THE BANACH-SAKS THEOREM
IN NUCLEAR SPACES

by

John J. Benedetto

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Let M be a T_2 locally convex topological vector space and let the directed system $y_\alpha \to 0$ weakly in M. It follows from the Hahn-Banach theorem that the weak and strong closures of the convex hull of $\{y_\alpha\}$ are the same. Thus there is $\{x_\beta\}$ contained in the convex hull of $\{y_\alpha\}$ such that $x_\beta \to 0$ in M. In general we have no knowledge about the x_β except that they are linear combinations of the y_α with non-negative coefficients whose sum is 1. In particular, we have no 'Toeplitz' condition guaranteeing the convergence to 0 of a particular 'column' of these coefficients as β 'increases'.

Banach and Saks [1] showed that for $M = L^p$, $1 < p < \infty$, we have: $y_m \to 0$ weakly implies there is $\{x_k\} \subseteq \{y_n\}$ such that

$$\frac{x_1 + \dots + x_m}{m} \xrightarrow{m} 0 \text{ in } L^p.$$

Similar results have been proved for some other normed spaces, although the result is not true in L^1. We mention the work of S. Mazur [2] and the references given there.

We shall discuss such matters for nuclear spaces. Recall that M is <u>nuclear</u> if, when $\{p_t\}$ is a system of semi-norms describing the topology of M, we have the following situation [4]: for each k there is t, $\{T_n\} \subseteq M'$, $\{c_n\} \in \ell^1$, $\{\mu_n\} \subseteq M$ so that for all $x \in M$

$$| T_n(x) | < p_t(x), \qquad p_k(\mu_n) \leq 1$$

and

$$x = \sum_{n=1}^{\infty} c_n T_n(x) \mu_n$$

in the topology of p_k. Also, a T_2 locally convex space M with semi-norms $\{p_t\}$ is general nuclear if the above conditions for nuclearity hold, except possibly $p_k(\mu_n) \leq 1$, and if the topology on M is weaker than that given by the inner products

$$H_k(x,y) \equiv \sum_{n=1}^{\infty} c_n T_n(x) \overline{T_n(y)} .$$

Note, of course, that H_k defines an inner product since we can take $c_n \geq 0$ if we replace T_n by $T_n e^{i \arg c_n}$; and that $\{H_k\}$ and $\{p_k\}$ define equivalent topologies on general nuclear spaces since

$$H_k(x,x) \leq \sum_n c_n p_t(x)^2 .$$

We prove

Theorem Assume M is a metrizable general nuclear space and let $y_m \to 0$ weakly. Then there is $\{x_k\} \subseteq \{y_n\}$ such that

$$\frac{x_1 + \dots + x_m}{m} \longrightarrow 0 \text{ in } M.$$

It is, of course, true that if M is semi-Montel, weak convergence of $\{x_n\}$ implies convergence in the topology of M. Further, bounded sets are totally bounded in nuclear spaces, so that if a nuclear space is quasi-complete it is semi-Montel. Thus, a completeness criterion on a nuclear space yields the semi-reflexivity of the space, and this property was also vital in the Banach-Saks proof for L^p, $p > 1$. In fact, weak sequential convergence in any nuclear space implies convergence since the completion of any nuclear space is nuclear. Even more, weak convergence implies convergence in any Schwartz space. Considering the above theorem and the weak convergence properties of nuclear spaces it is natural to ask the relation between nuclear and general nuclear; in fact, Pietsch [3] has recently proved the two concepts equivalent. We present the following proof since it does not use the Pietsch result and because it shows that the "$p_k(\mu_n)$ boundedness" condition is not necessary in nuclear space arguments - as indeed is the case by Pietsch's theorem.

<u>Proof</u> (of theorem). Let $\{p_k\}$ be an increasing sequence of semi-norms determining the topology of M.

We shall use a Moore-Smith argument to show that $y_m \to 0$ weakly in M implies $y_m \to 0$ weakly for each inner product.

