Annali Scuola Normale Superiore - Pisa
Classe di Scienze
Serie IV - Vol. VI, n. 2 (1979)

Zeta Functions for Idelic Pseudo-Measures.

J. J. BENEDETTO (*)

In [Benedetto, 1973] I introduced the notion of an «idelic pseudo-measure» in the context of investigating non-synthesizable phenomena. The Fourier analysis program that I envisaged for these pseudo-measures, and that I began in the above reference, stalled because I did not know enough about their structure. The theory of idelic pseudo-measures that I now develop is meant to correct this shortcoming. [Benedetto, 1976] complements the present paper and provides relations with some fundamental issues in analytic number theory.

Let $A'(\boldsymbol{T})$ be the space of pseudo-measures on $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ with norm $\| \;\|_{A'}$. As is well-known, Hecke L-functions for the field $\boldsymbol{Q}$ of rationals can be formed from characters of the locally compact abelian idele group $J_{\boldsymbol{Q}}$; and if these characters belong to the subgroup $\boldsymbol{Q}^{\times\perp} \subseteq \hat{J}_{\boldsymbol{Q}}$ of idele class characters, then the associated L-functions comprise the usual Dirichlet zeta functions and have functional equations for purposes of analytic continuation. The Riemann ζ-function is the L-function associated with the identity $1 \in \boldsymbol{Q}^{\times\perp}$.

We shall choose certain elements $\gamma \in \hat{J}_{\boldsymbol{Q}}$ for which there is not only an associated L-function $L(s, \gamma)$, $s = \sigma + i\tau$, but also a distribution T_s, $\sigma \leqslant 1$, on $\boldsymbol{T}$. These distributions are called *idelic distributions* and if $T_s \in A'(\boldsymbol{T})$ then T_s is an *idelic pseudo-measure*; IPM designates the space of such pseudo-measures. We shall prove the following analytic continuation and approximation result. There is a sequence $\{\gamma_n\} \subseteq \hat{J}_{\boldsymbol{Q}} \setminus \boldsymbol{Q}^{\times\perp}$ and a sequence $\{T_{n,s}\} \subseteq \text{IPM}$, $\sigma > \frac{1}{2}$, of corresponding idelic pseudo-measures having the following properties:

a) For each n and each $s_0 \in \boldsymbol{C}$, $\sigma_0 > \frac{1}{2}$, there is a sequence $\{A_m\} \subseteq A'(\boldsymbol{T})$ such that $T_{n,s} = \sum A_m (s - s_0)^m$ converges in the $A'(\boldsymbol{T})$ norm in some neighborhood of s_0;

(*) University of Maryland, College Park, Maryland.
Pervenuto alla Redazione il 7 Gennaio 1978.

b) $\{\gamma_n\}$ converges to 1 in $\hat{J}_Q$ and $\|T_{n,s}\|_{A'} = \|T_{n+1,s}\|_{A'}$ for each n.

The result is also valid for $L^p(\boldsymbol{T})$, $p < \infty$, cf., *Example* 2.3; although a major interest in these problems that I've posed is to construct pseudo-measures which are not measures. Part *a* is given in sect. 3 and somewhat more than part *b* is given in sect. 4.

Section 1 establishes criteria for determining elements of IPM and section 2 provides a selection of examples indicating the effectiveness of the criteria and the relation with number theoretic estimates. The setting of this paper is not intrinsically idelic and so it is necessary to explain our idelic language beyond the analytic reasons given in [Benedetto, 1973]. From a number theoretic point of view, it is natural to ask if knowledge about idele characters which are not idele class characters can ever provide any information about idele class characters. The results of sections 3 and 4 can be viewed in this context with the idea that IPM is an effective analytic space to study. In fact, we can construct idele characters γ which are not idele class characters such that the Riemann hypothesis is valid if and only if $T_1 \in$ IPM.

Acknowledgement. In section 2 we have used D. Aharanov's development of the Lebedev-Milin inequalities.

1. – Criteria for idelic pseudo-measures.

