Proof.

$$D_h \Psi_\varphi(f) = \frac{d}{du}(\hbar(u) + f, \varphi)|_{u=0} = (f, \varphi)\frac{d}{du}(\hbar(u), \varphi)|_{u=0}$$

$$= (f, \varphi)\frac{d}{du}\, e^{iuh(\varphi)}|_{u=0} = i h(\varphi)\,(f, \varphi). \qquad \text{q.e.d.}$$

Let $C^\infty(\Gamma)$ be the vector space of functions $\Psi: \Gamma \to C$ such that $\forall B \subseteqq \hom(G, R)$, norm bounded, and for every non-negative integer N

$$p_{B,N}(\Psi) \equiv \sup\{\|D_{h_1} \dots D_{h_k} \Psi\|_\infty : h_j \in B, k \leqq N\} < \infty.$$

Clearly, $\mathcal{T} \subseteqq C^\infty(\Gamma)$ and the set of $p_{B,N}$ defines a localy convex topology on $C^\infty(\Gamma)$.

Proposition 3.2. *$C^\infty(\Gamma)$ is metrizable.*

Proof. Let $B_n \equiv \{h \in \hom(G, R): \|h\|_{\hom} \leqq n\}$.
Given $B \subseteqq \hom(G, R)$ take $B_n \supseteqq B$.

Then

$$V_\varepsilon(B_n, N) \subseteqq V_\varepsilon(B, N)$$

where, for example,

$$V_\varepsilon(B, N) = \{\Psi \in C^\infty(\Gamma): p_{B,N}(\Psi) < \varepsilon\}.$$

Consequently, $C^\infty(\Gamma)$ is metrizable. q.e.d.

We define the following subspaces of $C^\infty(\Gamma)$:

$C^\infty_{\mathcal{T}}(\Gamma)$ – The completion of $\mathcal{T}$ in the $C^\infty(\Gamma)$ topology;
$C^\infty_F(\Gamma)$ – The elements $\Psi \in C^\infty(\Gamma)$ which are represented by

$$\sum_{\varphi \in G} \hat{\Psi}(\varphi)\,(f, \varphi) \tag{3.1}$$

in the $C^\infty(\Gamma)$ topology.

Remark. $\forall \Psi \in C^\infty(\Gamma)$, we can show supp $\hat{\Psi}$, to be σ-compact, and hence there is a sequence $\{\varphi_j\} \subseteqq G$, depending on Ψ, so that $\hat{\Psi}(\varphi) = 0$ whenever $\varphi \notin \{\varphi_j : j = 1, \dots\}$. As such the Fourier series formula (3.1) is meaningful.

We give $C^\infty_{\mathcal{T}}(\Gamma)$ and $C^\infty_F(\Gamma)$ the induced topology from $C^\infty(\Gamma)$. Let $C^\infty_A(\Gamma)$ be the completion of $\mathcal{T}$ when the topology on $\mathcal{T}$ is defined by the family of norms

$$\varrho_{B,N}(\Psi) \equiv p_{B,N}(\Psi) + \sum_{\varphi \in G} |\hat{\Psi}(\varphi)|.$$

Clearly $C^\infty_F(\Gamma)$ is dense in $C^\infty_{\mathcal{T}}(\Gamma)$, and of course, $C^\infty_A(\Gamma) \subseteqq C^\infty_{\mathcal{T}}(\Gamma)$.

Proposition 3.3. *$C^\infty_A(\Gamma)$ and $C^\infty_{\mathcal{T}}(\Gamma)$ are Fréchet spaces.*

Proof. Let $\{\Psi_n\}$ be Cauchy in $C^\infty_A(\Gamma)$. Thus, there is $\Psi \in C^\infty_{\mathcal{T}}(\Gamma)$ such that $p_{B,N}(\Psi_n - \Psi) \to 0$ for all B, N; and there is $\Phi \in A(\Gamma)$ such that

$$\sum_{\varphi} |\hat{\Psi}_n(\varphi) - \Phi(\varphi)| \to 0. \tag{3.2}$$

Clearly, $\Psi = \Phi$ since (3.2) implies $\Psi_n \to \Phi$ uniformly on Γ. q.e.d.

The strong duals of $C^\infty(\Gamma)$, $C^\infty_{\mathscr{T}}(\Gamma)$, $C^\infty_F(\Gamma)$, and $C^\infty_A(\Gamma)$ are $D'(\Gamma)$, $D'_{\mathscr{T}}(\Gamma)$, $D'_F(\Gamma)$, and $D'_A(\Gamma)$, respectively; and we have the imbeddings

$$D'_{\mathscr{T}}(\Gamma) = D'_F(\Gamma) \subseteqq D'_A(\Gamma) \tag{3.3}$$

since $\bar{\mathscr{T}} = C^\infty_{\mathscr{T}}(\Gamma)$.

