The theory of frames is due to Duffin and Schaeffer [22], and it was developed to address problems in non-harmonic Fourier series. Prior to [22], these problems were concerned with finding criteria on real sequences $\{t_n\}$ so that the closed linear span, $\overline{span}\{e_{t_n}\}$, of exponentials $e_{t_n}(f) = e^{2\pi j t_n f}$ would be equal to the space $L^2[-\Omega, \Omega]$ of finite energy signals defined on $[-\Omega, \Omega]$. The origins of these problems go back at least to G.D. Birkhoff (1917) and J.L. Walsh (1921), and the basic and highly non-trivial theory associated with such problems was established by Paley and Wiener in [35]. The theory was further developed in remarkable ways by Levinson [30], Pollard [36], Beurling and Malliavin [14], [15], and others, see [38], [48]. The impact of the ideas in [22] to address issues in signal reconstruction was not fully appreciated until the work of Daubechies, Grossmann, and Meyer [21], as well as subsequent work by Daubechies in her book [20]. The 1990s have seen a plethora of contributions to the theory of frames, e.g., see the Duffin memorial issue of the Journal of Fourier Analysis and Applications (Volume 3, 1997).

The formulation of irregular sampling algorithms in terms of so-called Fourier frames (Definition 10) is a natural topic in the theory of frames, see [7] from 1990; and there are several expositions on the subject including [2] and [23], cf., Marvasti's book [33] for many classical irregular sampling results and a comprehensive bibliography. The relationship between sampling theory and other types of signal decompositions, including wavelet and Gabor frames, has also become a highly developed area, see [5].

The mammalian auditory system possesses excellent abilities to detect, separate, and recognize speech and environmental sounds. In recent decades, these capabilities have been the subject of theoretical and experimental research, particularly with a view towards applying auditory functional principles in the design of man-machine communication links, e.g., [16], [18], [24], [41], [45]. As indicated above, we shall use frames and irregular sampling methods to construct a wavelet and Fourier frame based mathematical model for the mammalian auditory system. It is called the Wavelet Auditory Model (WAM) [9]. Our purpose is to present the mathematical underpinnings of WAM more fully than in [8] with a goal of gaining a deeper understanding of the role of irregular sampling in the reconstruction of speech signals.

After introducing the wavelet transform and some notation in Section 2, we shall describe WAM in Section 3. In fact, we shall trace the processing of a speech signal in a mammalian auditory system, and construct a corresponding mathematical model, viz., WAM. This model will exhibit some of the mathematical structure associated with wavelet frames, and in the process of exploiting this structure we shall see the role of irregular sampling in reconstructing a signal $y$.

Because of the point of view developed in Section 3, we shall present the elements of the theory of frames in Section 4 and related results about irregular sampling and Fourier frames in Section 5.

We begin Section 4 with the definition of frames (Definition 1). The Frame Decomposition Theorem (Theorem 2) which follows is reformulated in Proposition 4 as a reconstruction formula for a signal $y$ in terms of its "sampled values" $Ly$. This reformulation gives rise to a reconstruction algorithm (Algorithm 8), which implements a Gram operator whose entries can theoretically be stored off-line. Algorithm 8 is a perfect reconstruction theorem, and its implementation on WAM data guarantees excellent speech signal reconstruction under ideal conditions. Because of the inherent wavelet frame structure in WAM which emerged in Section 3, we close Section 4 with a wavelet frame calculation (Example 11) that will play a role in Section 7. The calculation also shows how Fourier frames can arise in this context, thereby giving explicit motivation for the material in Section 5.

In Theorem 12 of Section 5, we give basic criteria for irregularly spaced modulates and translates of a signal to be a frame for $L^2(\mathbb{R})$, the space of finite energy signals on $\mathbb{R}$. Then we use a corollary of this result to prove the Yao-Thomas irregular sampling formula in Theorem 14. The Yao-Thomas Theorem is really a result about so-called exact frames (Definition 1).

A critical feature in the success of mammalian auditory systems is the fact that the cochlea has the equivalent of a sophisticated and effective filter bank on its basilar membrane. This filter bank is discussed in Section 3, and we shall use the Paley-Wiener Logarithmic Integral Theorem in Section 6 to design the corresponding filters in WAM. We shall then see that the wavelet and Fourier frame approach to mammalian auditory systems corroborates the fact that white noise is often naturally reduced in such systems using such filters during speech processing. We shall also include related calculations for some other noises.

In our final section, Section 7, we shall integrate the modelling of Section 3, the theoretical results of Sections 4 and 5, and the construction of Section 6 to apply WAM to a typical problem in speech coding. The mathematical success of this application is illustrated, and its potential practical value is the subject of [9].

## 2. MATHEMATICAL BACKGROUND

Let $L^2(\mathbb{R})$ be the space of complex-valued finite energy signals defined on the real line $\mathbb{R}$. The *Fourier transform* $Y$ or $\hat{y}$ of $y \in L^2(\mathbb{R})$ is

$$Y(f) = \hat{y}(f) = \int y(t)e^{-2\pi jtf}dt$$

for $f \in \hat{\mathbb{R}}(=\mathbb{R})$, where integration is over $\mathbb{R}$. The Fourier pairing between $y$ and $Y$ is designated by $y \leftrightarrow Y$. If $Y$ is defined on $\hat{\mathbb{R}}$, then formally one has

$$y(t) = Y^{\vee}(t) = \int Y(f)e^{2\pi jtf}df,$$

where integration is over $\hat{\mathbb{R}}$, see [3] for conditions for the validity of this formula. $Y^{\vee}$ is the *inverse Fourier transform* of $Y$.

For $s > 0$, the $L^2$-*dilation* operator $D_s$ is defined by $D_s y(t) = s^{1/2}y(st)$ for $y \in L^2(\mathbb{R})$, and the Fourier transform of $D_s y$ is

$$(D_s y)^{\wedge}(f) = s^{-1/2}\hat{y}(s^{-1}f) = D_{1/s}\hat{y}(f).$$

For $u \in \mathbb{R}$, the *translation* operator $\tau_u$ is defined by $\tau_u y(t) = y(t-u)$ for $y \in L^2(\mathbb{R})$. As such, $(\tau_u y)^{\wedge}(f) = e^{-2\pi juf}\hat{y}(f) = e_{-u}(f)\hat{y}(f)$, where

$$e_{-t}(f) = e^{-2\pi jtf}.$$

The *convolution* of $x, y \in L^2(\mathbb{R})$ is

$$x * y(t) = \int x(t-u)y(u)du = \int x(u)y(t-u)du.$$

$x{*}y$ is an absolutely convergent Fourier transform; and the *inner product* of $x$ and $y$ is $\langle x, y\rangle = \int x(t)\overline{y(t)}dt$.

For a fixed $g \in L^2(\mathbb{R})$, the *wavelet transform* of $y \in L^2(\mathbb{R})$ is the function

$$W_g y(t,s) = (y * D_s g)(t) \tag{2.1}$$

defined on the *time-scale plane* $t \in \mathbb{R}$ , $s > 0$. By a straightforward calculation, we obtain

$$W_g y(t,s) = \langle y, \tau_t D_s \tilde{g}\rangle,$$

where $\tilde{g}$ is the *involution* of $g$ defined as $\tilde{g}(u) = \overline{g}(-u)$. If the derivative $\partial_t g$ is an element of $L^2(\mathbb{R})$, we define $W_{\partial_t g}y$ analogously to the definition of $W_g y$ in (2.1). In this case, if $W_{\partial_t g}y$ converges uniformly on time intervals for each fixed scale $s > 0$, and if a mild smoothness condition is satisfied, then

$$\partial_t W_g y(t,s) = s W_{\partial_t g} y(t,s). \tag{2.2}$$

These hypotheses for the validity of (2.2) can be weakened; and (2.2) is true generally for the causal filters $g$ and signals $y$ under consideration in this chapter.

Notationally, we follow standard notation in mathematical analysis, e.g., [42]. In particular, $L^2[-\Omega,\Omega]$ is the space of finite energy signals defined on the interval $[-\Omega,\Omega]$, and $PW_\Omega$ is the *Paley-Wiener space*, defined as

$$PW_\Omega = \{y \in L^2(\mathbb{R}) : supp\, \hat{y} \subseteq [-\Omega,\Omega]\},$$

where $supp\, \hat{y}$ is the support of $\hat{y}$. Let

$$d_{2\pi\Omega}(t) = \frac{sin 2\pi\Omega t}{\pi t}, \quad t \in \mathbb{R},$$

where "*d*" is for Dirichlet. In the case $\Omega = 1/2$, $d_\pi$ is the sinc function. $\mathbf{1}_{(\Omega)}$ denotes the characteristic function of the interval $[-\Omega,\Omega] \subseteq \hat{\mathbb{R}}$. Thus, we have the Fourier pairing $d_{2\pi\Omega} \leftrightarrow \mathbf{1}_\Omega$.

Finally, $l^2(\mathbb{Z})$ is the space of finite energy sequences, where $\mathbb{Z}$ is the ring of integers.