For each $\gamma \in \hat{J}_Q$ there is a function $k\colon P \to \boldsymbol{R}$ and a conductor $f \in \boldsymbol{N}$ (P is the set of primes and $\boldsymbol{N} = \{1, 2, 3, \ldots\}$) which determine the ***Hecke-L series***

$$L(s, \gamma) = \sum_{\substack{n=1 \\ (n,f)=1}}^{\infty} \frac{1}{n^s} e^{ik(n)},$$

where $k(1) = 0$ and $k\colon \boldsymbol{N} \to \boldsymbol{R}$ is defined by the rule $k(pq) = k(p) + k(q)$ for each $p, q \in P$. If $k(P) \subseteq \boldsymbol{Z}$ then the ***idelic distribution*** T_s (on $\boldsymbol{T}$) corresponding to $\gamma \in \hat{J}_Q$ is

$$T_s \sim \sum \hat{T}_s(n)\, e^{inx}$$

where

$$\hat{T}_s(n) = \sum_{\substack{k(m)=n \\ m \geq 1,\, (m,f)=1}} 1/m_s\,.$$

T_s is ***unbounded*** if $\hat{T}_1(0) = \infty$ and $T_s \in$ IPM if $\sup_n |\hat{T}_s(n)| < \infty$. We shall write $\gamma \sim k \sim T_s$ to designate the relation between idele character, k-function, and distribution. The above definitions and following result are developed in [Benedetto, 1976]. We shall assume that k is an increasing function $P \to N \cup \{0\}$; the conditions on k can be weakened at the expense of a more technical presentation.

PROPOSITION 1.1. *Take* $\sigma > \frac{1}{2}$ *and* $\gamma \sim k \sim T_s$. $T_s \in \mathrm{IPM}$ *if and only if the power series expansion of* $\exp\left(\sum_{p\in P} z^{k(p)}/p^{\sigma}\right)$ *has bounded coefficients.*

REMARK 1. Given $\gamma \sim k \sim T_s$ and take $\sigma_0 \in (\frac{1}{2}, 1)$. Set $j_n = \inf\{p \in P: k(p) = n\}$ and $m_n = \sup\{p \in P: k(p) = n\}$. Assume that

$$\frac{m_n - (j_n - 1)}{\log j_n} \leq \frac{j_n^{\sigma_0}}{n}$$

for each $n \geqslant 1$. Then $T_s \in \mathrm{IPM}$ for every $\sigma \in [\sigma_0, 1]$.

REMARK 2. Given $\gamma \sim k \sim T_s$ and take $\sigma \in (\frac{1}{2}, 1)$. Define the trigonometric series

$$f_\sigma \sim \sum_{p\in P} \frac{1}{p^\sigma} e^{ik(p)x}.$$

If $\sum \|f_\sigma^n\|_{A'}/n! < \infty$ then $T_s \in \mathrm{IPM}$.

2. – Examples of idelic pseudo-measures.

EXAMPLE 2.1. *a*) Let $k(p) = [\log p]$, $p \in P$. Then $T_s \in \mathrm{IPM}/M(\boldsymbol{T})$ for $s = 1 + i\tau$, e.g., [Benedetto, 1973, Example 4.1; 1976, Example 2.3; Holzsager, 1975].

b) Using the number theoretic techniques necessary to prove part *a* we see that $T_1 \notin \mathrm{IPM}$ if $k(p) = [\log^\alpha p]$ for $\alpha < 1$.

c) If $k(p)$ increases faster than $\log p$ then $T_s \in \mathrm{IPM}$, $s = 1 + i\tau$.

EXAMPLE 2.2. Let $k(p) = [\log^2 p]$. To see that $T_\sigma \notin \mathrm{IPM}$, $\sigma < 1$, by means of *Prop.* 1.1 and the fact that $\sum_0 z^n = \exp\left(\sum_1 z^n/n\right)$, note that

$$\sum_{k(p)=n} 1/p^\sigma \geq \left(\pi(\exp\sqrt{n+1}) - \pi(\exp\sqrt{n})\right)/\exp\left(\sigma\sqrt{n+1}\right)$$

and apply the prime number theorem $\left(\pi(x) = \mathrm{card}\,\{p \in P: p \leqslant x\}\right)$. In particular, if $k(p) = [\log p]$ then $T_\sigma \notin \mathrm{IPM}$ for $\sigma < 1$.