For given k we choose $\{T_n\}$, p_t, $\{c_n\}$ by general nuclearity and consider the sum

$$s(r,m) = \sum_{n=1}^{r} c_n T_n(y_m) \overline{T_n(y)}$$

where y is any element of M; for this y and $\varepsilon > 0$ we shall find $M > 0$ so that $m > M$ implies

$$H_k(y_m, y) < \varepsilon . \tag{1}$$

Since $y_m \to 0$ weakly we have for each r that

$$\lim_{m \to \infty} s(r,m) = 0 .$$

Also, for each m, $\lim_{r \to \infty} s(r,m)$ exists and equals $H_k(y_m, y)$.

Note that

$$(2) \qquad \left| \sum_{n=r}^{\infty} c_n T_n(y_m) \overline{T_n(y)} \right| \leq p_t(y) p_t(y_m) \sum_{n=r}^{\infty} |c_n| .$$

$p_t(y_m)$ is bounded independent of m since the weak convergence of the sequence $\{y_m\}$ yields its weak boundedness; hence, by Mackey's theorem, $\{y_m\}$ is bounded in M.

Thus, the right hand side of (2) converges to 0 independent of m; that is, $\lim_{r \to \infty} s(r,m)$ exists uniformly for all m.

Therefore, (1) follows by the Moore-Smith theorem; and we have that weak convergence in M yields weak convergence for each inner product H_k.

Further, since subsequences of weakly convergent sequences also converge weakly, we have that these subsequences converge weakly for each H_k.

Let $< M, H_k >$ be the inner product space M with inner product H_k; also, we write $y_m > y_n$ if $m \geq n$, and $a \cdot y_n \longrightarrow 0$ if the arithmetic means of y_m converge to 0.

We now give the procedure for choosing $\{x_m\}$.

By a variation of the Banach-Saks proof, which we describe in detail below, there is $\{y_{1,m}\} \subseteq \{y_m\}$ such that $a \cdot y_{1,m} \xrightarrow{m} 0$ in $< M, H_1 >$. This follows since $y_m \longrightarrow 0$ weakly in $< M, H_1 >$. Also, there is no loss of generality in letting $y_{1,1} = y_1$.

$y_{1,m} \xrightarrow{m} 0$ weakly in $< M, H_2 >$ since $\{y_{1,m}\} \subseteq \{y_m\}$. We choose a subsequence $\{y_{2,m}\} \subseteq \{y_{1,m}\}$ such that

$$(3) \qquad a \cdot y_{2,m} \xrightarrow{m} 0 \quad \text{in} \quad < M, H_2 >$$

and

(4) for each m, $y_{2,m} > y_{1,m}$.

We proceed as follows to find such a sequence $\left\{y_{2,m}\right\}$. Let $y_{2,1} = y_{1,1}$ so that $y_{2,1} > y_{1,1}$. Let $y_0 \in \left\{y_{1,m}\right\}$ have the property that for all $y_n \in \left\{y_{1,m}\right\}$ such that $y_n > y_0$ we have $|H_2(y_{2,1}, y_n)| \leq 1$; then let $y_{2,2}$ be one of these y_n with the further property that $y_{2,2} > y_{1,2}$. We use the weak convergence of $\left\{y_{1,m}\right\}$ in $< M, H_2 >$ to do this.

Similarly, to choose $y_{2,m+1}$ we let $y_0 \in \left\{y_{1,m}\right\}$ have the property that for all $y_n \in \left\{y_{1,m}\right\}$ with $y_n > y_0$ we have

(5) $$|H_2(y_{2,1}, y_n)| \leq 1/m, \ \ldots, \ |H_2(y_{2,m}, y_n)| \leq 1/m .$$

Again, we do this by the weak convergence of $\left\{y_{1,m}\right\}$ to 0 in $< M, H_2 >$. We then let $y_{2,m+1}$ be one of these y_n with the further property that $y_{2,m+1} > y_{1,m+1}$.

In this manner we clearly find $\left\{y_{2,m}\right\}$ so that (4) is satisfied. We can prove (3) using (5) by an inequality which we indicate below for a number of other cases and because of the weak boundaries of $\left\{y_m\right\}$.