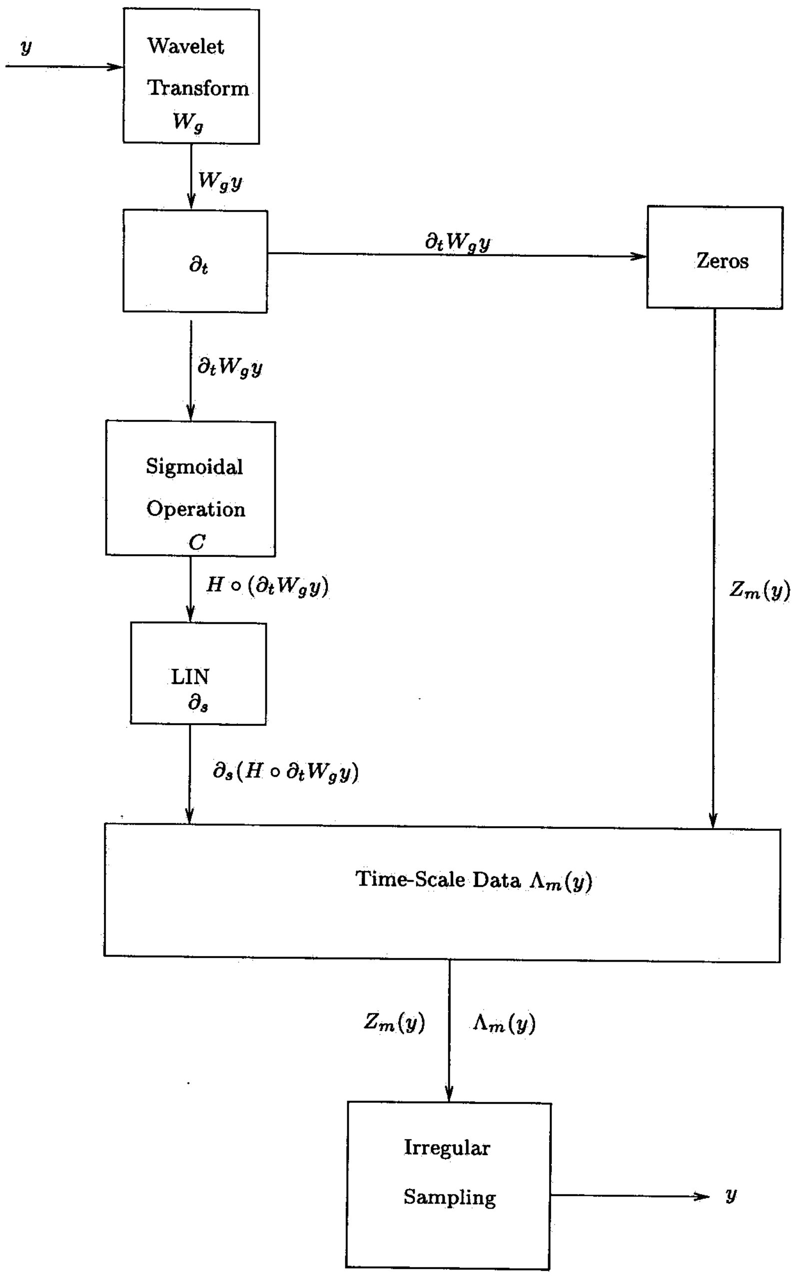
# 3. WAVELET AUDITORY MODEL

## 3.1 SETTING

In this section we shall describe Figure 1.1.

In a mammalian auditory system an acoustic signal or sound wave $y$ induces vibrations in the ear drum, which travel through the middle ear to the cochlear fluid of the inner ear. These vibrations then cause traveling waves on the basilar membrane of the cochlea. As the waves propogate into the spiral shaped cochlea, they produce a pattern of displacements $W$ of the basilar membrane at different locations for different frequencies. Displacements for high frequencies occur at the basal end; for low frequencies they occur at the wider apical end inside the spiral, e.g., [24]. The basilar membrane records frequency responses between 200 and 20,000 Hz. For comparison, telephone speech bandwidth deals with the range 300-4,000 Hz. The cochlea analyzes sound in terms of these traveling waves much like a parallel bank of filters - in this case a band with 30,000 channels. The impulse responses of these filters along most of the interior length of the cochlea are related by dilation. Consequently, their transfer functions are invariant except for frequency

*Figure 1.1* Wavelet Auditory Model




$$Z_m(y) = \{(t(n; s_m) : \partial_t W_g y(t(n; s_m), s_m) = 0\}$$

$$\Lambda_m(y) = \{\partial_s \partial_t W_g y(t(n; s_m), s_m) : t(n; s_m) \in Z_m(y)\}$$

translation along the approximately logarithmic axis of the cochlea, e.g., [39], [40].

Mathematically, this dilational relationship between impulse responses can be expressed by the assertion that there is a function $g : \mathbb{R} \mapsto \mathbb{C}$ such that the set of impulse responses is of the form $\{D_s g : s \in [s_1, s_2] \subseteq (0, \infty)\}$. Thus, we identify the displacements $W$, due to the stimulus $y$, with the output of the cochlear filter bank having the impulse responses $\{D_s g\}$, i.e., we set $W = W_g y(t, s)$, where $g$ is a fixed causal impulse response. Specifically, because of the structure of the cochlear filter bank, we fix $a > 1$ and set $s_m = a^m, m \in \mathbb{Z}$. As such, in WAM the signal $y$ is first transformed into a pattern of displacements,

$$W_g y(t, s_m), \quad m \in \mathbb{Z},$$

for a discrete set of points $(t, s_m)$ in the time-scale plane, where $t$ is written more explicitly in (3.1), see the box labelled Wavelet Transform in Figure 1.1.

The shape of $|\hat{g}|$ is critical for the effectiveness of the auditory model. Generally, $\hat{g}$ should be a causal filter that has a "shark-fin" shaped amplitude. The design problems for such filters are dealt with in Section 6. In the case of properly designed filters, the high frequency edges of the cochlear filters $D_{1/s}\hat{g}$ act as abrupt "scale" delimiters. Thus, a sinusoidal stimulus will propogate up to the appropriate scale and die out beyond it.

The auditory system does not receive the wavelet transform directly, but rather a substantially modified version of it. In fact, the output of each cochlear filter is effectively highpassed by the velocity coupling between the cochlear membrane and the cilia of the hair cell transducers that initiate the electrical nervous activity by a shearing action on the tectorial membrane. Hence, the mechanical motion of the basilar membrane is converted to a receptor potential in the inner hair cells. It is reasonable to approximate this stage by a time derivative, obtaining the ouput $\partial_t W_g y(t, s_m)$, see the box labelled $\partial_t$ in Figure 1.1. The extrema of the wavelet transform $W_g y(t, s_m)$ become the zero-crossings of the new function $\partial_t W_g$; and so one output of the auditory process is

$$\forall m \in \mathbb{Z}, \quad Z_m = \{t(n; s_m) : \partial_t W_g y(t(n; s_m), s_m) = 0\}, \tag{3.1}$$

i.e., the box labelled Zeros in Figure 1.1.

## 3.2 SIGMOIDAL OPERATION

At the next step in the auditory process, an instantaneous sigmoidal non-linearity $R$ is applied, followed by a low pass filter with impulse response $h$. These operations model the threshold and saturation that occur in the hair cell channels and the leakage of electrical current throughout the membranes of these cells [34], [39]. The resulting cochlear output,

$$C_{h,R}(t,s) = (R \circ \partial_t W_g y(\cdot, s)) * h(t),$$

where "$\circ$" is composition and convolution is with respect to time, is a planar auditory nerve pattern sent to the brain along the scale-ordered array of auditory channels $(t, \cdot)$. Typically, the composition by $R$ can be represented by functions

$$R_T(x) = \frac{e^{Tx}}{1 + e^{Tx}},$$

parameterized by $T$. Obviously, $\lim_{T\to\infty} R_T = H$, the Heaviside function. Approximations to the Heaviside function are reasonable since the nerve fibers from the inner hair cells to the auditory nervous system fire at positive rates, and since this action cannot process above a certain limit, i.e., the aforementioned saturation. For computational convenience in WAM, we take $R$ to be $H$ and set $h = \delta$, the Dirac $\delta$-measure, even though $\delta$ does not give rise to a low pass filter. Thus, $C_{h,R}(t,s)$ above is replaced by the *cochlear output,*

$$C(t,s) = H \circ \partial_t W_g y(t,s),$$

i.e., the output of the box labelled Sigmoidal Operation $C$ in Figure 1.1.

## 3.3 LATERAL INHIBITORY NETWORK

The mammalian auditory nerve patterns determined by $W, \partial W$, and $Z_m$ are now processed by the brain in ways that are not completely understood. One processing model, the lateral inhibitory network (LIN), has been closely studied with a view to extracting the spectral pattern of the acoustic stimulus [34], [40], [46]; and we shall implement it in our alogorithm. Scientifically, it reasonably reflects proximate frequency channel behavior, and mathematically it is relatively simple.

For a given acoustic signal $y$, constant $a > 1$, and properly designed causal filter $\hat{g}$, we generate $Z_m$ and the set

$$\Lambda_m = \{\partial_s \partial_t W_g y(t(n; s_m), s_m) : t(n; s_m) \in Z_m\} \tag{3.2}$$

for each $m \in \mathbb{Z}$, i.e., the box labelled Time-Scale Data $\Lambda_m(y)$ in Figure 1.1. The scaling partial $\partial_s$ reflects the action of LIN, reflected by the box in Figure 1.1 called LIN $\partial_s$, see [8] for a derivation of the formula resulting in the coefficients (3.2).

## 3.4 THE WAM PROBLEM AND SOLUTION

Let $\hat{g}$ be a properly designed finite energy causal filter and let $a > 1$. Suppose an unknown acoustic signal $y$ has generated the set

$$\Lambda_m = \{\partial_s \partial_t W_g y(t(n; s_m), s_m) : t(n; s_m) \in Z_m\}$$

and that the receiver has knowledge of this set or of some subset. This irregularly spaced array in the $t - s$ plane is called *WAM data*. It is natural for the receiver to attempt to reconstruct the signal $y$ in terms of this data. The *WAM problem* is to effect this reconstruction by means of irregular sampling formulas. We shall outine the solution given in [8], but stress the mathematical development more than appeared there. In particular, we shall present irregular sampling formulas developed by Benedetto and Heller, e.g., [2], [7], and give perspective in terms of the theory of frames and other irregular sampling criteria. The fact that the WAM problem can be solved theoretically by means of such formulas leaves open the problem of effective implementation, see Section 7.