EXAMPLE 2.3. We shall show that (for most cases) if $k(p)$ is strictly increasing and $T_s \in \text{IPM}$ for each $\sigma > \frac{1}{2}$ then $T_s \in L^2(\boldsymbol{T})$. Besides *Prop.* 1.1 we also need the inequality

$$|D_n|^2 \leq \exp\left(\sum_{k=1}^{n} k|A_k|^2 - \sum_{k=1}^{n} \frac{1}{k}\right), \qquad D_0 = 1\,,$$

where $\exp\left(\sum A_k z^k\right) = D_k z^k$ (Lebedev and Milin). The boundedness of $\{|D_n|\}$ is assured if

$$\sum_{j=1}^{n} jA_j^2 \leqslant C \log n\,,$$

where $A_j = 1/p^\sigma$ if $j = k(p)$ and $A_j = 0$ otherwise. Since $\sigma > \frac{1}{2}$ we can apply *Prop.* 1.1 if there at most $C \log n$ elements in $\{k(p): k(p) \leqslant n\}$. This latter condition essentially demands that $k(p_j) \geqslant Me^j$ where p_j is the j-th prime. If $k(p_j) \geqslant Me^j$ then each $n = \prod p^r$ has at most one representation $\sum rk(p)$. Consequently, $T_s \in L^2(\boldsymbol{T})$ for each $\sigma > \frac{1}{2}$.

EXAMPLE 2.4. *a*) Let p_j be the j-th prime and note that T_s, $\sigma \leqslant 1$, is not in $L^\infty(\boldsymbol{T})$ even if

$$k(p_{j+1}) > k(p_1) + \ldots + k(p_j)\,. \tag{2.1}$$

This is checked directly using the Fourier series criterion of $\| \|_\infty$-boundedness of Fejér partial sums, cf. part *d* which is weaker.

b) On the other hand if k satisfies (2.1) then an application of the Hausdorff-Young theorem shows that

$$\forall \sigma \in (\tfrac{1}{2}, 1] \quad \text{and} \quad \forall q < 1/(1-\sigma)\,, \qquad T_\sigma \in L^q(\boldsymbol{T})\,;$$

and, of course, $T_s \in L^2(\boldsymbol{T})$ for each $\sigma > \frac{1}{2}$.

c) The significance of (2.1) is that the frequencies of

$$S_s(x) = \prod_{p\in P} \left(1 + \frac{e^{ik(p)x}}{p^s}\right)$$

are distinct, noting that $T_s = S_s \varphi_s$ where

$$\varphi_s(x) = \prod_{p\in P} \left(\sum_{j=0}^{\infty} (e^{ik(p)x}/p^s)^{2j}\right) \in A(\boldsymbol{T})\,.$$

If $k(p_j) = 2^{j-1}$ then $\boldsymbol{N} \cup \{0\}$ is the set of frequencies of S_s.

d) Set $k(p_j) = 2^{j-1}$ and note that S_s, $\sigma \in (\frac{1}{2}, 1]$, does not have bounded variation on $\boldsymbol{T}$ since the Fourier coefficients of its distributional derivative are unbounded.

e) For perspective, recall that for each $x \in (0, 2\pi)$, $\sum (1/n^s) e^{inx}$ is an analytic function in the region $\sigma > 0$.

3. – The analytic continuation problem.

Given a function $k: P \to N \cup \{0\}$ extended to N as in section 1. For each n, define

$$H(n) = \{m \geqslant 1: k(m) = n \text{ and } p|m \Rightarrow k(p) \neq 0\},$$

and $m_n = \inf\{m: m \in H(n)\}$. Note that if $\{q_1, \ldots, q_s\} \subseteq P$ is the zero set of k and if $\lim_p k(p) = \infty$ then card $H(n) < \infty$ for each n. To prove this let $k(p') > n$ and set

$$m_1 = \prod \{q^j: q < p', q \in P \diagdown \{q_1, \ldots, q_s\} \text{ and } j \in N \cup \{0\}\}.$$

where $jk(q) > k(p')$. Take $m = \prod_{p \neq q_j} p^r \geqslant m_1$, where the product is the prime decomposition of m. If some $p > p'$ (in this product), then $rk(p) \geqslant k(p') > n$ since k is increasing. If each $p < p'$ (in this product) then some $p = q$ has the property that $r \geqslant j$; thus, $k(m) \geqslant rk(p) \geqslant jk(p) = jk(q) > k(p') > n$. Therefore there are at most m_1 elements in $H(n)$.

