

AN ENDPOINT $(1, \infty)$ BALIAN-LOW THEOREM

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ABSTRACT. It is shown that a $(1, \infty)$ version of the Balian-Low Theorem holds. If $g \in L^2(\mathbb{R})$, $\Delta_1(g) < \infty$ and $\Delta_\infty(\widehat{g}) < \infty$, then the Gabor system $\mathcal{G}(g, 1, 1)$ is not a Riesz basis for $L^2(\mathbb{R})$. Here, $\Delta_1(g) = \int |t| |g(t)|^2 dt$ and $\Delta_\infty(\widehat{g}) = \sup_{N>0} \int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma$.

1. INTRODUCTION

Given a square integrable function $g \in L^2(\mathbb{R})$, and constants $a, b > 0$, the associated *Gabor system*, $\mathcal{G}(g, a, b) = \{g_{m,n}\}_{m,n \in \mathbb{Z}}$, is defined by

$$g_{m,n}(t) = e^{2\pi i a m t} g(t - bn).$$

Gabor systems provide effective signal decompositions in a variety of settings ranging from eigenvalue problems to applications in communications engineering. Background on the theory and applications of Gabor systems can be found in [16], [12], [13], [3].

We shall use the Fourier transform defined by $\widehat{g}(\gamma) = \int g(t) e^{-2\pi i \gamma t} dt$, where the integral is over \mathbb{R} . We let $\mathbf{1}_S(t)$ denote the characteristic function of a set $S \subseteq \mathbb{R}$, and let S^c denote the complement of $S \subseteq \mathbb{R}$. Depending on the context, $|\cdot|$ will denote either the Lebesgue measure of a set, or the modulus of a function or complex number.

The Balian-Low Theorem is a classical manifestation of the uncertainty principle for Gabor systems.

Theorem 1.1 (Balian-Low). *Let $g \in L^2(\mathbb{R})$. If*

$$\int |t|^2 |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^2 |\widehat{g}(\gamma)|^2 d\gamma < \infty,$$

then $\mathcal{G}(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The Balian-Low Theorem has a long history and some of the original references include [1], [19], [2]. The theorem still holds if “orthonormal basis” is replaced by “Riesz basis”. For this and other generalizations of the Balian-Low Theorem, we refer the reader to the survey articles [6], [9], as well as [4], [5], [7], [8], [10], [14], [17]. The issue of sharpness in the Balian-Low Theorem was investigated in [5], where the following was shown.

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Theorem 1.2. *If $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p, q < \infty$, and $d > 2$, then there exists a function $g \in L^2(\mathbb{R})$ such that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and*

$$\int \frac{1 + |t|^p}{\log^d(2 + |t|)} |f(t)|^2 dt < \infty \quad \text{and} \quad \int \frac{1 + |\gamma|^q}{\log^d(2 + |\gamma|)} |\widehat{f}(\gamma)|^2 d\gamma < \infty.$$

When $(p, q) = (2, 2)$, this says that the Balian-Low Theorem no longer holds if the weights (t^2, γ^2) are weakened by appropriate logarithmic terms. In view of Theorem 1.2, it is also natural to ask if there exist versions of the Balian-Low Theorem for the general (p, q) case corresponding to the weights (t^p, γ^q) . The best that is known is the following.

Theorem 1.3. *Suppose $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p < \infty$ and let $\epsilon > 0$. If*

$$\int |t|^{(p+\epsilon)} |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^{(q+\epsilon)} |\widehat{g}(\gamma)|^2 d\gamma < \infty$$

then $\mathcal{G}(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The above theorem follows by combining Theorem 4.4 of [11] and Theorem 1 in [15]. The $\epsilon > 0$ can, of course, be removed in the case $(p, q) = (2, 2)$, by the Balian-Low Theorem.

This note shows the existence of a Balian-Low Theorem in the case $(p, q) = (1, \infty)$, and thus extends Theorems 1.1 and 1.3. To define what this means, let $g \in L^2(\mathbb{R})$ and $1 \leq p < \infty$ and set

$$\Delta_p(g) = \int |t|^p |g(t)|^2 dt \quad \text{and} \quad \Delta_\infty(g) = \sup_{N > 0} \int |t|^N |g(t)|^2 dt.$$

With this notation, the classical Balian-Low Theorem says that if $\Delta_2(g) < \infty$ and $\Delta_2(\widehat{g}) < \infty$ then $\mathcal{G}(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

Our main result of this note is the following theorem.

Theorem 1.4. *Let $g \in L^2(\mathbb{R})$ and suppose that $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$. Then*

$$\Delta_1(g) = \infty \quad \text{or} \quad \Delta_\infty(\widehat{g}) = \infty.$$

This yields the following $(1, \infty)$ version of the classical Balian-Low Theorem.

Corollary 1.5. *Let $g \in L^2(\mathbb{R})$ and suppose*

$$\Delta_1(g) < \infty \quad \text{and} \quad \Delta_\infty(\widehat{g}) < \infty.$$

Then $\mathcal{G}(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

2. BACKGROUND

A collection $\{e_n\}_{n \in \mathbb{Z}} \subseteq L^2(\mathbb{R})$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants $0 < A \leq B < \infty$ such that

$$\forall f \in L^2(\mathbb{R}), \quad A \|f\|_{L^2(\mathbb{R})}^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, e_n \rangle|^2 \leq B \|f\|_{L^2(\mathbb{R})}^2.$$

A and B are the *frame constants* associated to the frame. If $\{e_n\}_{n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, but is no longer a frame if any element is removed, then we say that $\{e_n\}_{n \in \mathbb{Z}}$ is a *Riesz basis* for $L^2(\mathbb{R})$. Riesz bases are also known as *exact frames* or *bounded unconditional bases*, e.g., see [3]. The Zak transform is an important tool for studying Riesz bases given by Gabor systems.