## 3.5 WAM WAVELET FRAME

Observe that for $t(n; s_m) \in Z_m$,

$$\partial_s \partial_t W_g y(t(n; s_m), s_m)$$

$$\approx \frac{\partial_t W_g y(t(n; s_m), s_{m+1}) - \partial_t W_g y(t(n; s_m), s_m)}{s_{m+1} - s_m}.$$

The second term in the numerator of the right side vanishes, since $t(n; s_m) \in Z_m$. Hence,

$$\frac{\partial_t W_g y(t(n; s_m), s_{m+1}) - \partial_t W_g y(t(n; s_m), s_m)}{s_{m+1} - s_m} = \frac{\partial_t W_g y(t(n; s_m), s_{m+1})}{s_{m+1} - s_m}$$

$$= \frac{s_{m+1} W_{\partial_t g} y(t(n; s_m), s_{m+1})}{s_{m+1} - s_m},$$

where the last equality follows from (2.2). Writing this approximation as an equality, we have

$$\partial_s \partial_t W_g y(t(n; s_m), s_m) = \frac{s_{m+1}}{s_{m+1} - s_m} W_{\partial_t g} y(t(n; s_m), s_{m+1})$$

$$= -\frac{1}{a-1} \langle y, \tau_{t(n; s_m)} D_{s_{m+1}} (\partial_t \tilde{g})(u) \rangle.$$

Because of this equation and the frame-theoretic point of view of the next section (Section 4), we define

$$\psi_{m,n} = -\frac{1}{a-1}\tau_{t(n;s_m)}D_{s_{m+1}}(\partial_t\tilde{g}) \tag{3.3}$$

and the mapping

$$L : \mathbb{H} \to l^2(\mathbb{Z}\times\mathbb{Z})$$
$$y \mapsto \{\langle y, \psi_{m,n}\rangle\},$$

where $\mathbb{H}$ is a Hilbert subspace of $L^2(\mathbb{R})$ containing the class of acoustic signals to be analyzed. Each function $\psi_{m,n}$ corresponds to an element $t(n; s_m) \in Z_m$. Note that (3.3) can be rewritten as

$$\psi_{m,n} = -\frac{1}{a-1}D_{s_{m+1}}\tau_{s_{m+1}t(n;s_m)}(\partial_t\tilde{g}).$$

In particular, $\{\psi_{m,n}\}$ depends on a given acoustic signal $y$ and the known filter $\hat{g}$.

The discussion in the previous paragraph leads naturally into the theory of wavelet frames, that we shall develop in Section 4. The dependence of $\{\psi_{m,n}\}$ on $y$ is not amenable to a global theory of frames but such a theory is not essential for our purpose. The degree to which the sequence $\{\psi_{m,n}\}$ can be considered as a wavelet frame will be analyzed in Example 11.

## 4. THEORY OF FRAMES

In this section we review the theory of frames that was introduced by Duffin and Schaeffer [22], see also [20], [48], and Chapters 3 and 7 of [6]. Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle x, y\rangle$ and norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$.

**Definition 1 Frames**

a. A sequence $\{x_n : n \in \mathbf{Z}^d\} \subseteq \mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist $A, B > 0$ such that

$$\forall y \in \mathbb{H}, \quad \mathrm{A}\|y\|^2 \leq \sum |\langle y, x_n\rangle|^2 \leq B\|y\|^2.$$

$A$ and $B$ are *frame bounds*, and a frame is *tight* if $A = B$. A frame is *exact* if it is no longer a frame whenever any one of its elements is removed.

b. The *frame operator* of the frame $\{x_n\}$ is the function $S : \mathbb{H} \to \mathbb{H}$ defined as $Sy = \sum\langle y, x_n\rangle x_n$ for all $y \in \mathbb{H}$ .

c. It is easy to see that from part $b$ that if $\{x_n\} \subseteq \mathbb{H}$ is a frame, then $S = L^*L$, where

$$L : \mathbb{H} \to l^2(\mathbb{Z}^d)$$
$$x \mapsto \{\langle x, x_n \rangle\}$$

is called the *Bessel map* associated with $\{x_n\}$ and where the adjoint map

$$L^* : l^2(\mathbb{Z}) \to \mathbb{H}$$

of $L$ is the *reconstruction map.*

**Theorem 2 Frame Decomposition Theorem**

Let $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ be a frame for $\mathbb{H}$ with frame bounds $A$ and $B$.

a. The frame operator $S$ is a topological isomorphism with inverse $S^{-1} : \mathbb{H} \to \mathbb{H}$ . $\{S^{-1}x_n\} \subseteq \mathbb{H}$ is a frame with frame bounds $B^{-1}$ and $A^{-1}$, and

$$\forall y \in \mathbb{H}, \quad y = \sum \langle y, S^{-1}x_n \rangle x_n = \sum \langle y, x_n \rangle S^{-1}x_n \quad \text{in } \mathbb{H}.$$

$\{S^{-1}x_n\}$ is called the *dual frame* of $\{x_n\}$, and it is easy to see that $S^{-1}$ is the frame operator of $\{S^{-1}x_n\}$.

b. If $\{x_n\}$ is a tight frame for $\mathbb{H}$ , if $\|x_n\| = 1$ for all $n$, and if $A = B = 1$, then $\{x_n\}$ is an orthonormal basis for $\mathbb{H}$ , see [2] for the original formulation of this result by Vitali.

c. If $\{x_n\}$ is an exact frame for $\mathbb{H}$, then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthonormal, i.e.,

$$\forall m, n, \quad \langle x_m, S^{-1}x_n \rangle = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise,} \end{cases}$$

and $\{S^{-1}x_n\}$ is the unique sequence in $\mathbb{H}$ which is biorthonormal to $\{x_n\}$.

d. If $\{x_n\}$ is an exact frame for $\mathbb{H}$ , then the sequence resulting from the removal of any one element is not complete in $\mathbb{H}$ , i.e., the linear span of the resulting sequence is not dense in $\mathbb{H}$ .

**Theorem 3 Characterization of Frames**

a. A sequence $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ is a frame for $\mathbb{H}$ with frame bounds $A$ and $B$ if and only if the map

$$L : \mathbb{H} \to l^2(\mathbb{Z}^d)$$
$$y \mapsto \{\langle y, x_n \rangle\}$$

is a topological isomorphism of $\mathbb{H}$ onto a closed subspace of $l^2(\mathbb{Z}^d)$. In this case,

$$\|L\| \leq B^{\frac{1}{2}} \quad \text{and} \quad \|\mathrm{L}^{-1}\| \leq \mathrm{A}^{-\frac{1}{2}},$$

where $L^{-1}$ is defined on the range $L(\mathbb{H})$. Thus, in the case of a frame, $L$ is the associated Bessel map.

b. A sequence $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ is a frame for $\mathbb{H}$ if and only if there is $C > 0$ such that for all $y \in \mathbb{H}$

$$\sum |\langle y, x_n \rangle|^2 < \infty,$$

$$\exists c_y = \{c_n\} \in l^2(\mathbb{Z}^d), \quad \text{such that} \quad y = \sum c_n x_n \in \mathbb{H}\,,$$

and

$$\|c_y\|_{l^2(\mathbb{Z}^d)} \leq C\,\|y\|\,.$$

Part $a$ is proved in Theorem 7.15 of [6]; and part $b$ is proved in Remark 3.9 of [6], cf., the treatment of part $b$ in [10].

If $\{x_n\}$ is a frame with Bessel map $L$ and given data $c \in l^2(\mathbb{Z}^d)$ is of the form $Ly = \{\langle y, x_n \rangle\} = c$, where $y$ is not explicitly known, then $c$ can be thought of as "sampled data" of some signal $y$, which must then be reconstructed in terms of $c$. The following result is the first step in developing this point of view.

**Proposition 4 Frame Reconstruction Formula**

Let $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ be a frame for $\mathbb{H}$ with frame operator $S$, frame bounds $A$ and $B$, and Bessel map $L : \mathbb{H} \to l^2(\mathbb{Z}^d)$. Then

$$\forall y \in \mathbb{H}, \quad y = (S^{-1}L^*)Ly, \tag{4.1}$$

cf., (4.6).

Equation 4.1 can be viewed as a reconstruction formula for signals $y$ in which discrete "sampled data" $Ly$ is given, as indicated in the remark introducing Proposition 4. In fact, using the hypotheses of Proposition 4, Equation (4.1) and the Neumann expansion

$$S^{-1} = \frac{2}{A+B} \sum_{k=0}^{\infty} \left( I - \frac{2}{A+B} S \right)^k$$

can be combined to provide an iterative reconstruction algorithm for signal reconstruction, see Algorithm 8.

**Definition 5 Gram Operator**

Let $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ be a frame for $\mathbb{H}$ with frame operator $S$, frame bounds $A$ and $B$, and Bessel map $L : \mathbb{H} \to l^2(\mathbb{Z}^d)$.

a. The *Gram operator* associated with $\{x_n\}$ is the map $R = LL^* : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$.

b. Let $L'$ and $R'$ denote the Bessel map and Gram operator, respectively, associated with the dual frame $\{S^{-1}x_n\}$.

c. It is easy to check that $R$ restricted to $L(\mathbb{H})$ is a bijection onto $L(\mathbb{H})$. If $R^{-1}$ denotes the inverse defined on $L(\mathbb{H})$, then we can extend $R^{-1}$ to $l^2(\mathbb{Z}^d)$ by defining the *pseudo-inverse* $R^\dagger$ of $R$ as

$$R^\dagger = R^{-1} P_{L(\mathbb{H})} : l^2(\mathbb{Z}^d) \to L(\mathbb{H}) \subseteq l^2(\mathbb{Z}^d),$$

where $P_{L(\mathbb{H})}$ is the orthogonal projection operator onto the image of $L$.

**Lemma 6**

Let $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ be a frame for $\mathbb{H}$, with Gram operator $R$, frame bounds $A$ and $B$, and Bessel map $L : \mathbb{H} \to l^2(\mathbb{Z}^d)$. If $0 < \lambda < 2/B$, then $\|I - \lambda R\|_{L(\mathbb{H})} < 1$. We may take $\lambda = 2/(A+B)$.