THEOREM 3.1. *Given* $\sigma \in (\frac{1}{2}, 1]$ *and* $k(p) \subseteq N \cup \{0\}$ *where* $k(1) = k(2) = k(3) = k(5) = k(7) = 0$ *and* card $\{p \in P: k(p) = 0\} < \infty$. *Assume that the power series expansion,* $\sum a_n t^n$, *of*

$$\exp\left(\sum_p \frac{\log p}{p^\sigma} t^{k(p)}\right)$$

has bounded coefficients, and that

$$\text{(3.1)} \qquad \exists K_\sigma \text{ such that } \forall n \geq 0, \quad \frac{\operatorname{card} H(n)}{m_n^\sigma} < K_\sigma$$

(*in particular,* card $H(n) < \infty$). *Extend* k *additively to* $N \cup \{0\}$ *as in section* 1. *Then*

$$\text{(3.2)} \qquad \forall \sigma > 1/2, \quad \sup\left\{\left|\sum_{k(m)=n} \frac{\log m}{m^s}\right|: n \geq 0\right\} < \infty.$$

PROOF. Clearly, $\sum_0 z^j \geq e^z \geq 1 + z$; and so

$$\prod_p \left(\sum_{j=0}^{\infty} z_p^j\right) \geq \exp\left(\sum_p z_p\right) \geq \prod_p (1 + z_p) = \sum c_n t^n, \tag{3.3}$$

where $z_p = ((\log p)/p^\sigma)\, t^{k(p)}$. $\{c_n\}$ is bounded since $\{a_n\}$ is bounded. The left hand side of (3.3) can be written as

$$\prod_p (1 + z_p) \prod_p (1 + z_p^2 + z_p^4 + \ldots) = \sum d_n t^n .$$

Some straight forward estimates and an occasional use of the prime number theorem yield the convergence of $\sum b_n$, where b_n is defined by $\prod_p (1 + z_p^2 + \ldots) = \sum b_n t^n$. Combining these observations we see that $\{d_n\}$ is bounded. The bound on (3.2) is then obtained in terms of $\{d_n\}$. The hypotheses in the statement that we have not mentioned in this outline of proof arise in the rather extensive technical verification of several of the above claims [Benedetto, 1974]. *q.e.d.*

Take $\gamma \in \hat{J}_Q$ whose corresponding k-function takes values in $N \cup \{0\}$ and satisfies (3.2). For each $\hat{f} = \varphi \in A(T)$, we define

$$\forall \sigma > \tfrac{1}{2}, \qquad F_\varphi(s) = \langle T_s, \varphi \rangle = \sum_{n \in Z} f(n) \hat{T}_s(n),$$

and we see that F_φ *is analytic in the half-plane* $\sigma > \frac{1}{2}$ since its derivative

$$-\sum_{n \in Z} f(n) \left(\sum_{k(m)=n} \frac{\log m}{m^s}\right)$$

is an absolutely convergent series. In this case we write for a fixed s_0, $\sigma_0 > \frac{1}{2}$,

$$F_\varphi(s) = \sum_{n \geq 0} A_n(\varphi)(s - s_0)^n . \tag{3.4}$$

THEOREM 3.2. *Given* $\gamma \sim k \sim T_s$ *and assume that* k *satisfies* (3.2). *Then for any fixed* $s_0 \in C$, $\sigma_0 > \frac{1}{2}$, *there exists a sequence* $\{A_n : n = 0, 1, \ldots\} \subseteq A'(T)$ *depending on* s_0 *such that*

$$T_s = \sum_{n=0}^{\infty} A_n (s - s_0)^n \tag{3.5}$$

converges in the $\| \|_{A'}$*-norm in some neighborhood of* s_0; *and*

$$\sum_{n=0}^{\infty} \|A_n\|_{A'} < \infty \tag{3.6}$$

if $\sigma_0 > \frac{3}{2}$.