4. $\mathscr{D}_X$ and $\mathscr{D}'_X$

For every $k \geqq 1$ define

$$X_k \equiv \{\Psi \in C^\infty_A(\Gamma) : |G_\Psi| \leqq k\}$$

where

$$G_\Psi \equiv \{\varphi \in G : \hat{\Psi}(\varphi) \neq 0\}$$

and

$$|G_\Psi| \equiv \sup\{\|\varphi\|_G : \varphi \in G_\Psi\}\,.$$

Proposition 4.1. *$\forall k \geqq 1$, X_k is a closed subspace of $C^\infty_A(\Gamma)$ and thus is a Fréchet space with the induced topology from $C^\infty_A(\Gamma)$.*

Proof. Take $\Psi_1, \Psi_2 \in X_k$ and let $\alpha_1 \hat{\Psi}_1(\varphi) + \alpha_2 \hat{\Psi}_2(\varphi) \neq 0$. Then $\hat{\Psi}_j(\varphi) \neq 0$ for some j and so $\|\varphi\|_G \leqq k$. Let $\Psi_n \to \Psi$ in $C^\infty_A(\Gamma)$, $\Psi_n \in X_k$. For each $\varphi \in G_\Psi$ we show $\|\varphi\|_G \leqq k$. Note that

$$|\hat{\Psi}_n(\varphi) - \hat{\Psi}(\varphi)| \leqq \int_\Gamma |\Psi_n - \Psi| \to 0\,.$$

Consequently, $\exists N$ such that $\forall n \geqq N$, $\hat{\Psi}_n(\varphi) \neq 0$ and so $\|\varphi\|_G \leqq k$. q.e.d.

Observe that

$$X_1 \subseteqq X_2 \subseteqq \cdots,$$

and we set

$$\mathscr{D}_X \equiv \bigcup_1 X_k\,,$$

taken with the inductive limit topology. Thus $\mathscr{D}_X$ *is* an (LF)-*space* and, in particular, *is complete* and *is not metrizable.* Recall that $\mathscr{D}(R^n)$, à la Schwartz, is (LF). The fact that $\mathscr{D}_X$ is not metrizable follows from Baire's theorem and the fact that $\mathscr{D}_X$ is complete is an application of Köthe's theorem on the completeness of strict inductive limits of complete spaces.

We let $\mathscr{D}'_X$ be the strong dual of $\mathscr{D}_X$ and this will be our space of *distributions.*

Proposition 4.2. a) *$\mathscr{T} \subseteqq \mathscr{D}_X \subseteqq C^\infty_A(\Gamma) \subseteqq A(\Gamma)$.*

b) *$t \in \mathscr{D}'_X \Leftrightarrow t$ is continuous on each X_k, taken with its induced topology from $C^\infty_A(\Gamma)$.*

c) *$A'(\Gamma) \subseteqq D'_A(\Gamma) \subseteqq \mathscr{D}'_X$.*

Proof. a) follows from definition and b) is a property of inductive limits.

c) Let $t \in D'_A(\Gamma)$ so that $t \in X'_k$, $k \geqq 1$, where X_k has the induced topology from $C^\infty_A(\Gamma)$. Thus $t \in \mathscr{D}'_X$ since

$$\bar{\mathscr{T}} = C^\infty_A(\Gamma)\,.$$

To show $A'(\Gamma) \subseteqq D'_A(\Gamma)$ we need only note that $\bar{\mathscr{T}} = A(\Gamma)$ which is clear by the definition of $A(\Gamma)$. q.e.d.

(LF)-spaces are of course barrelled and bornological, and this latter property implies $\mathscr{D}'_X$ is complete. Also the topology induced on X_k from $\mathscr{D}_X$ is precisely that of the indiced topology from $C^\infty_A(\Gamma)$.

Note that if $\Psi \in \mathscr{D}_X$ and $\hat{\Psi}(\varphi) \neq 0$ then there is n_φ such that $\forall n > n_\varphi$

$$\int_\Gamma \Psi(f)(f, \varphi)^n \, df = 0;$$

this follows since $\Psi \in X_k$ and $\|n\varphi\|_G > k$ for n large.

Proposition 4.3. a) $\forall k$, $\overline{\mathscr{T} \cap X_k} = X_k$.

b) $\bar{\mathscr{T}} = \mathscr{D}_X$.

Proof. a) Since $C^\infty_A(\Gamma)$ is a complete metric space, $\operatorname{int} X_k \neq 0$, where X_k is considered as a subset of $C^\infty_A(\Gamma)$.

Take $\Psi \in \operatorname{int} X_k$.

Because $\bar{\mathscr{T}} = C^\infty_A(\Gamma)$ we let $\Psi_n \to \Psi$, $\Psi_n \in \mathscr{T}$; hence we can take $\Psi_n \in X_k$.

Now let $\Psi \in X_k$ and $\Phi_n \to \Psi$, $\Phi_n \in \operatorname{int} X_k$.

$\forall n$, take $\Psi_n \in \mathscr{T} \cap X_k$ such that $\varrho(\Phi_n - \Psi_n) < 1/n$ where ϱ is the metric corresponding to the topology on $C^\infty_A(\Gamma)$.

Thus $\Psi_n \to \Psi$ since $\varrho(\Psi - \Psi_n) \leqq \varrho(\Psi - \Phi_n) + \varrho(\Phi_n - \Psi_n)$.

b) Assume $\bar{\mathscr{T}} \neq \mathscr{D}_X$ and take $\Psi \in \mathscr{D}_X - \bar{\mathscr{T}}$.

From the Hahn-Banach theorem let $t \in \mathscr{D}'_X$ have the properties that $\langle t, \bar{\mathscr{T}} \rangle = 0$ and $\langle t, \Psi \rangle = 1$.

By definition, $\Psi \in X_k$, some k, and thus, from a), there is $\{\Psi_n\} \subseteqq \mathscr{T} \cap X_k$ such that $\Psi_n \to \Psi$.