**Proof**

Since $(L')^*$ is surjective, for any $y \in \mathbb{H}$ there is a $c \in L'(\mathbb{H})$ so that $y = (L')^* c$. This together with the fact that $\{S^{-1}x_n\}$ is a frame for $\mathbb{H}$ yields

$$B^{-1}\langle c, R'c\rangle \leq \langle c, (R')^2 c\rangle \leq A^{-1}\langle c, R'c\rangle.$$

Letting $c = (R')^\dagger d$ for some $d \in l^2(\mathbb{Z}^d)$, we have $B^{-1}\langle Rd, d\rangle \leq \langle d, d\rangle \leq A^{-1}\langle Rd, d\rangle$. For all nonzero $d \in L'(\mathbb{H})$ this means

$$A \leq \frac{\langle Rd, d\rangle}{\langle d, d\rangle} \leq B.$$

Thus, we have for $\lambda > 0$ that

$$1 - \lambda B \leq \frac{\langle (I - \lambda R)d, d\rangle}{\langle d, d\rangle} \leq 1 - \lambda A,$$

and, since $I - \lambda R$ is self-adjoint,

$$\|I - \lambda R\|_{L(\mathbb{H})} = \sup_{c \in L(\mathbb{H})} \frac{|\langle (I - \lambda R)c, c\rangle|}{\langle c, c\rangle} \leq \max\{|1 - \lambda A|, |1 - \lambda B|\}. \tag{4.2}$$

We would like to find $\lambda$ such that $\|I - \lambda R\|_{(L(\mathbb{H})} < 1$. This condition is satisfied for all $\lambda \in (0, 2/B)$. In particular, if $\lambda = 2/(A + B)$ then $|1 - \lambda A| = |1 - \lambda B| = (B - A)/(A + B) < 1$. For this choice of $\lambda$ we have proved that $\|I - \lambda R\|_{L(\mathbb{H})} < 1$. □

**Proposition 7 Frame Reconstruction Formula**

Let $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ be a frame for $\mathbb{H}$, with frame operator $S$, Gram operator $R$, frame bounds $A$ and $B$, and Bessel map $L : \mathbb{H} \to l^2(\mathbb{Z}^d)$. If $\lambda \in (0, 2/B)$, e.g., if $\lambda = 2/(A + B)$, then

$$\forall y \in \mathbb{H}, \quad y = \lambda \sum_{i=0}^{\infty} L^*(I - \lambda R)^i Ly, \tag{4.3}$$

where $L^*c = \sum c_n x_n$ for $c = \{c_n\} \in l^2(\mathbb{Z}^d)$.

**Proof**
Since $\langle Lx, c\rangle = \langle x, L^*c\rangle$ and $\langle Lx, c\rangle = \sum \overline{c_n}\langle x, x_n\rangle$, we obtain the formula for $L^*c$.
Because of the Neumann expansion

$$S^{-1} = \frac{2}{A + B} \sum_{k=0}^{\infty} \left(I - \frac{2}{A + B}S\right)^k$$

and the fact that $S = L^*L$, it is sufficient to prove

$$\lambda \sum_{i=0}^{\infty} L^*(I - \lambda R)^i Ly = \sum_{i=0}^{\infty} (I - \lambda L^*L)^i (\lambda L^*L)y, \tag{4.4}$$

where the sums are well-defined by Lemma 6. The $i = 0$ terms are clearly the same in (4.4). Assume

$$\lambda L^*(I - \lambda R)^i Ly = (I - \lambda L^*L)^i (\lambda L^*L)y. \tag{4.5}$$

Then, using (4.5), compute

$$\begin{aligned}\lambda L^*(I-\lambda R)^{i+1}Ly &= \lambda L^*(I-\lambda R)^i Ly - \lambda L^*(I-\lambda R)^i \lambda R(Ly)\\ &= \lambda(I-\lambda L^*L)^i L^*Ly - \lambda(I-\lambda L^*L)^i L^*L(\lambda L^*Ly)\\ &= \lambda(I-\lambda L^*L)^i(I-\lambda L^*L)L^*Ly = \lambda(I-\lambda L^*L)^{i+1}L^*Ly\,,\end{aligned}$$

and the result follows by induction. □

Proposition 7 leads directly to the following theorem (Algorithm 8), which provides an iterative reconstruction procedure for the recovery of a signal $y$ from its "sampled values" $Ly$. This iterative procedure converges at an exponential rate.

**Algorithm 8 Frame Reconstruction Algorithm**

Let $\{x_n : n \in \mathbb{Z}^d\} \subseteq \mathbb{H}$ be a frame for $\mathbb{H}$, with Gram operator $R$, frame bounds $A$ and $B$, and Bessel map $L$. Let $y \in \mathbb{H}$ and set $c_{(0)} = Ly \in l^2(\mathbb{Z}^d)$, $y_0 = 0$, $\lambda = 2/(A+B)$, and $\alpha = \|I - \lambda R\|_{L(\mathbb{H})} < 1$. Define $y_m, u_m \in \mathbb{H}$, and $c_{(m)} \in L(\mathbb{H})$, $m = 0, 1, ...,$ recursively, as

$$u_m = \lambda L^* c_{(m)}, \quad c_{(m+1)} = c_{(m)} - Lu_m,$$

and

$$y_{m+1} = y_m + u_m.$$

Then

$$\forall m \in N, \quad \|y - y_m\| < \alpha^m \frac{B}{A}\|y\|,$$

and, in particular, $\lim_{m\to\infty} y_m = y$ in $\mathbb{H}$ .

**Proof**

i. An elementary induction argument shows that

$$\forall m = 0, 1, ..., \quad y_{m+1} = \lambda L^* \left(\sum_{i=0}^{m} (I-\lambda R)^i\right) c_{(0)}.$$

Consequently, by Proposition 7, we have $\lim_{m\to\infty} y_m = y$ in $\mathbb{H}$.

ii. For any fixed $m \geq 0$ and for any $k \in N$,

$$\begin{aligned}\|y - y_m\| &= \|y + (y_{m+1} - y_m) + ... + (y_{m+k+1} - y_{m+k}) - y_{m+k+1}\|\\ &\leq \|y - y_{m+k+1}\| + \sum_{i=0}^{k} \|y_{m+i+1} - y_{m+i}\|\,.\end{aligned}$$

Using part $i$ and taking the lim sup as $k \to \infty$, e.g., page 278 of [3], we obtain

$$\|y - y_m\| \le \sum_{k \ge m} \|y_{k+1} - y_k\|,$$

from which we compute, using part $i$ for the first step and (4.2) for the last, that

$$\|y - y_m\| \le \sum_{k \ge m} \left\|\lambda L^*(I - \lambda R)^k L y\right\|$$

$$\le \sum_{k \ge m} \lambda \|L^*\| \left\|(I - \lambda R)^k\right\|_{L(\mathbb{H})} \|L\| \, \|y\|$$

$$\le \lambda B (\sum_{k \ge m} \alpha^k) \|y\| = (\frac{\alpha^m}{1-\alpha}) \lambda B \|y\| \le \alpha^m \frac{B}{A} \|y\|.$$

□

Algorithm 8 underscores the importance of the discrete nature of the Gram operator $R$ in the reconstruction process. Also, formally, we may rewrite (4.3) as

$$y = (L^* R^{-1}) L y, \tag{4.6}$$

cf., (4.1). It should be pointed out that the implementation of Algorithm 8, in terms of finite matrices approximating $R$, is sometimes difficult, see [17], [26], [43].

A crucial element in the proof of Algorithm 8 is the fact that the sampled data $c_{(0)}$ has the form $c_{(0)} = Ly$. If $c_{(0)}$ is not entirely in $L(\mathbb{H})$, then the algorithm will not converge. An analysis of this latter situation is found in Section 6 of [44], and it is related to noise reduction.

### Definition 9 Wavelet Systems and Frames

Let $\psi \in L^2(\mathbb{R})$. The *affine system* or *wavelet system* for $\psi$ is the sequence $\{\psi_{m,n} : (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$, where, for $a > 1$,

$$\psi_{m,n}(t) = a^{m/2} \psi(a^m t - n).$$

The Fourier transform of $\psi_{m,n}$ is computed to be

$$\hat{\psi}_{m,n}(f) = a^{-m/2} e^{-2\pi j n (f/a^m)} \hat{\psi}(f/a^m) = a^{-m/2} (e_{-n} \hat{\psi})(f/a^m).$$

If the wavelet system $\{\psi_{m,n}\}$ for $\psi$ is a frame, then it is a *wavelet frame.*

**Definition 10 Fourier Frames**

a. Let $\{b_m\} \subseteq \mathbb{R}$. If $\{e_{b_m}\}$ is a frame for $L^2[-T,T]$, it is called a *Fourier frame* for $L^2[-T,T]$. In this situation we also say that $\{b_m\}$ is a Fourier frame for $PW_T \subseteq L^2(\hat{\mathbb{R}})$.

b. Given $g \in L^2(\mathbb{R})$ and sequences $\{a_n\}, \{b_m\} \subseteq \mathbb{R}$. If $\{e_{b_m}\tau_{a_n}g\}$ is a frame for $L^2(\mathbb{R})$ it is called a *weighted Fourier frame* for $L^2(\mathbb{R})$ with weight $g$ or a *Gabor frame* for $L^2(\mathbb{R})$ depending on the window function $g$.

**Example 11 WAM Wavelet Frame Analysis**

The following calculation illustrates to what extent $\{\psi_{m,n}\}$, defined in (3.3), can be considered a wavelet frame for some sufficiently robust Hilbert space $\mathbb{H} \subseteq L^2(\mathbb{R})$; cf., the critical observation in Section 3 that $\{\psi_{m,n}\}$ depends on $y$. We first compute

$$\sum |\langle y, \{\psi_{m,n}\}\rangle|^2 = \frac{1}{(a-1)^2} \sum \left| \langle \overline{\hat{y} D_{s_{m+1}^{-1}} (\partial \tilde{g})^\wedge}, e_{-t(n;s_m)} \rangle \right|^2 .$$

Then, we assume that for each $m \in \mathbb{Z}, \{-e_{-t(n;s_m)} : n \in \mathbb{Z}\}$ is a Fourier frame with frame bounds $A_m, B_m$. Thus,

$$\sum_m A_m \left\| \hat{y} D_{s_{m+1}^{-1}} (\partial \tilde{g})^\wedge \right\|^2 \leq (a-1)^2 \sum_m \sum_n |\langle y, \psi_{m,n}\rangle|^2$$

$$\leq \sum_m B_m \left\| \hat{y} D_{s_{m+1}^{-1}} (\partial \tilde{g})^\wedge \right\|^2 .$$

Consequently, if we suppose that

$$0 < A \leq \frac{1}{(a-1)^2} A_m \leq \frac{1}{(a-1)^2} B_m \leq B < \infty,$$

for some $A, B$, then by a simple calculation and Plancherel's theorem, we have

$$A \left( \inf_f \sum_m \left| D_{s_{m+1}^{-1}} (\partial \tilde{g})^\wedge (f) \right|^2 \right) \|y\|^2 \leq \sum |\langle y, \psi_{m,n}\rangle|^2$$

$$\leq B \left\| \sum_m \left| D_{s_{m+1}^{-1}} (\partial \tilde{g})^\wedge (f) \right|^2 \right\|_\infty \|y\|^2 . \tag{4.7}$$

The inequalities in (4.7) lead to frame properties of $\{\psi_{m,n}\}$ if

$$G(f) = \sum_m \left| D_{s_{m+1}^{-1}} (\partial \tilde{g})^{\wedge}(f) \right|^2 \tag{4.8}$$

is bounded above and bounded below away from 0. In any case, the function in (4.8) must be quantified to obtain effective frame decompositions by means of Theorem 2; and it should be noted that the scaling constant $a$ plays a role in (4.8). The mammalian cochlear filter $\hat{g}$ satisfies (4.8), see the examples in [8].