PROOF. Because of (3.4),

$$\forall \varphi \in A(\boldsymbol{T}), \qquad A_n(\varphi) = \frac{1}{2\pi i} \int_C \frac{\langle T_s, \varphi \rangle}{(s - s_0)^{n+1}} ds,$$

where $\sigma_0 > \frac{1}{2}$, C is the circle $s_0 + re^{i\theta}$, and $\sigma_0 - r > \frac{1}{2}$. In particular, $A_n \colon A(\boldsymbol{T}) \to \boldsymbol{C}$ is linear and depends on s_0 (and not on r). Then,

$$|A_n(\varphi)| = \frac{1}{2\pi} \left| \int_0^{\pi 2} \frac{F_\varphi(s_0 + re^{i\theta}) re^{i\theta} d\theta}{(re^{i\theta})^{n+1}} \right| \leq \left(\|\varphi\|_A \sup_{s \in C} \|T_s\|_{A'} \right) / r^n .$$

By the definition of T_s we have

$$\|A_n\|_{A'} \leq \|T_{\sigma_0 - r}\|_{A'} / r^n . \tag{3.7}$$

Thus, if $s \in \boldsymbol{C}$ with $\sigma > \frac{1}{2}$ has the property that $|\sigma - \sigma_0|/r < 1$ then

$$\sum_{n=0}^{\infty} \|A_n\|_{A'} |s - s_0|^n < \infty .$$

This yields (3.5). (3.6) is clear from (3.7) and the requirements that $r > 1$ and $\sigma_0 - r > \frac{1}{2}$. *q.e.d.*

EXAMPLE 3.1. Given $\gamma \sim k \sim T_s$ and assume that k satisfies (3.2). Then

$$\forall \sigma > \tfrac{1}{2}, \quad \lim_{|n| \to \infty} \hat{T}_s(n) = 0 .$$

In fact,

$$\left| \sum_{k(m)=n} \frac{1}{m^\sigma} \right| \leq \frac{1}{\log m_n} \left| \sum_{k(m)=n} (\log m)/m^\sigma \right| \leq \frac{k}{\log m_n},$$

where m_n is the smallest m for which $k(m) = n$. Clearly $\lim m_n = \infty$.

4. – A topological-arithmetic property of idele characters.

Given integers j, $J \geqslant 1$, and a finite set Y having at least $([j^{J/j}]+1)^j$ elements. The elements of Y will consist of j-tuples of integers satisfying the following property: if $y = (y_1, \ldots, y_j)$ and $y' = (y'_1, \ldots, y'_j)$ are in Y and $y \neq y'$ then $y - y'$ or $y' - y$ is a j-tuple of *positive* integers.

EXAMPLE 4.1. Define $Y = \{y_1, \ldots\}$ in either of the following two ways

a) $y_1 = (1, 1, \ldots, 1)$, $y_2 = (2, 2, \ldots, 2)$, $y_3 = (3, 3, \ldots, 3)$, ...

b) Given $x = (x_1, \ldots, x_j)$ where x_h is a positive integer. Define $y_h = hx$ for $h = 1, 2, \ldots$.

The following elementary result in diophantine approximation depends only on the Dirichlet pigeon-hole principle.

LEMMA. *Given j, J, and Y as above. There are elements y, $y' \in Y$ such that if $k = y' - y > 0$, where $k = (k(p_1), k(p_2), \ldots, k(p_j))$ and $k(p_h) \geqslant 1$, then for each j-tuple $(n_{p_1}, \ldots, n_{p_j}) \in \boldsymbol{Z} \times \ldots \times \boldsymbol{Z}$ (j-times) with $|n_p| \leqslant [J^{1/j}]$, there is $N = N(p_1, \ldots, p_j)$ for which*

$$\Big| \sum_{p \leqslant p_j} k(p) n_p - 2\pi N \Big| < 1/j\,.$$

PROOF. Let $M = [J^{1/j}]$ so that there are on the order of $M^j = J$ j-tuples $(n_{p_1}, \ldots, n_{p_j})$ of integers for which $|n_p| \leqslant M$ if $p \leqslant p_j$. Consider the $J \times j$ matrix $(\alpha_{r,c})$ of numbers $n/2\pi$ where $n \in \boldsymbol{Z}$ and $|n| \leqslant M$.