Consequently, $\langle t, \Psi_n \rangle \neq 0$ for some n, a contradiction. q.e.d.

5. Remarks

Let t be a linear functional on $\mathscr{T}$. We associate the Fourier series

$$t \sim \sum_{\varphi \in G} \hat{t}(\varphi)(f, \varphi)$$

where

$$\hat{t}(\varphi) \equiv \langle t, \Psi_{-\varphi} \rangle. \tag{5.1}$$

If $\{\hat{t}(\varphi) : \varphi \in G\}$ is bounded then t induces an element of $A'(\Gamma)$ by defining

$$\langle t, \Phi \rangle \equiv \Sigma \, \hat{t}(\varphi) \, \hat{\Phi}(\varphi) \tag{5.2}$$

for all $\Phi \in A(\Gamma)$. (5.2) is well-defined since $\hat{\Phi}$ vanishes off a countable set, $\sum_\varphi |\hat{\Phi}(\varphi)| < \infty$, and $\hat{t}$ is bounded; clearly $t \in A'(\Gamma)$.

Also, $\forall T \in A'(\mathring{\Gamma})$ we define a linear functional on $\mathscr{T}$: $\forall \varphi \in G$ set

$$\langle t, \Psi_\varphi \rangle \equiv \langle T, e^{i\varphi} \rangle. \tag{5.3}$$

In fact, $j' \Psi_\varphi(\lambda) = \Psi_\varphi(j\lambda) = e^{i\varphi(\lambda)}$, and $\varphi \in G \subseteqq A(\mathring{\Gamma})$, real-valued, gives $e^{i\varphi} \in A(\mathring{\Gamma})$. If t so defined uniquely determines an element of $\mathscr{D}'_X$ we write $t \in \mathscr{D}'_X$.

Clearly, for any linear functional $t: \mathscr{T} \to C$ we can define its Fourier coefficients by (5.1).

Assume that the norm $\| \ \|_G \equiv \| \ \|_\infty$.

Proposition 5.1. *Let* $b \subseteqq G$ *be* $\| \ \|_\infty$ *bounded by* 1 *(as functions on* $\mathring{\Gamma}$*). Then* b *as a subset* $\hom(G, R)'$ *is bounded by* 1.

Proof. For $\varphi \in \hom(G, R)'$,

$$\|\varphi\|' \equiv \sup\{|\langle \varphi, h\rangle| : \|h\|_{\hom} \leqq 1\}.$$

Let $\varphi \in b$, $\varepsilon > 0$. There is $h \in \hom(G, R)$, $\|h\|_{\hom} \leqq 1$, so that

$$\|\varphi\|' - \varepsilon \leqq |\langle \varphi, h\rangle| \equiv |h(\varphi)|.$$

Now $\|h\|_{\hom} \leqq 1$ if and only if $\sup\{|h(\psi)| : \|\psi\|_\infty \leqq 1, \psi \in G\} \leqq 1$. Thus, $|h(\varphi)| \leqq 1$ and so $\|\varphi\|' \leqq 1$. q.e.d.

Proposition 5.2. *Let* $b \subseteqq G$ *be* $\| \ \|_\infty$ *bounded by* 1. $\forall t \in \mathscr{D}'_X\ \exists K_t > 0$ *such that* $\forall \varphi \in b$

$$|\hat{t}(-\varphi)| \leqq K_t.$$

Proof. $t \in \mathscr{D}'_X$ implies t is continuous on each X_p. Taking $p = 1$, fixed, we can therefore find $C_t > 0$, $B \subseteqq \hom(G, R)$ bounded, and N so that $\forall \Psi \in X_1$,

$$|\langle t, \Psi\rangle| \leqq C_t\, \varrho_{B,N}(\Psi).$$

$B \subseteqq \hom(G, R)$ and $b \subseteqq \hom(G, R)'$ bounded (from Prop. 5.1) imply

$$\sup\{|\langle \varphi, h\rangle| : \varphi \in b, h \in B\} \equiv M < \infty.$$

Thus, for any $h_1, \ldots, h_k \in B$, we have

$$|D_{h_1} \ldots D_{h_k} \Psi_{-\varphi}(f)| \leqq M^k$$

$\forall \varphi \in b$ and $\forall f \in \Gamma$ (from Prop. 3.1).

Consequently, since $\sum_\psi |\hat{\Psi}_\varphi(\psi)| = 1$ and $\{\Psi_\varphi : \varphi \in b\} \subseteqq X_1$,

$$\forall \varphi \in b, \ |\hat{t}(-\varphi)| = |\langle t, \Psi_\varphi\rangle| \leqq C_t(M^N + 1) \equiv K_t. \quad \text{q.e.d.}$$

Note that $M(\Gamma) \subseteqq D'_{\mathscr{T}}(\Gamma)$. In fact, if $\mu \in M(\Gamma)$, μ is defined on $C^\infty_{\mathscr{T}}(\Gamma) \subseteqq C(\Gamma)$, and if $\Phi_n \to 0$ in $C^\infty_{\mathscr{T}}(\Gamma)$ then $\Phi_n \to 0$ in $C(\Gamma)$.

6. Sets of Strong Spectral Resolution

Assume that $\forall F \in \mathscr{F}$ *G contains a* $\psi_F \in C^\infty(\mathring{\Gamma})$ where $0 \leqq \psi_F \leqq 1$, $\psi_F \equiv 1$ on a neighborhood of F, and $\psi_F \equiv 0$ on a neighborhood of $E - F$.