## 5. IRREGULAR SAMPLING AND FOURIER FRAMES

In this section we shall state and prove an irregular sampling expansion by frame methods. We begin with the following result, e.g., [2], [7].

**Theorem 12 Fourier and Gabor Frames**

Let $g \in PW_\Omega$ for a given $\Omega > 0$. Assume that $\{a_n\} \subseteq \mathbb{R}$, $\{b_m\} \subseteq \hat{\mathbb{R}}$ are real sequences for which

$$\{e_{a_n}\} \text{ is a Fourier frame for } L^2[-\Omega, \Omega],$$

and that there exist $A, B > 0$ such that

$$0 < A \leq G(f) \leq B < \infty \quad a.e. \text{ on } \hat{\mathbb{R}},$$

where

$$G(f) = \sum |\hat{g}(f - b_m)|^2.$$

Then $\{e_{a_n} \tau_{b_m} \hat{g}\}$ is a frame for $L^2(\hat{\mathbb{R}})$ with frame operator $S$; and $\{e_{a_n} \tau_{b_m} \hat{g}\}$ is a tight frame for $L^2(\hat{\mathbb{R}})$ if and only if $\{e_{a_n}\}$ is a tight frame for $L^2[-\Omega, \Omega]$ and $G$ is constant a.e. on $\hat{\mathbb{R}}$.

**Corollary 13**
Let us assume the hypotheses and notation of Theorem 12, and set $I_m = [-\Omega, \Omega] + b_m$. Then, for each fixed $m$, $\{\tau_{b_m} e_{a_n}\}$ is a frame for $L^2(I_m)$ with frame operator $S_m$, and

$$\forall y \in L^2(\mathbb{R}), \quad S\hat{y} = \sum (\tau_{b_m} \hat{g}) S_m (\hat{y} \tau_{b_m} \bar{\hat{g}}) \quad \text{in } L^2(\hat{\mathbb{R}}).$$

**Proof**
We compute

$$
\begin{aligned}
S\hat{y} &= \sum_m \sum_n \langle \hat{y}, e_{a_n}\tau_{b_m}\hat{g}\rangle e_{a_n}\tau_{b_m}\hat{g} \\
&= \sum_m (\tau_{b_m}\hat{g})(\sum_n \langle \hat{y}, e_{a_n}\tau_{b_m}\hat{g}\rangle e_{a_n})\mathbf{1}_{I_m} \\
&= \sum_m (\tau_{b_m}\hat{g})(\sum_n \langle \hat{y}\tau_{b_m}\bar{\hat{g}}, e_{a_n}\rangle_{I_m} e_{a_n}\mathbf{1}_{I_m}) \\
&= \sum_m (\tau_{b_m}\hat{g}) S_m(\hat{y}\tau_{b_m}\bar{\hat{g}}).
\end{aligned}
$$

□

We can now prove the Yao-Thomas irregular sampling theorem in terms of exact frames, see [47].

**Theorem 14 Yao-Thomas Formula**

Let $\{e_{a_n}\}$ be an exact frame for $L^2[-\Omega,\Omega]$ for a given $\Omega > 0$ and sequence $\{a_n\} \subseteq \mathbb{R}$. Define the sampling function $s_n$ in terms of its involution $\tilde{s}_n(t) = \overline{s_n}(-t)$, where

$$
\forall t \in \mathbb{R}, \quad \tilde{s}_n(t) = \int_{-\Omega}^{\Omega} \overline{\hat{h}_n}(f) e^{2\pi j t f}\, df,
$$

and where $\{h_n\} \subseteq L^2[-\Omega,\Omega]$ is the unique sequence for which $\{e_{a_n}\}$ and $\{h_n\}$ are biorthonormal. (In particular, $\tilde{s}_n \in PW_\Omega$.) If $t_n = -a_n$ , then

$$
\forall y \in PW_\Omega, \quad y = \sum y(t_n) s_n \quad \text{in } L^2(\mathbb{R}).
$$

**Proof**
Let $g = (2\Omega)^{-\frac{1}{2}} d_{2\pi\Omega}$, and set $b_m = 2m\Omega$. Since $\{e_{a_n}\}$ is a frame we can apply Theorem 12 and, hence, $\{e_{a_n}\tau_{b_m}\hat{g}\}$ is a frame for $L^2(\hat{\mathbb{R}})$ with frame operator $S$. In particular,

$$
\forall h \in L^2(\mathbb{R}), \quad \hat{h} = \sum \langle h, e_{a_n}\tau_{b_m}\hat{g}\rangle S^{-1}(e_{a_n}\tau_{b_m}\hat{g}) \quad \text{in } L^2(\hat{\mathbb{R}}). \tag{5.1}
$$

We obtain

$$
\langle \hat{y}, e_{a_n}\tau_{b_m}\hat{g}\rangle = \begin{cases} (2\Omega)^{\frac{-1}{2}} y(-a_n) & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \tag{5.2}
$$

for $y \in PW_\Omega$. By means of Corollary 13 we can then verify that

$$
S^{-1} = 2\Omega {S_0}^{-1} \quad \text{on} \quad L^2[-\Omega,\Omega],
$$

where $S_0$ is defined in Corollary 13. Thus, since $g \in PW_\Omega$, we compute

$$S^{-1}(e_{a_n}\tau_{b_0}\hat{g}) = (2\Omega)^{1/2}{S_0}^{-1}(e_{a_n}\mathbf{1}_{(\Omega)}),$$

so that, by the exactness hypothesis and part $a$ of Theorem 2, the right side is

$$(2\Omega)^{1/2}\sum_m \langle e_{a_n}, h_m\rangle_{[-\Omega,\Omega]}h_m = (2\Omega)^{1/2}h_n.$$

Combining these two equalities with (5.1) and (5.2) gives the reconstruction,

$$\forall y \in PW_\Omega, \quad \hat{y} = \sum_n (2\Omega)^{-1/2}y(-a_n)(2\Omega)^{1/2}h_n \quad \text{in}\, L^2(\mathbb{R}),$$

and the result follows. □

**Remark 15 Perspective on Fourier Frames and Irregular Sampling**

a. The Yao-Thomas result, formulated in terms of exact frames, is an irregular sampling theorem in that the coefficients of the decomposition are really sampled values. The assertion about biorthonormality in Theorem 14 shows the relation betwen Fourier frames and the Yao-Thomas decomposition. The in-depth study of the Fourier frame case is due to Beurling, e.g., [13], and Landau [29], and is treated in terms of multidimensional irregular sampling in [11], [12].

b. The three assertions in this remark deal with a density criterion measuring the uniform distance between $n/(2\Omega)$, $n \in \mathbb{Z}$, and elements $a_n$ of the sampling set. This particular idea is due to Duffin and Schaeffer [22] for frames, but goes back to Wiener (1927) for determining the closure of linear spans. Although seemingly weaker than obtaining frame decompositions, some of the most formidable analysis of the 20th century is associated with such closure issues, e.g., [35], [30], [14], [15]. A beautiful exposition of the following assertions is due to Young [48].

   i. Let $\{a_n\} \subseteq \mathbb{R}$. Levinson [30] proved that if

$$\sup\left|a_n - \frac{n}{2\Omega}\right| \leq \frac{1}{4}(\frac{1}{2\Omega}),$$

then $\overline{span}\{e_{a_n}\} = L^2[-\Omega,\Omega]$ .

ii. Further, Kadec's $\frac{1}{4}$-Theorem (1964) asserts that if

$$\sup\left|a_n - \frac{n}{2\Omega}\right| \le L < \frac{1}{4}(\frac{1}{2\Omega}),$$

then $\{e_{a_n}\}$ is an exact frame for $L^2[-\Omega,\Omega]$ .

iii. Kadec's $\frac{1}{4}$-Theorem is sharp in the sense that there exists a sequence $\{a_n\} \subseteq \mathbb{R}$ such that

$$sup\left|a_n - \frac{n}{2\Omega}\right| = \frac{1}{4}(\frac{1}{2\Omega}),$$

and therefore $\{e_{a_n}\}$ is complete in $L^2[-\Omega,\Omega]$, but $\{e_{a_n}\}$ is not an exact frame for $L^2[-\Omega,\Omega]$.

c. If $\{t_n\} \subseteq \mathbb{R}$ be a strictly increasing sequence for which $\lim_{n\to\pm\infty} t_n = \pm\infty$, and for which

$$\exists d > 0 \quad \text{such that} \quad \forall m \neq n, \quad |t_m - t_n| \ge d,$$

then $\{t_n\} \subseteq \mathbb{R}$ is said to be *uniformly discrete.* A uniformly discrete sequence $\{t_n\}$ is *uniformly dense* with *uniform density* $\Delta > 0$ if

$$\exists L > 0 \quad \text{such that} \quad \forall n \in \mathbb{Z}, \left|t_n - \frac{n}{\Delta}\right| \le L.$$

d. Using a theorem due to Duffin-Schaeffer [22] for one direction, Jaffard [28] has provided the following characterization of frames $\{e_{-t_n}\}$ for $L^2[-\Omega,\Omega]$. *Let $\{t_n\} \subseteq \mathbb{R}$ be a strictly increasing sequence for which $\lim_{n\to\pm\infty} t_n = \pm\infty$, and let $I \subseteq \mathbb{R}$ denote an interval.*

1. *The following two assertions are equivalent:*
   (a) *There is $I \subseteq \mathbb{R}$ for which $\{e_{-t_n}\}$ is a frame for $L^2(I)$.*
   (b) *The sequence $\{t_n\}$ is a disjoint union of a uniformly dense sequence with uniform density $\Delta$ and a finite number of uniformly discrete sequences.*
2. *In the case assertion b of part 1 holds, then $\{e_{-t_n}\}$ is a frame for $L^2(I)$ for each $I \subseteq \mathbb{R}$ for which $|I| < \Delta$.*

The Classical Sampling Theorem, often associated with the names Whittaker, Kotel'nikov, Shannon, et al., goes back to Cauchy (1841), e.g., [4], and it provides a sampling formula of the form

$$y = \sum T y(nT)\tau_{nT}\theta$$

when $y \in PW_\Omega, 2T\Omega \leq 1$, and the sampling function $\theta$ satisfies some natural conditions, see Theorem 3.10.10 of [3]. Theorem 17 was proved with Heller in [7], and gives an analogue of the Classical Sampling Theorem for irregularly spaced sampling sets $\{t_n\}$.