For each $y = (y_1, \ldots, y_j) \in Y$ consider the J-tuple $w_y = (w_1, \ldots, w_J)$ defined by

$$w_r = w_r(y) = \sum_{c=1}^{j} \alpha_{r,c} y_c\,, \qquad r = 1, \ldots, J\,.$$

For any such w_y, let $(x_1, \ldots, x_J) \in \boldsymbol{Z} \times \ldots \times \boldsymbol{Z}$ (J-times) have the property that $\omega_y = (x_1 - w_1, \ldots, x_J - w_J) \in [0, 1) \times \ldots \times [0, 1_p$ (J-times).

Divide $[0, 1)$ into j intervals $[h/j, (h+1)/j)$, $h = 0, 1, \ldots, j-1$; and in this way partition $[0, 1) \times \ldots \times [0, 1)$ (J-times) into j^J cubes.

We have

$$([j^{J/j}] + 1)^j > (j^{J/j})^j = j^J\,.$$

Since card $Y \geqslant ([j^{J/j}] + 1)^j$ there is at least one of the j^J cubes, call it C, that has at least two of the ω_y ($y \in Y$) in it. Say ω_y, $\omega_{y'} \in C$, where $y \neq y'$.

Because C is a cube of side $1/j$ we have

$$\frac{1}{j} > |(x_r - w_r) - (x_r' - w_r')| = \left| \sum_{c=1}^{j} \alpha_{r,c}(y_c' - y_c) - (x_r' - x_r) \right| \quad \text{for } r = 1, \dots, J .$$

By our hypothesis we can assume without loss of generality that $y_c' - y_c > 0$ for each $c = 1, \dots, j$. Thus, finally, we can define $k(p_c) = y_c' - y_c$, $1 \leqslant c \leqslant j$. *q.e.d.*

Choose $\gamma \in \hat{J}_Q$ so that if $\gamma \sim k \sim T_s$ then $T_s \in \mathrm{IPM}$ for each $\sigma > \frac{1}{2}$. If $j \geqslant 1$ define $k_j(p) = jk(p)$ for $p \in P$, and write $\gamma_j \sim k_j \sim T_{s,j}$. By definition, we have

PROPOSITION 4.1. *Given $T_{s,j} \in A'(\boldsymbol{T})$ where j is a positive integer.*

a) Take $n \in \boldsymbol{Z}$. If $j \nmid n$, then $\hat{T}_{s,j}(n) = 0$; and if $j \mid n$, then $\hat{T}_{s,j}(n) = \hat{T}_{s,1}(n/j)$.

b) $\|T_{s,j}\|_{A'} = \|T_{s,j+1}\|_{A'}$.

EXAMPLE 4.2. *a)* Define $k_j(p) = 0$ if $p \leqslant p_j$ and $k_j(p) = k(p) > 0$ *if* $p > p_j$, where $\gamma \sim k \sim T_s \in \mathrm{IPM}$ for all $\sigma \in (\frac{1}{2}, 1]$. Then

$$\hat{T}_{j,\sigma}(0) = \sum_{k(m)=0} 1/m^\sigma \geq \frac{1}{p_1^\sigma} + \dots + \frac{1}{p_j^\sigma}$$

and so $\hat{T}_{j,\sigma}(0)$ increases indefinitely as $j \to \infty$. Thus, $\lim_{j\to\infty} \|T_{j,\sigma}\|_{A'} = \infty$. On the other hand, it is easy to compute that $\hat{T}^{\gamma,\sigma}(0) < \infty$ for such σ.

b) For the k_j defined in part *a* we let $\gamma_j \in \hat{J}_Q$ be the corresponding idele character; and note that $\{\gamma_j\}$ converges to 1 in $\hat{J}_Q$.