Proposition 6.1. *Let* $T \in A'(E)$ *and assume that* $\exists M > 0$ *such that* $\forall F \in \mathscr{F}$

$$|\langle T, e^{i\psi_F}\rangle| < M. \tag{6.1}$$

Then $T \in M(E)$.

Proof. Let $F \in \mathscr{F}$. $\forall \lambda \in \mathring{\Gamma}$ we write

$$e^{i\psi_F(\lambda)} = \sum_0^\infty \frac{i^k}{k!} (\psi_F(\lambda))^k.$$

It is then easy to see that

$$\langle T, e^{i\psi_F}\rangle = \langle T, 1\rangle + \sum_1^\infty \frac{i^k}{k!} \langle T, \psi_F^k\rangle. \tag{6.2}$$

There are several ways to prove (6.2). For example, if we write $f' = T$, $f = \Sigma\, k_m\, \chi_{I_m} \in L^1(\Gamma)$, we calculate

$$\begin{aligned}\langle T, e^{i\psi_F}\rangle = -i\langle f, e^{i\psi_F}\psi_F'\rangle &= -\sum_0^\infty \frac{i^{k+1}}{k!} \langle f, \psi_F' \psi_F^k\rangle \\ &= \sum_0^\infty \frac{i^k}{k!} \langle T, \psi_F^k\rangle\end{aligned}$$

using Lebesgue's convergence theorem.

Now, supp $T \subseteqq E$ and $\psi_F = 1$ on a neighborhood of F, $\psi_F = 0$ on a neighborhood of $E - F$ gives

$$\forall k \geqq 1, \ \langle T, \psi_F^k\rangle = \langle T, \psi_F\rangle.$$

Thus,

$$\langle T, e^{i\psi_F}\rangle - \langle T, 1\rangle = \langle T, \psi_F\rangle > \sum_1^\infty \frac{i^k}{k!},$$

and so since $c \equiv e^i - 1 = \sum_1 (i^k/k!) \neq 0$

$$|\langle T, \psi_F\rangle| \leqq \frac{1}{|c|} |\langle T, 1\rangle| + \frac{1}{|c|} |\langle T, e^{i\psi_F}\rangle|.$$

Hence, by (6.1), T, as a finitely additive set function on $\mathscr{F}$, is bounded; therefore we apply [1; 2, Chapter 2] and have $T \in M(E)$. q.e.d.

Theorem 6.1. *Let $T \in A'(E)$. $T \in M(E)$ if and only if $t \in \mathscr{D}'_X$ (where t is defined in* (5.3)*).*

Proof. Let $T \in M(E)$. $\mathscr{T}$ is dense in $C(\Gamma)$ since G separates the points of Γ and by Stone-Weierstrass. If $\Psi_n \to \Psi$ in $C(\Gamma)$ topology on $\mathscr{T}$ then $j'\Psi_n \to j'\Psi$ in $C(\mathring{\Gamma})$ topology by Prop. 2.1; and so $\langle T, j'\Psi_n - j'\Psi\rangle \to 0$.

Since $\langle t, \Phi\rangle = \langle T, j'\Phi\rangle$ we have $t \in M(\Gamma) \subseteqq \mathscr{D}'_X$, where the last inclusion follows from § 5 or Prop. 4.2c.

Conversely, let $t \in \mathscr{D}'_X$.

$b \equiv \{\psi_F : F \in \mathscr{F}\}$ is sup norm bounded by 1 and so is a bounded set of $\hom(G, R)'$ (Prop. 5.1).

Also $\hat{t}$ is bounded on b (Prop. 5.2).

Consequently, from (5.3) and Prop. 6.1 we have $T \in M(E)$. q.e.d.

Corollary. *Let t correspond to $T \in M(\mathring{\Gamma})$ by* (5.3). *Then $t \in M(\Gamma)$.*

7. Conditions for Strong Spectral Resolution

Assume $G \subsetneqq C^\infty(\mathring{\Gamma})$.

Let $T \in A'(E)$ and suppose, without loss of generality, that $\hat{T}(0)=0$. There is $\{k_m\} \subseteqq C$ such that, for

$$T_n \equiv \sum_1^n k_m(\delta_{\lambda_m} - \delta_{\gamma_m}),$$

$$\forall \varphi \in C^1(\mathring{\Gamma}), \quad \lim \langle T_n, \varphi \rangle = \langle T, \varphi \rangle$$

and

$$\Sigma\, e^{r|k_m|} \varepsilon_m < \infty, \text{ some } r > 0$$

[2, Chapter 2]. For each n we associate $t_n \in M(jE)$ defined by

$$\forall \Psi \in C(\Gamma), \quad \langle t_n, \Psi \rangle \equiv \langle T_n, j' \Psi \rangle .$$

Thus

$$t_n = \sum_1^n k_m(\delta_{j\lambda_m} - \delta_{j\gamma_m}) .$$

Consider the expression

$$\sup_n \sup_{\varphi \in G_\Psi} \left| \sum_1^n k_m(e^{i\varphi(\lambda_m)} - e^{i\varphi(\gamma_m)}) \right| \tag{7.1}$$

defined for $\Psi \in A(\Gamma)$ and for all $\varphi \in G_\Psi$.