**Lemma 16**

Given $y, y_n \in L^2(\mathbb{R})$, and assume $y = \sum y_n$ in $L^2(\mathbb{R})$. If $x \in L^\infty(\mathbb{R})$ then $xy = \sum xy_n$ in $L^2(\mathbb{R})$.

**Theorem 17 An Irregular Sampling Theorem**

Suppose $\Omega > 0$ and $\Omega_1 > \Omega$, and let $\{t_n\} \subseteq \mathbb{R}$ have the property that $\{e_{-t_n}\}$ is a Fourier frame for $L^2[-\Omega_1, \Omega_1]$ with frame operator $S$. Further, let $\theta \in L^2(\mathbb{R})$ have the properties that $\hat{\theta} \in L^\infty(\mathbb{R})$,

$$supp\, \hat{\theta} \subseteq [-\Omega_1, \Omega_1]\ ,$$

and $\hat{\theta} = 1$ on $[-\Omega, \Omega]$. Then

$$\forall y \in PW_\Omega, \quad y = \sum c_n(y) \tau_{t_n} \theta \quad \text{in } L^2(\mathbb{R}), \tag{5.3}$$

where

$$c_n(y) = \langle S^{-1}(\hat{y}\mathbf{1}_{[-\Omega_1,\Omega_1]}), e_{-t_n} \rangle. \tag{5.4}$$

**Proof**
Since $\{e_{-t_n}\}$ is a Fourier frame for $L^2[-\Omega_1, \Omega_1]$ and $supp\, \hat{y} \subseteq [-\Omega, \Omega]$, we have

$$\hat{y} = \hat{y}\mathbf{1}_{(\Omega_1)} = \sum \langle S^{-1}(\hat{y}\mathbf{1}_{(\Omega_1)}), e_{-t_n} \rangle_{[-\Omega_1,\Omega_1]} e_{-t_n} \mathbf{1}_{(\Omega_1)} \quad \text{in } L^2(\hat{\mathbb{R}}). \tag{5.5}$$

In this expression, we note that $S^{-1}$, being positive, is self-adjoint so that the frame expansion in Theorem 2 gives rise to (5.5). Also, the convergence in $L^2[-\Omega_1, \Omega_1]$ from our frame hypothesis can be taken in $L^2(\hat{\mathbb{R}})$ by extending all functions to be zero outside $[-\Omega_1, \Omega_1]$.

We have $\hat{y} = \hat{y}\hat{\theta}$ on $\hat{\mathbb{R}}$ since $\hat{\theta} = 1$ on $[-\Omega, \Omega]$ and $\hat{y} = 0$ off of $[-\Omega, \Omega]$. Further,

$$\begin{aligned} &\hat{\theta} \sum \langle S^{-1}(\hat{y}\mathbf{1}_{(\Omega_1)}), e_{-t_n} \rangle_{[-\Omega_1,\Omega_1]} e_{-t_n} \mathbf{1}_{(\Omega_1)} \\ &= \sum \langle S^{-1}(\hat{y}\mathbf{1}_{(\Omega_1)}, e_{-t_n} \rangle_{[-\Omega_1,\Omega_1]} e_{-t_n} \mathbf{1}_{(\Omega_1)} \hat{\theta} \qquad \text{in } L^2(\hat{\mathbb{R}}) \end{aligned}$$

by Lemma 16. Thus, since $supp\, \hat{\theta} \subseteq [-\Omega_1, \Omega_1]$, we obtain

$$\hat{y} = \hat{y}\hat{\theta}$$

$$= \sum_n \langle S^{-1}(\hat{y}\mathbf{1}_{(\Omega_1)}), e_{-t_n}\rangle_{[-\Omega_1,\Omega_1]} e_{-t_n}\hat{\theta} \qquad \text{in } L^2(\hat{\mathbb{R}}).$$

Taking the inverse Fourier transform gives (5.3). □

**Remark 18**

Let $\{e_{-t_n}\}$ be a Fourier frame for $L^2[-\Omega_1, \Omega_1]$ with frame bounds $A$ and $B$ and frame operator $S$. In general we cannot write $c_n(y) = y(t_n)$ in (5.3). However, Theorem 17 is an irregular sampling theorem in the sense that the coefficients $c_n(y)$ can be described in terms of values of $y$ on the irregularly spaced sampling set $\{t_n\}$. From the Neumann expansion,

$$S^{-1} = \frac{2}{(A+B)} \sum_{k=0}^{\infty} \left(I - \frac{2}{(A+B)}S\right)^k,$$

we have

$$c_n(y) = \frac{2}{(A+B)} \sum_{k=0}^{\infty} \langle \left(I - \frac{2}{(A+B)}\right)^k (\hat{y}\mathbf{1}_{[-\Omega_1,\Omega_1]}), e_{-t_n}\rangle. \tag{5.6}$$

If we truncate this expression after the $k = 0$ term we obtain the sampled value

$$\frac{2}{(A+B)} y(t_n) \tag{5.7}$$

as an approximation of (5.6).

**Remark 19**

In the case of regular sampling we can use a frame analysis similar to the proof of Theorem 17 to prove the formula

$$\forall y \in L^2(\mathbb{R}), \quad y = T \sum_{m,n} \langle \hat{y}, e_{nT}\tau_{mb}\hat{\theta}\rangle \tau_{-nT}(e_{mb}s) \quad \text{in } L^2(\mathbb{R}), \tag{5.8}$$

where $T, \Omega > 0$ are constants for which $0 < 2T\Omega \leq 1$, $\theta \in PW_{1/(2T)}$ has the properties that $\hat{\theta} \in L^\infty(\hat{\mathbb{R}})$ and $\hat{\theta} = 1$ on $[-\Omega, \Omega]$, and, in case $2T\Omega < 1$, $\hat{\theta}$ is continuous and

$$\left|\hat{\theta}\right| > 0 \quad \text{on } (-\frac{1}{2T}, -\Omega] \cup [\Omega, \frac{1}{2T}).$$

In dealing with high frequency information, with close fluctuations, it is necessary to sample closely in order to capture all of the fluctuations.

By definition, then, in the case with very high frequencies, thought of as "infinite frequencies", and hence nonbandlimited, we can not reconstruct the function with a discrete set of samples. However, the frame reconstruction formula (5.8) gives the Classical Sampling Theorem for bandlimited functions, as well as giving signal representation for nonbandlimited functions. In this latter case, there is added complexity in the coefficients necessary to deal with "infinite frequencies". Equation (5.8) also allows us to interpret aliasing in a quantitative way, see [2], [7].

# 6. FILTER DESIGN

## 6.1 COCHLEAR FILTERS

As mentioned in Section 3, the shape of $|\hat{g}|$, where $g$ is the impulse response of the cochlear system, is critical for the effectiveness of the auditory process, and generally $\hat{g}$ has an asymmetrical "shark-fin" shaped amplitude with faster rate of decay on the high frequency side than on the low frequency side. All realizable systems, such as our filter bank with "shark-fin" shaped amplitudes, are necessarily causal. In particular, the cochlear filter bank cannot characterize (reconstruct) future utterances in terms of known (present) speech signals. As such, we design *causal filters* $\hat{g} \in L^2(\hat{\mathbb{R}})$, i.e., *supp* $g \subseteq [0, \infty)$, for which $\hat{g}$ has the required "shark-fin" shaped amplitude consistent with mammalian auditory models. Our point of view is that such filters provide a realistic mathematical model for the cochlear filters described in Section 3, and are therefore the proper filters for optimizing the reconstruction process inherent in WAM.

The starting point for the design of such causal filters is the Paley-Wiener Logarithmic Integral Theorem, i.e., Theorem X11 of [35].