Generally, then, we have $y_{k,m} \xrightarrow{m} 0$ weakly in $< M, H_{k+1} >$ and we can choose $\left\{y_{k+1,m}\right\} \subseteq \left\{y_{k,m}\right\}$ so that $y_{k+1,1} = y_{k,1}$, and, for each m,

(6) $|H_{k+1}(y_{k+1,1}, y_{k+1,m+1})| \leq 1/m, \ldots, |H_{k+1}(y_{k+1,m}, y_{k+1,m+1})| \leq 1/m$

and

$$y_{k+1,m} > y_{k,m}.$$

As before we use (6) to prove $a \cdot y_{k+1,m} \xrightarrow{m} 0$ in $< M, H_{k+1} >$.

Let $x_m = y_{m,m}$. We shall show $a \cdot x_m \xrightarrow{m} 0$ in each $< M, H_k >$ and this will prove the result. We do this by first showing that for each k

(7) $$|H_k(x_n, x_{m+1})| \leq 1/m, \qquad k \leq n \leq m.$$

We have $|H_n(y_{n,n}, y_{n,m+1})| \leq 1/m$ so that since $y_{m+1,m+1} > y_{n,m+1}$,

$$|H_n(x_n, x_{m+1})| \leq 1/m ;$$

further, $n \geq k$ implies $H_n \geq H_k$ since the sequence $\{p_k\}$ was chosen (without loss of generality) to be increasing, and therefore (7) follows.

For $k = 1$ we have the semi-norm $[H_1(x,x)]^{\frac{1}{2}}$ and

$$\left| H_1\left(\frac{x_1 + \ldots + x_m}{m}, \frac{x_1 + \ldots + x_m}{m}\right)\right| \leq$$

(8)

$$\frac{1}{m^2}\left[2 \cdot 1 + 4 \cdot \frac{1}{2} + \ldots + 2(m-1)\frac{1}{m-1} + \sum_{n=1}^{m} H_1(x_n, x_n)\right] \leq \frac{M_1 + 2}{m}$$

where $|H_1(x_n, x_n)| \leq M_1$ since $\{y_m\}$ is weakly bounded and therefore bounded in M.

Because of (8) we have $a \cdot x_m \xrightarrow{m} 0$ in $< M, H_1 >$.

For $k \geq 2$ we proceed similarly to the $k = 1$ case although there is an additional technicality with which we must deal.

For such a k we have

$$\left| H_k\left(\frac{x_1 + \dots + x_m}{m}, \frac{x_1 + \dots + x_m}{m}\right) \right| \leq$$

(9)

$$\frac{1}{m^2}\left[2 \cdot 1 + 4 \cdot \frac{1}{2} + \dots + 2(m-1)\frac{1}{m-1} + \sum_{n=1}^{m} H_k(x_n, y_n) + 2p(m-1)M_k\right].$$

Here p is the number of integers less than k and M_k is a bound in $< M, H_k >$ for $\{y_m\}$. Thus (9) becomes less than or equal to

$$\frac{1}{m^2}\left[2(m-1) + mM_k + 2p(m-1)M_k\right] \leq \frac{2 + M_k(1 + 2p)}{m};$$

and the right hand side converges to 0 as $m \to \infty$ – that is, $a \cdot x_m \xrightarrow{m} 0$ in $< M, H_k >$ and we are done.

qed

1. S. Banach and S. Saks "Sur la convergence forte dans les champs L^p", Studia Mathematica 2(1930)51-57.

2. S. Mazur "Über konvexe Mengen in linearen normierten Räumen", Studia Mathematica 4(1933)70-84.

3. A. Pietsch "Absolut p-summierende Abbildungen in normierten Räumen", Studia Mathematica 28(1967)333-353.

4. M. De Wilde "Espaces de fonctions à valeurs dans un espace linéaire à semi-normes", Extrait des Mémoires de la Société Royale des Sciences de Liège 13 (Fasc. 2) (1966)1-198.