Theorem 7.1. *Given* $T \in A'(E)$ *with corresponding* T_n, t_n, k_m. *Assume that for each* $\Psi \in A(\Gamma)$, (7.1) *is bounded. Then*

a) $t \in A'(\Gamma)$.

b) $T \in M(E)$.

Proof. b) follows from Theorem 6.1, a) and the fact that $A'(\Gamma) \subseteqq \mathscr{D}'_X$.

a) Take $\Psi \in A'(\Gamma)$. Then

$$\begin{aligned} \langle t_n, \Psi \rangle &= \sum_1^n k_m \left[\sum_{\varphi \in G_\Psi} \hat{\Psi}(\varphi)\, (e^{i\varphi(\lambda_m)} - e^{i\varphi(\gamma_m)}) \right] \\ &= \sum_{\varphi \in G_\Psi} \hat{\Psi}(\varphi) \left(\sum_1^n k_m(e^{i\varphi(\lambda_m)} - e^{i\varphi(\gamma_m)}) \right). \end{aligned} \tag{7.2}$$

By our hypothesis and the fact that $\Psi \in A(\Gamma)$, $\lim \langle t_n, \Psi \rangle \equiv \langle s, \Psi \rangle$ exists for each $\Psi \in A(\Gamma)$.

Consequently we have $s \in A'(\Gamma)$ by the Banach-Steinhaus theorem.

Take Ψ_φ. Then $\langle t, \Psi_\varphi \rangle \equiv \langle T, e^{i\varphi} \rangle = \lim \langle T_n, e^{i\varphi} \rangle = \lim \langle t_n, \Psi_\varphi \rangle = \langle s, \Psi_\varphi \rangle$. Therefore, $t = s$ and $t \in A'(\Gamma)$. q.e.d.

In the following result, part a) may seem weaker than Theorem 7.1a but, because we need only consider $\Psi \in \mathscr{D}_X$, we have the more useful criterion that $\sup\{\|\varphi\|_\infty : \varphi \in G_\Psi\} < \infty$ for applications.

Theorem 7.2. *Given* $T \in A'(E)$ *with corresponding* T_n, t_n, k_m. *Assume that for each* $\Psi \in \mathscr{D}_X$, (7.1) *is bounded. Then*

a) $t \in \mathscr{D}'_X$.

b) $T \in M(E)$.

Proof. Once again b) is clear from Theorem 6.1 and a).

a) We have (7.2) for $\Psi \in \mathscr{D}_X$; so that since $\varphi \in C^\infty(\mathring{\Gamma})$ and $\Sigma |\hat{\Psi}(\varphi) < \infty$, $\lim\langle t_n, \Psi\rangle \equiv \langle s, \Psi\rangle$ exists for each $\Psi \in \mathscr{D}_X$. Because $\mathscr{D}_X$ is barrelled we can again use the Banach-Steinhaus theorem and so $s \in \mathscr{D}'_X$.

Now $s = t$ on $\mathscr{T}$ and therefore $t \in \mathscr{D}'_X$ since $\bar{\mathscr{T}} = \mathscr{D}_X$. q.e.d.

Remark. In the representation of $T \in A'(E)$ as f', $f = \Sigma k_m \chi_{I_m}$, we make the following calculation with regard to Prop. 6.1:

$$\langle T, e^{i\psi}\rangle = -i\langle f, e^{i\psi}\psi'\rangle = \sum_0 [i^k/(k+1)k!]\left(\sum_1^m (k_{n_{2j}} - k_{n_{2j-1}})\right)$$

where $\lambda_{n_{2j-1}} < \lambda_{n_{2j}} < \lambda_{n_{2j+1}}$. Thus, if

$$\left|\sum_1^m (k_{n_{2j}} - k_{n_{2j-1}})\right|$$

is bounded for all partitions then we can show that T is a measure, and, in fact, that the converese is true.

8. A Particular G

Given E we shall now construct G so that Theorem 7.2 is applicable. First, take $\lambda' \in (\lambda_0, \gamma_0)$ (§ 1).

Recall that in § 7 G is a real vector space (under addition) of real-valued elements of $C^\infty(\mathring{\Gamma})$, and that the following conditions are satisfied:

G a) $\forall \lambda, \gamma \in \mathring{\Gamma}, \lambda \neq \gamma, \exists \varphi \in G$ such that $\forall k \in Z\, \varphi(\lambda) - \varphi(\gamma) \neq 2\pi k$;

G b) $\forall F \in \mathscr{F}, \exists \varphi \in G$ such that $\varphi = 1$ on a neighborhood of F, $\varphi = 0$ on a neighborhood of $E - F$, and $0 \leq \varphi \leq 1$.