**Theorem 20 Paley-Wiener Logarithmic Integral Theorem**

Let $A \in L^2(\hat{\mathbb{R}})\backslash\{0\}$ be non-negative on $\hat{\mathbb{R}}$. $A(f) = |\hat{g}(f)|$ *a.e.* for some causal filter $\hat{g} \in L^2(\hat{\mathbb{R}})$ if and only if

$$\int \frac{|logA(f)|}{1+f^2} df < \infty. \qquad (6.1)$$

Let $A \in L^2(\hat{\mathbb{R}})$ satisfy (6.1), and define

$$\phi(x,f) = \frac{1}{\pi}\int \frac{x log A(\lambda)}{x^2 + (f-\lambda)^2} d\lambda.$$

Clearly, $\phi$ is harmonic in the half-plane $x > 0$. If $\theta$ is a conjugate harmonic function of $\phi$, then it is unique up to an additive constant; and we shall construct a particular $\theta$ in (6.4). The functions $\phi$ and $\theta$ satisfy the Cauchy-Riemann equations, and $K(z) = \phi(x,f) + j\theta(x,f)$, $z = x + jf$, is an analytic fucntion in the half-plane $x > 0$. We let

$$p(f) = \frac{1}{\pi}\frac{1}{(1+f^2)}$$

and consider the $L^1$-dilation (by $1/x$),

$$p_{\frac{1}{x}}(f) = \rho(x,f) = \frac{1}{\pi}\frac{x}{x^2+f^2}, \quad x > 0.$$

Thus, $\lim_{x\to 0} p_{1/x} = \delta$ distributionally, in fact, in the $\sigma(M_b, C_0)$ topology, where $C_0$ is the space of continuous functions vanishing at $\pm\infty$ and $M_b$ is the space of bounded Radon measures on $\hat{\mathbb{R}}$, e.g., [1]. By the definition of $\phi$ we have

$$\phi(x+jf) = p_{\frac{1}{x}} * (log A)(f), \quad x > 0, \tag{6.2}$$

and, because of the approximate identity $p_{\frac{1}{x}}$, a classical calculation yields

$$\lim_{x\to 0^+} \phi(x+jf) = log A(f) \quad a.e.; \tag{6.3}$$

see e.g., [27], [37], [42].

The harmonic function

$$K(x,f) = \frac{-1}{\pi}\frac{f}{x^2+f^2}, \quad x > 0$$

is a conjugate harmonic function of $\rho$ and so the Cauchy-Riemann equations, $\partial_x\rho = \partial_f K$ and $\partial_f\rho = -\partial_x K$, are valid in the half-plane $x > 0$. Using (6.2), the equations,

$$\partial_x\phi = (\partial_x\rho)*_\lambda log A$$

and

$$\partial_f\phi = (\partial_f\rho)*_\lambda log A, \quad x > 0,$$

follow from (6.3), where "$*_\lambda$" designates convolution in the second variable of $\rho$. Thus, we define

$$\theta = K *_\lambda log A, \quad x > 0. \tag{6.4}$$

The function

$$G(z) = e^{K(z)}, \quad z = x + jf,$$

is analytic in the half-plane $x > 0$, and provides the solution asserted in Theorem 20 in the following sense. By (6.3), we formally compute

$$G(jf) = A(f)e^{j\theta(0,f)} \quad a.e., \tag{6.5}$$

and note, by (6.4), that

$$\theta(0, f) = \frac{-1}{\pi} \int \frac{log A(\lambda)}{f - \lambda} d\lambda \tag{6.6}$$

is formally the Hilbert transform $\mathcal{H}(-log A)$ of $-log A$. It turns out that condition (6.1) allows us to assert the existence of a causal filter $\hat{g} \in L^2(\hat{\mathbb{R}})$ for which $\hat{g}(f) = G(jf)\, a.e.$. The actual filter design is a consequence of (6.5) and (6.6), and is formulated in the following result.

**Theorem 21 Construction of Causal Filter**

Let $A \in L^2(\hat{\mathbb{R}})\backslash\{0\}$ be non-negative on $\hat{\mathbb{R}}$, and assume condition (6.1). Then the function

$$\hat{g} = Ae^{-j\mathcal{H}(log A)} \tag{6.7}$$

is a causal filter in $L^2(\hat{\mathbb{R}})$, i.e., $g \in L^2(\mathbb{R})$ and $supp\, g \subseteq [0, \infty)$.

**Example 22 WAM Filter**

Take $F(f) = mf\mathbf{1}_{[0,\Gamma)}(f)$. Let $\rho \geq 0$ be compactly supported with the property that $\int \rho(f)df = 1$. Then consider the nonnegative function $A_\rho = F * \rho$. The cochlear filters for WAM use Theorem 21 and $A_\rho$ in the following way. Note that $suppF = [0, \Gamma]$, and choose $N >> \Gamma$. Let $\epsilon(f)$ be an even function on $(-\infty, -N] \cup [N, \infty)$ defined by

$$\forall f \geq N, \quad \epsilon(f) = e^{f/(log^2 f)}.$$

Then, for the function $A$ in Theorem 21, we set $A = A_\rho$ on $[0, \Gamma)$, and let $A(f) = A_\rho(0)$ on $[-N, 0] \cup [\Gamma + \epsilon, N]$, where $\epsilon > 0$ is small. Finally, let $A(f) = e^{-f/log^2|f|}$ if $|f| > N$. Clearly, $A \in L^2(\hat{\mathbb{R}})$ and (6.1) is valid. Thus, the causal cochlear filter $\hat{g}$ can be defined by (6.7) in Theorem 21.

## 6.2 OTHER FILTERS

The cochlear filter suppresses white noise in a way we shall comment on in Section 7, but other filter designs can be implemented to reduce other types of noises. Given a signal $y$ and a set $\{\psi_{m,n}\}$ of functions, we say that $y$ is coherent or "non-noisy" with respect to $\{\psi_{m,n}\}$ if $y$ may be effectively approximated by a linear combination of a relatively small number of elements of $\{\psi_{m,n}\}$, cf., [32]. With this point of view, noise in a signal is that part which lacks coherence with respect to $\{\psi_{m,n}\}$. If $\{\psi_{m,n}\}$ is a frame for a large enough space of finite energy signals, then this view of coherent signal versus noise admits a nonlinear thresholding algorithm, inspired by mammalian auditory systems, which allows for the recovery of a coherent signal embedded in noise.

For simplicity, we shall assume additive noise, i.e., if $y_0$ is coherent with respect to $\{\psi_{m,n}\}$ and $N$ is noise, then we express the signal $y$ as $y_N = y_0 + N$. We interpret the norm equivalence property of frames as an approximate energy preservation between the signal $y_N$ and its digitized version or sampled values $Ly_N = Ly_0 + LN$, where $L$ is the Bessel map associated with $\{\psi_{m,n}\}$, i.e., $Ly = \{\langle y, \psi_{m,n}\rangle\}$ and

$$\langle y, \psi_{m,n}\rangle = -\frac{1}{a-1} W_{\partial_t g} y(t(n; s_m), s_{m+1}).$$

With this interpretation and the fact that $\{\psi_{m,n}\}$ depends on $g$, we have the following problem. For a given noise $N$ construct a filter $\hat{g}$ so that the coefficients $\{LN(m,n)\}$ are small.

For a given time $t$, scale $s$, and parameter $a > 1$, we have

$$\begin{aligned} -\frac{1}{a-1} W_{\partial_t g} y(t,s) &= \frac{\sqrt{s}}{1-a} \int y(v)(\partial g)(s(t-v))dv \qquad (6.8) \\ &= C_s(y)(t). \end{aligned}$$

Thus, for a noise $N$, we can use the Plancherel-Parseval Theorem to compute

$$C_s(N)(t) = \frac{2\pi j}{1-a}\frac{1}{s} \int f\hat{N}(f)(D_{s^{-1}}\hat{g})(f)e^{2\pi jtf}df. \qquad (6.9)$$

We shall consider noises $N_\alpha$ which are $\frac{1}{f}$-*processes*, e.g. [19]. These are noises whose generalized power spectra $S_\alpha(f)$ satisfy

$$\frac{b}{|f|}\alpha \leq S_\alpha(f) \leq \frac{c}{|f|}\alpha, \qquad (6.10)$$

where $\alpha \in [0,2]$ is fixed. In a very simple model, if $b = c = 1$, then $N_0$ is white noise and $N_2$ is Brownian motion.

### 6.2.1 $\frac{1}{f}$-noises.

Let $y_{N_\alpha} = y_0 + N_\alpha$, where $N_\alpha$ is a $\frac{1}{f}$-*process* defined by Equation (6.10). We assume that $N_\alpha$ is uncorrelated with respect to the signal $y_0$, and, hence, $y_0 * \tilde{N}_\alpha$ is small, where $\tilde{N}_\alpha$ is the involution of $N_\alpha$. Thus, recalling that $\{\psi_{m,n}\}$ is dependent on $g$, our goal is to construct $\hat{g}$ so that the coefficients

$$C_{s_{m+1}}(N_\alpha)(t(n; s_m)), \quad m, n \in \mathbb{Z}, \tag{6.11}$$

defined by Equation (6.9), are small. Such a construction presents theoretical difficulties, and so, because of the above assumption on the noncorrelation of $y_0$ and $N_\alpha$, we shall solve the design of $\hat{g}$ problem in the case that the coefficients

$$C_{s_{m+1}}(N_\alpha * \tilde{N}_\alpha)(t(n; s_m)) \tag{6.12}$$

are small. A minimization of (6.11) yields noise suppression for $N_\alpha$-noise contaminating signals $y_0$. Our minimization of (6.12) provides an approximation to the desired minimization of (6.11). Note that

$$C_s(N_\alpha * \tilde{N}_\alpha)(t) = \frac{2\pi j}{1-a}\frac{1}{s}\int f \frac{1}{|f|^\alpha}(D_{s^{-1}}\hat{g})(f)e^{2\pi j t f}df$$

$$= \frac{2\pi j}{1-a}\frac{1}{\sqrt{s}}[[(sgn f)|f|^{1-\alpha}]^\vee(u) * g(us)](t). \tag{6.13}$$

For the case of white noise $\alpha = 0$, the term $(sgn f)|f|^{1-\alpha}$, on the right side of Equation (6.13), becomes $f$ and we have the Fourier pairing,

$$\frac{1}{2\pi j}\delta'(t) \leftrightarrow f,$$

where $\delta'$ is the dipole at the origin. Thus,

$$C_{s_{m+1}}(N_0 * \tilde{N}_0)(t) = \frac{1}{(1-a)\sqrt{s_{m+1}}}(\delta'(u) * g(us_{m+1}))(t)$$

$$= \frac{1}{1-a}D_{s_{m+1}}g'(t). \tag{6.14}$$

We want to construct $g$ so that

$$C_{s_{m+1}}(N_0 * \tilde{N}_0)(t(n;s_m)) \tag{6.15}$$

is small whenever

$$W_{\partial_t g'}(N_0 * \tilde{N}_0)(t(n;s_m), s_m) = 0. \tag{6.16}$$

The points $t(n;s_m)$ are defined by Equation (6.16). By Equation (6.8), this means we want the quantity defined in (6.15) to be small whenever

$$C_{s_m}(N_0 * \tilde{N}_0)(t(n;s_m)) = 0. \tag{6.17}$$

The left side of Equation (6.17) is

$$\frac{1}{1-a} D_{s_m} g'(t(n;s_m))$$

by Equation (6.14). Thus, if $D_{s_m} g'(t) = a^{m/2} g'(a^m t) = 0$, we want to conclude that

$$\frac{1}{1-a} D_{s_{m+1}} g'(t) = \frac{a^{\frac{(m+1)}{2}}}{1-a} g'(a^{m+1} t)$$

is small, and this is the criterion used to define $g$.