Given $\gamma \sim k \sim T_s \in \mathrm{IPM}$ for some $\sigma \in (\frac{1}{2}, 1]$, where $k(p) > 0$ for each $p \in P$. Define k_j as we did before *Prop.* 4.1. Then

THEOREM 4.1. *a) There is a subsequence of $\{\gamma_j\}$ which converges to 1 in $\hat{J}_Q$.*

b) Along with Prop. 4.1, we also have that $\lim_{j\to\infty} T_{j,s} = 1$ in the $\sigma(A', A)$ topology; in fact, for each $j\hat{T}_{j,s}(0) = 1$ and for each $n \neq 0$ $\lim_{j\to\infty} \hat{T}_{j,s}(n) = 0$.

PROOF. a.i. Let $|\ |_p$ be the p-adic metric on the multiplicative group $\boldsymbol{Q}_p^\times$ of the p-adic completion of $\boldsymbol{Q}$; and let U_p be the p-adic units. We define

$$B_{p,n} = \left\{ r_p \in \boldsymbol{Q}_p^\times : \frac{1}{n} \leq |r_p|_p \leq n \right\},$$

$$K_n = \prod_{0 \leqslant p \leqslant p_n} B_{p,n} \times \prod_{p > p_n} U_p ,$$

and

$$N_n = \{\gamma \in \hat{J}_Q : \forall x \in K_n,\ |(\gamma, x) - 1| < 1/n\} \quad (\text{where } Q_0^\times = R^\times).$$

From the definition of the topology in $\hat{J}_Q$ we must verify that for each N_n there is J_n so that

$$(4.1) \qquad \forall x = \{r_p\} \in K_n, \quad |(\gamma_j, x) - 1| < 1/n \quad \text{for each } j > J_n.$$

We know that

$$(\gamma_j, x) = \prod_p \exp i \frac{k_j(p)}{\log p} \log |r_p|_p,$$

where $n_p \in Z$, $-\log n \leqslant -n_p \log p \leqslant \log n$, and $\log |r_p|_p = -n_p \log p$; in particular, $|n_p| \leqslant \log n / \log p$.

Taking $x \in K_n$, we have $r_p \in U_p$ if $p > p_n$ and so $|r_p|_p = 1$ if $p > p_n$; thus $n_p = 0$ for each $p > p_n$.

Consequently, if $x \in K_n$, then

$$(\gamma_j, x) = \prod_{p \leqslant p_n} \exp - i k_j(p) n_p, \qquad |n_p| \leqslant \log n / \log p .$$

Using the notation of the *Lemma*, we set $M = [\log n]$ since

$$\max_{p \leqslant p_n} ([\log n / \log p]) \leqslant [\log n] ;$$

and we define $J = M^n$.

Note that if $x \in K_n$, then, by the above, the n-tuple $(n_{p_1}, \ldots, n_{p_n})$ corresponding to x has the property that $|n_{p_i}| \leqslant M$ for each i.

Consequently, we apply the *Lemma* (and raise to the e^i power) to obtain (4.1).

b follows from *Prop* 4.1. *q.e.d.*

The analogous result holds for $L^q(T)$.

BIBLIOGRAPHY

J. Benedetto, *Idele characters in spectral synthesis on* $R/2\pi Z$, Ann. Inst. Fourier, **23** (1973) pp., 45-64.

J. Benedetto, *Zeta functions for idelic pseudo-measures*, U. of Maryland Math. Dept. Technical report 74-55 (1974), pp. 1-46.

J. Benedetto, *Idelic pseudo-measures and Dirichlet series*, Symposia mathematica (Istituto Nazionale di Alta Matematica, Roma). Conference in harmonic analysis, Academic Press, New York, 1976.

R. Holzsager, *A boundedness condition*, J. London Math. Soc., **11** (1975), pp. 179-180.

J. Tate, *Fourier analysis in number fields and Hecke's zeta functions* (1950), *Algebraic number theory* (ed. J. Cassels and A. Fröhlich), Academic Press, New York, 1967.