We designate any collection satisfying G b) by $b_{\mathscr{F}}$. We need the following –

Lemma 8.1. *Let $\alpha > 0$ and define*

$$\theta(x) = e \exp\{-\alpha^2/(\alpha^2 - x^2)\}, \qquad x \in [0, \alpha).$$

a) *θ is C^∞ on $[0, \alpha]$, $\theta^{(n)}(0) = \theta^{(n)}(\alpha) = 0$ for each n, and $\exists r > 0$, independent of α, such that*

$$\sup_{x \in [0,\alpha]} |\theta'(x)| = \frac{r}{\alpha}.$$

b) *Let $\alpha_m < \varepsilon_m$ and let $\varphi \in C^\infty(\mathring{\Gamma})$, real-valued, have the form*

$$\varphi(x) = c\, e \exp\{-\alpha_m^2/(\alpha_m^2 - (x - \lambda'_m)^2)\}, \qquad c \in R,$$

on $[\lambda'_m, \lambda'_m + \alpha_m] \subsetneqq (\lambda_m, \gamma_m)$. Then

$$\sup\{|\varphi'(x)| : x \in [\lambda'_m, \lambda'_m + \alpha_m]\} = \frac{|c|\, r}{\alpha_m}.$$

Proof. b) is clear from a).

a)

$$\theta'(x) = -\frac{2x\alpha^2 e}{(\alpha^2 - x^2)^2} \exp\{-\alpha^2/(\alpha^2 - x^2)\}$$

and

$$\theta''(x) = -2e\alpha^2 \frac{\alpha^4 - 3x^4}{(\alpha^2 - x^2)^4} \exp\{-\alpha^2/(\alpha^2 - x^2)\}.$$

We then compute the sup of θ'. q.e.d.

For a given E we now construct G. Take I_j, I_k, $\lambda_j < \gamma_k$, $j \neq k$, and assume $\lambda' \notin (\lambda_j, \gamma_k)$. Let $\varphi \in C^\infty(\mathring{\Gamma})$, real-valued, satisfy

Ea) $\varphi \equiv 1$ on some (γ^j, λ^k) containing $[\gamma_j, \lambda_k]$.

Eb) $\varphi \equiv 0$ on $[0, \lambda^j) \cup (\gamma^k, 2\pi]$ where $\lambda_j < \lambda^j < \gamma^j$ and $\lambda^k < \gamma^k < \gamma_k$.

Ec) φ has the "form" of θ (in Lemma 8.1) in $[\lambda^j, \gamma^j]$ and $[\lambda^k, \gamma^k]$.

We designate such a φ by φ_{jk} for the moment. If $\lambda' \in (\lambda_j, \gamma_k)$ we further consider φ of the form

$$\varphi \equiv \varphi_{j,0} + \varphi_{0,k}$$

where we stipulate that

$$\lambda^0 < \lambda' < \gamma^0.$$

Also when $\lambda' \in (\lambda_j, \gamma_k)$ we take those φ which are equal to 1 and 0 in Ea) and Eb), respectively, as well as being $C^\infty(\mathring{\Gamma})$, real-valued, and satisfying Ec); then we include the corresponding φ for the case $\lambda' \notin (\lambda_j, \gamma_k)$.

Let B_E be the set of all such φ for all j, k and all permissible $\lambda^j, \gamma^j, \lambda^k, \gamma^k$; and let G be the real vector space generated by B_E.

Ga) is satisfied since we are taking all possible λ^j, γ^j, λ^k, γ^k. For Gb) we need the following lemma which is clear (e.g., [2, Prop. 2.1]) from our definitions and a compactness argument.

Lemma 8.2. *$\forall \psi_F$ there is $\varphi \in G$ such that $\varphi = 1$ on a neighborhood of F, $\varphi = 0$ on a neighborhood of $E - F$, and $0 \leqq \varphi \leqq 1$.*

9. The (*LF*)-Spaces $\mathscr{D}_Y$ and $\mathscr{D}_Z$ for Estimating Pseudo-Measures

Take the G of § 8.

Define

$$Y_k \equiv \{\Psi \in C_A^\infty(\Gamma) : |G_\Psi| \leqq k \quad \text{and} \quad \forall \varphi \in G_\Psi, \forall m \geqq k, \varphi' \equiv 0 \quad \text{on} \quad I_m\}.$$

$$Z_k \equiv \{\Psi \in C_A^\infty(\Gamma) : |G_\Psi| \leqq k \quad \text{and} \quad \forall \varphi \in G_\Psi, \varphi' \neq 0 \quad \text{for at most} \quad k\, I_m\text{'s}\}.$$

Thus, for each $k \geqq 1$

$$Y_k \subseteqq Z_k \subseteqq X_k.$$

As with Prop. 4.1 the following is trivial –

Proposition 9.1. *Each Y_k (resp., Z_k) is a Fréchet space with the induced topology from $C_A^\infty(\Gamma)$.*

We have

$$Y_1 \subseteqq Y_2 \subseteqq \cdots, \quad Z_1 \subseteqq Z_2 \subseteqq \cdots;$$

and we form the (LF)-spaces

$$\mathscr{D}_Y \equiv \bigcup_1^\infty Y_k, \quad \mathscr{D}_Z \equiv \bigcup_1^\infty Z_k$$

and corresponding "distribution" spaces, $\mathscr{D}'_Y$ and $\mathscr{D}'_Z$.

Remark. In a similar manner we can define the space $\mathscr{D}_U$ in terms of

$$U_k \equiv \{\Psi \in C_A^\infty(\Gamma) : \Psi \in Y_k \quad \text{and} \quad |\varphi'| \leqq k \ \forall \varphi \in G_\Psi\}.$$

As such $\mathscr{D}_U$ has the property that

$$j' \mathscr{D}_U \subseteqq C^1(\mathring{\Gamma}) \subseteqq A(\mathring{\Gamma})$$

since

$$i \Sigma \hat{\Psi}(\varphi) \varphi'(\lambda) e^{i\varphi(\lambda)}$$

converges uniformly. We shall not investigate $\mathscr{D}'_U$ at this time.