### 6.2.2 Noise Reduction for $\frac{1}{f}$-noise.

For the case of $\frac{1}{f}$-*noise* $\alpha = 1$, the term $(sgnf)|f|^{1-\alpha}$, on the right side of Equation (6.13), becomes $sgnf$ and we have the Fourier pairing,

$$-\frac{1}{\pi j} pv(\frac{1}{t}) \leftrightarrow sgnf,$$

where $pv(\frac{1}{t})$ is the first order distributional principal value. Thus,

$$C_{s_{m+1}}(N_1 * \tilde{N}_1)(t)$$

is the Hilbert transform $\mathcal{H}$ of $D_{s_{m+1}} g(u)$:

$$C_{s_{m+1}}(N_1 * \tilde{N}_1)(t) = \frac{2}{a-1} pv \int \frac{(D_{s_{m+1}} g)(u)}{t-u} du. \tag{6.18}$$

As before, we want to construct $g$ so that the right side of Equation (6.18) is small whenever

$$pv \int \frac{(D_s g)(u)}{t-u} du = 0.$$

To attack this problem, we can use the bounds of the form

$$|\mathcal{H}F(t)| \leq \frac{4}{\pi}(log2)\sqrt{Mm},$$

where $m = \left\|F'\right\|_{L^\infty(\mathbb{R})}$ and $\left|\int_b^c F(t)dt\right| \leq M$ for all $b, c$ [31].

# 7. WAM IMPLEMENTATION AND AN APPLICATION TO SPEECH CODING

## 7.1 WAM IMPLEMENTATION

Now that we have established some results from the theory of frames, as well as proving a Fourier frame based irregular sampling formula, we can address the WAM problem posed in Section 3.4. Given a cochlear system with impulse response $g$, let $y$ be a speech signal to be processed. From (3.1) and (3.2) the corresponding WAM data is

$$\Lambda_m = \{\partial_s \partial_t W_g y(t(n; s_m), s_m) : t(n; s_m) \in Z_m\},$$

where

$$Z_m = \{t(n; s_m) : \partial_t W_g y(t(n; s_m), s_m) = 0\}.$$

Also, from Section 3.5 we have

$$\begin{aligned}\partial_s \partial_t W_g y(t(n; s_m), s_m) &= \frac{-1}{a-1}\langle y, \tau_{t(n;s_m)} D_{s_{m+1}}(\partial_t \tilde{g})\rangle \\ &= \langle y, \psi_{m,n}\rangle,\end{aligned}$$

where

$$\psi_{m,n} = -\frac{1}{a-1}\tau_{t(n;s_m)} D_{s_{m+1}}(\partial_t \tilde{g}).$$

Thus, if we assume $\{\psi_{m,n}\}$ to be a frame for $\mathbb{H}$ with Bessel map $L$, then $Ly = \langle y, \psi_{m,n}\rangle = \partial_s \partial_t W_g y(t(n, s_m), s_m)$.

By Proposition 4,

$$y = (S^{-1}L^*)Ly,$$

i.e., the signal $y$ can be reconstructed from the discrete WAM data $Ly$. As mentioned in the remark following Proposition 4,

$$S^{-1} = \frac{2}{A+B}\sum_{k=0}^{\infty}\left(I - \frac{2}{A+B}S\right)^k,$$

and, hence,

$$y = (S^{-1}L^*)Ly = \sum_{i=0}^{\infty}(I - \lambda S)^i(\lambda S)y,$$

where $\lambda = 2/(A+B)$ and $A, B$ are frame bounds. Moreover, Proposition 7 asserts that

$$y = \lambda \sum_{i=0}^{\infty} L^*(I - \lambda R)^i (Ly), \tag{7.1}$$

where $R$ is the Gram operator, i.e., $R = LL^*$.

We now apply Algorithm 8 to compute the signal $y$ when we are given the WAM data $Ly$. Thus, we let $c_{(0)} = Ly$ and $y_0 = 0$; and then define

$$u_m = \lambda L^* c_{(m)},$$

$$c_{(m+1)} = c_{(m)} - Lu_m,$$

and

$$y_{m+1} = y_m + u_m.$$

The algorithm gives

$$\lim_{m \to \infty} y_m = y,$$

and so we can reconstruct $y$ from $Ly$.

Since our data is irregularly spaced, Equation (7.1) is really an irregular sampling formula which we can think of in the following "one-dimensional" way. First, we can consider $\theta = D_{s_{m+1}}(\partial_t \tilde{g})$ as a sampling function and suppose that $\{e_{-t(n;s_m)}\}$ is a Fourier frame with frame operator $S$. Then we can apply Theorem 17 to obtain, as in (5.3),

$$\forall y \in PW_{\Omega}, \quad y = \sum c_{m,n} \tau_{t(n;s_m)}(D_{s_{m+1}} \partial_t \tilde{g}),$$

where $c_{m,n} = \langle S^{-1}(\hat{y}), e_{-t(n;s_m)} \rangle_{L^2[-\Omega_1, \Omega_1]}$ and $0 < \Omega < \Omega_1$. The coefficients can be computed by means of (5.4) and several algorithms have been developed for this purpose, e.g., [26],[43]. The coefficients $c_{m,n}$ are sampled values in the sense of the approximation (5.7).

It is more analytically precise to think of Equation (7.1) as a "two-dimensional" wavelet frame expansion using the calculations in Example 11. Be "two-dimensional", we mean that we are dealing with $\{Z_m : m \in \mathbb{Z}\}$ defined in (3.1) as a sampling set in two-dimensional $t - s$ space. In this case, the wavelet frame expansion, which is a consequence of Example 11, is a two-dimensional irregular sampling formula whose coefficients are sampled values of $y$ in the same way that (5.6) and (5.7) are related. In any case, the actual implementation is based on Proposition 7, i.e., Equation (7.1), and Algorithm 8.

## 7.2 AN APPLICATION TO SPEECH CODING

In this subsection, we shall show how to use WAM processing in speech coding. Let $y$ be an acoustic signal on a time interval $I$ of duration $|I|$, and let $Ly = \{\langle y, \psi_{m,n}\rangle\}$ be the corresponding WAM data defined in Sections 3.4 and 3.5. For this discussion we shall also refer to the WAM data as the set of WAM coefficients. A basic problem in speech coding is to designate a bit rate $b_r$ and a bit allocation $b_c$ for transmitting sets of coefficients corresponding to a speech signal $y$, and to reconstruct $y$ at this given bit rate and bit allocation. There are different goals for different versions of the problem, but one criterion of success is to obtain good reconstruciton using low bit rates and, even more, to obtain good reconstruciton of a given signal $y$ embedded in certain types of noises. Naturally, since we are dealing with WAM, the sets of coefficients to be transmitted will be WAM coefficients.

Because of the auditory modelling of Section 3, the reconstruction theory developed in Sections 4 and 5, and the properties of the mammalian auditory filter bank given in Section 6, it turns out that WAM coefficients solve the aforementioned coding problem at a mathematical level, cf., [9] for its practical implementation. In particular, signal reconstruction is theoretically perfect because of the frame decompositon theory and irregular sampling formulas we have given. Further, the structure of the mammalian auditory filter bank, coupled with the thresholding technique defined below, allow for signal reconstruction of signals embedded in certain levels of white noise, since the WAM coefficients of white noise tend to spread out over the $t - s$ plane and to have low amplitude.

In order to justify these claims, suppose we are given a bit rate of $b_r$ bits per second, and that we are also specified an allocation of $b_c$ bits per WAM coefficient $\langle y, \psi_{m,n}\rangle$. The signal $y$ is defined on $I$, but is only known to the receiver to the extent that it will receive the coefficients $\{\langle y, \psi_{m,n}\rangle\}$ at the bit rate $b_r$ with bit allocation $b_c$. Since $b_r$ and $b_c$ are fixed, we can define a fixed transmittal coefficient rate $c_r = b_r/b_c$, i.e., given $b_r$ and $b_c$, we send $c_r$ WAM coefficients per second to the reciever. Consequently, with the coefficient rate fixed and specified, the maximum number of coefficients $n_c$ that we are able to transmit for the function $y$ of duration $|I|$ is

$$n_c = c_r\,|I|\,,$$

i.e., $n_c$ is the maximum number of coefficients with which the signal $y$ may be represented. With respect to WAM data, $n_c$ can be related to a threshold value $\delta$ in the following way. We define the distribution function

$$\lambda(\delta) = card\{\langle y, \psi_{m,n}\rangle \geq \delta\}$$

for $\delta \in [0, M]$, where $M = sup\{|\langle y, \psi_{m,n}\rangle|\}$. Note that we have neglected negative coefficients. The distribution function $\lambda$ is monotonically decreasing and continuous from the left. As such, we may define an inverse $\lambda^{-1}$ as

$$\forall n \in \mathbb{N}, \quad \lambda^{-1}(n) = inf\{v \in [0, M] : \lambda(v) < n\}.$$

Hence, if we choose a threshold value $\delta$ as

$$\delta = \lambda^{-1}(n_c),$$

then we are within our bit rate and bit allocation constraint for encoding the signal $y$.

We now reconstuct $y$ by the WAM implementation method of Section 7.1 using the set

$$\{\langle y, \psi_{m,n}\rangle \geq \delta\}$$

of thresholded WAM coefficients as the initial sequence $c_{(0)}$ (in Section 7.1). Many examples demonstrating the effectiveness to reconstruct $y$ appear in [8].

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