The easy argument of Prop. 4.3 is general and gives

Proposition 9.2. *$\mathscr{T} \subseteqq \mathscr{D}_Y$, $\forall k$ $\overline{\mathscr{T} \cap Y_k} = Y_k$ and $\bar{\mathscr{T}} = \mathscr{D}_Y$ (resp., for Z_k and $\mathscr{D}_Z$).*

If t defined by (5.3) uniquely determines an element of $\mathscr{D}'_Y$ (resp., $\mathscr{D}'_Z$) we write $t \in \mathscr{D}'_Y$ (resp., $\mathscr{D}'_Z$).

Next note that

$$\mathscr{D}_Y \subseteqq \mathscr{D}_Z \subseteqq \mathscr{D}_X,$$

and because of general properties of inductive limits and various dense subspaces with which we are dealing, we extend (3.3) and Prop. 4.2c to

$$M(\Gamma) \subseteqq D'_{\mathscr{T}}(\Gamma) \subseteqq D'_A(\Gamma) \subseteqq \mathscr{D}'_X \subseteqq \mathscr{D}'_Z \subseteqq \mathscr{D}'_Y.$$

Theorem 9.1. *If $T \in A'(E)$ (resp., $A'_b(E)$) then the corresponding $t \in \mathscr{D}'_Y$ (resp., $\mathscr{D}'_Z$).*

Proof. Let $T \in A'(E)$ and define t by (5.3). We show that $\forall \Psi \in \mathscr{D}_Y$, (7.1) is bounded. This will prove $t \in \mathscr{D}'_Y$ by the same argument as in Theorem 7.2 since $\mathscr{D}_Y$ is barrelled.

If $\Psi \in \mathscr{D}_Y$ and $\varphi \in G_\Psi$ let

$$F_\varphi \equiv \{m \geqq 0 : \varphi' \not\equiv 0 \quad \text{on} \quad (\lambda_m, \gamma_m)\}.$$

Then card $F_\varphi \leqq 3$ for all $\varphi \in B_E$.

Also, $\Psi \in Y_k$, some k, and so

$$\forall \varphi \in G_\Psi, \ \forall m \geqq k, \ \varphi' \equiv 0 \quad \text{on} \quad I_m.$$

There are ξ'_m, ξ''_m such that (7.1) (for this $\Psi \in \mathscr{D}_Y$) equals

$$\sup_n \sup_{\varphi \in G_\Psi} \left| \sum_1^n k_m [-\varepsilon_m \sin \varphi(\xi'_m) \varphi'(\xi'_m) + i \varepsilon_m \cos \varphi(\xi''_m) \varphi'(\xi''_m)] \right|$$

$$2 \sup_n \sup_{\varphi \in G_\Psi} \sum_{\substack{m \in F_\varphi \\ m \leqq n}} |k_m| \varepsilon_m \sup_{x \in I_m} |\varphi'(x)|. \tag{9.1}$$

We must now show that we can take $\varphi \in G_\Psi$ with corresponding α_m (§ 8) satisfying $\alpha_m = \varepsilon_m/2$.

Given $\varphi \in G_\Psi$ and let $\psi \in G$ have the properties:

a) $\psi = \varphi$ on a neighborhood of E,

b) $\forall m \geqq k$, $\psi' \equiv 0$ on I_m,

c) $\forall m < k$ and $\lambda_\varphi^m, \gamma_\varphi^m$ corresponding to (as in § 8) let $\gamma_\psi^m - \lambda_\psi^m \equiv \alpha_m = \varepsilon_m/2$.

It is necessary to prove

$$\langle t_n, \Psi \rangle = \sum_{\varphi \in G_\Psi} \hat{\Psi}(\varphi) \left(\sum_1^n k_m (e^{i\psi(\lambda_m)} - e^{i\psi(\gamma_m)}) \right). \tag{9.2}$$

Now,

$$\langle t_n, \Psi \rangle = \langle T_n, j' \Psi \rangle = \langle T_n, j' \Sigma \hat{\Psi}(\varphi)(f, \varphi) \rangle , \tag{9.3}$$

$\mathscr{D}_Y \subsetneqq A(\Gamma)$, $\sum_1^n \hat{\Psi}(\varphi_j)(f, \varphi_j) \to \Psi(f)$ uniformly on Γ, and j' is a continuous map $C(\Gamma) \to C(\mathring{\Gamma})$ (Prop. 2.1).

Thus, the right hand side of (9.3) is

$$\sum_{\varphi \in G_\Psi} \hat{\Psi}(\varphi) \langle \mathrm{T}_n, \mathrm{e}^{i\varphi} \rangle = \sum_{\varphi \in G_\Psi} \hat{\Psi}(\varphi) \langle T_n, e^{i\psi} \rangle$$

by the definition of T_n. This gives (9.2).

Consequently, we can replace (9.1) by

$$K \sup_n \sup_{\varphi \in G_\Psi} \sum_{\substack{m \in F_\varphi \\ m \leqq n}} |k_m| \frac{\varepsilon_m}{\alpha_m}, \tag{9.4}$$

from the way we've chosen $\mathscr{D}_Y$.

Therefore, since $\sup\{m \in F_\varphi : \varphi \in G_\Psi\} \leqq k$ for our Ψ, (9.4) is bounded by $K' \sum_1^k |k_m|$ and so $t \in \mathscr{D}_Y'$.

For the $T \in A_b'(E)$ case we proceed in the same way replacing (9.1) by an estimate (corresponding to (9.4)) which is bounded by $C \sup \left\{ \sum_1^k |k_{m_j}| : m_j \in Z, m_j \geqq 1 \right\}$.

This procedure is possible by the definition of $\mathscr{D}_Z$ and we get the required boundedness of (7.1) (so that $t \in \mathscr{D}_Z'$) since $\{|k_m|\}$ is bounded. q.e.d.

Because of Theorem 9.1 we define the map

$$j : A'(E) \to \mathscr{D}_Y'$$

(and the canonical injection $A'(E)/\ker j \to \mathscr{D}_Y'$); it is easy to check that j is linear. Also, by the properties of bounded sets in $\mathscr{D}_Y$ and the proof of Theorem 9.1 we see that $t_n \to t$ in $\beta(\mathscr{D}_Y', \mathscr{D}_Y) \equiv \beta$. Now, if $S_n \to S$ in $A'(E)$ and $s_n \to t$ in β, with $jS_n = s_n$ and $S_n \in A'(E)$ (not necessarily in the form of § 7) then we have $jS = t$ on $\mathscr{T}$, and hence the graph of j is closed. Thus, we can use the lovely new closed graph theorems (Marc de Wilde's, say) noting that $\mathscr{D}_Y'$ is bornant since it is the dual of an LF-space. Consequently –

Corollary 9.1.1. *j is a continuous linear map from $(A'(E), \| \|_{A'})$ into $(\mathscr{D}'_Y, \beta)$.*

It is therefore interesting to note that *if E is Helson and $j\,A'(E)$ is a barrelled subspace of $(\mathscr{D}'_Y, \beta)$ then E is a spectral synthesis set.* For this line of thinking one is lead to consider the completion $C_A(\Gamma)$ of $\mathscr{T}$ with the "$\Sigma\,|\hat{\Psi}(\varphi)|$" norm along with

$$W_k \equiv \{\Psi \in C_A(\Gamma): \forall \varphi \in G_\Psi,\ \|\varphi\|_\infty \leqq k, \quad \text{and} \quad \forall m \geqq k,\ \varphi' \equiv 0 \quad \text{on} \quad I_m\}$$

and the corresponding LB-space $\mathscr{D}_W$. In fact, since $\mathscr{D}_W$ is a strict inductive limit of Banach spaces, $\mathscr{D}'_W$ is Fréchet; so that since the analogue to Theorem 9.1 still holds we have that *E is a spectral synthesis set if E is Helson and $j\,A'(E)$ is closed in $(\mathscr{D}'_W, \beta)$.* Note that if a closed set $F \subsetneqq \mathring{\Gamma}$ is Helson then $m(F) = 0$ [2, Chapter 7]. In any case we see that generally $j\,A'(E)$ is neither barrelled nor closed for otherwise we'd be able to argue that the measures with finite support in E are strongly dense in $M(E)$, a statement which is obviously not true for perfect E.

Remark 1. Since we are dealing with inductive limits and because $b_{\mathscr{F}} \nsubseteq Z_k$ (or Y_k) for any k we do not have $b_{\mathscr{F}}$ bounded in Z_k (or Y_k) and so we can't argue as in Theorem 6.1 to prove $T \in M(E)$. In fact, we can easily construct countable closed $F \subsetneqq \mathring{\Gamma}$ which support $T \in A'(E) - A'_b(E)$.

2. Theorem 9.1 is obviously true for a much larger class of distributions supported by E than $A'(E)$ (resp., $A'_b(E)$).

10. Projective Limit Spaces and Pseudo-Measures

Using inductive limits we saw that if $T \in A'(E)$ then the corresponding t was a canonical distribution but that $b_{\mathscr{F}}$ was not a bounded set in our spaces of test functions. We now note that $b_{\mathscr{F}}$ is a bounded set in certain natural projective limit spaces but that the corresponding dual spaces of distributions are too small to contain the images of pseudo-measures.

We again use G of § 8.

Let

$$S_{Zk} \equiv \{\varphi \in G: \|\varphi\|_\infty \leqq k \quad \text{and} \quad \varphi' \equiv 0 \quad \text{on all but at most} \quad k\,I_m\text{'s}\}$$

and

$$S_{Yk} \equiv \{\varphi \in G: \|\varphi\|_\infty \leqq k \quad \text{and} \quad \varphi' \equiv 0 \quad \text{on} \quad I_m \quad \text{for all} \quad m \geqq k\}.$$

Clearly $S_{Yk} \subseteqq S_{Zk}$ and we give $\bigcup Z_k$ and $\bigcup Y_k$ the projective limit topologies induced by the maps

$$F_{Zk}: \bigcup Z_j \to Z_k, \qquad F_{Yk}: \bigcup Y_j \to Y_k$$

where

$$F_k(\Psi)(f) \equiv \sum_{\varphi \in S_k} \hat{\Psi}(\varphi)\,(f, \varphi);$$

we denote the resulting topological vector spaces by $\mathscr{D}_{Z,p}$ and $\mathscr{D}_{Y,p}$. From the properties of projective limits it is easy to see that $b_{\mathscr{F}}$ is bounded in both

$\mathscr{D}_{Z,p}$, and $\mathscr{D}_{Y,p}$ but that, of course, the corresponding duals are too small to obtain a result like Theorem 9.1 – which would be necessary to use the technique of Theorem 6.1.

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