Given $g \in L^2(\mathbb{R})$, the *Zak transform* is formally defined by

$$\forall(t, \gamma) \in Q \equiv [0, 1]^2, \quad Zg(t, \gamma) = \sum_{n \in \mathbb{Z}} g(t - n)e^{2\pi i n \gamma}.$$

This defines a unitary operator from $L^2(\mathbb{R})$ to $L^2(Q)$. Further background on the Zak transform, as well as the next theorem, can be found in [3], [16].

Theorem 2.1. *Let $g \in L^2(\mathbb{R})$. $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ with frame constants $0 < A \leq B < \infty$ if and only if $A \leq |Zg(t, \gamma)|^2 \leq B$ for a.e. $(t, \gamma) \in Q$.*

A function $g \in L^2(\mathbb{R})$ is said to be in the *homogeneous Sobolev space* of order $s > 0$, denoted $\dot{H}^s(\mathbb{R})$, if $\|g\|_{\dot{H}^s(\mathbb{R})}^2 \equiv \int |\gamma|^{2s} |\widehat{g}(\gamma)|^2 d\gamma < \infty$. Since the condition $\Delta_1(g) < \infty$ in Theorem 1.5 is equivalent to $\widehat{g} \in \dot{H}^{1/2}(\mathbb{R})$, we shall need some results on $\dot{H}^{1/2}(\mathbb{R})$. The following alternate characterization of $\dot{H}^{1/2}(\mathbb{R})$ will be useful, e.g., [18].

Theorem 2.2. *If $f \in \dot{H}^{1/2}(\mathbb{R})$ then*

$$\|f\|_{\dot{H}^{1/2}(\mathbb{R})}^2 = \frac{1}{4\pi^2} \int \int \frac{|f(x) - f(y)|^2}{|x - y|^2} dx dy.$$

Given $f \in L^2(\mathbb{R})$, the *symmetric-decreasing rearrangement* f^* of f is defined by

$$f^*(t) = \int_0^\infty \mathbf{1}_{S_x}(t) dx,$$

where $S_x = (-s_x/2, s_x/2)$ and $s_x = |\{t : |f(t)| > x\}|$. An important property of a symmetric-decreasing rearrangement is that it decreases the $\dot{H}^{1/2}(\mathbb{R})$ norm of functions, [18].

Theorem 2.3. *If $f \in \dot{H}^{1/2}(\mathbb{R})$ then*

$$\|f\|_{\dot{H}^{1/2}(\mathbb{R})} \geq \|f^*\|_{\dot{H}^{1/2}(\mathbb{R})}.$$

This has the following useful corollary, [18].

Corollary 2.4. *If $S \subset \mathbb{R}$ is a measurable set of positive and finite measure then $\|\mathbf{1}_S\|_{\dot{H}^{1/2}(\mathbb{R})} = \infty$.*

3. PROOF OF THE $(1, \infty)$ BALIAN-LOW THEOREM

The proof of Theorem 1.4 requires the following preliminary technical theorem.

Theorem 3.1. *Let f be a non-negative measurable function supported in the interval $[-1, 1]$ and suppose that there exist constants $0 < A \leq B < \infty$ such that*

$$(3.1) \quad A \leq |f(x) \pm f(x-1)| \leq B, \quad \text{a.e. } x \in [-1, 1].$$

Then $\|f\|_{\dot{H}^{1/2}(\mathbb{R})} = \infty$.

Proof. We begin by defining the measurable sets

$$\begin{aligned} S &= \{x \in [0, 1] : f(x-1) \leq f(x)\}, \\ T &= S^c \cap [0, 1] = \{y \in [0, 1] : f(y) < f(y-1)\}, \end{aligned}$$

and note that (3.1) implies

$$(3.2) \quad A \leq f(x) - f(x-1), \quad \text{a.e. } x \in S,$$

$$(3.3) \quad A \leq f(y-1) - f(y), \quad \text{a.e. } y \in T.$$

We break up the proof into two cases depending on whether or not S is a proper non-trivial subset of $[0, 1]$.

Case I. We shall first consider the case where

$$(3.4) \quad 0 < |S| < 1,$$

and hence that $0 < |T| < 1$.

Define the following capacity type integral over the product set $S \times T$.

$$(3.5) \quad I = \int_S \int_T \frac{1}{|x - y|^2} dy dx.$$

Conditions (3.2) and (3.3) allow one to bound I in terms of the $\dot{H}^{\frac{1}{2}}(\mathbb{R})$ norm of f as follows.

$$\begin{aligned} I &\leq \frac{1}{4A^2} \int_S \int_T \frac{|f(x) - f(x-1) + f(y-1) - f(y)|^2}{|x - y|^2} dy dx \\ &\leq \frac{1}{2A^2} \left(\int_S \int_T \frac{|f(x) - f(y)|^2}{|x - y|^2} dy dx + \int_S \int_T \frac{|f(y-1) - f(x-1)|^2}{|x - y|^2} dy dx \right) \\ &\leq \frac{1}{A^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x - y|^2} dy dx \\ &= \frac{4\pi^2}{A^2} \|f\|_{\dot{H}^{1/2}}^2. \end{aligned}$$

It therefore suffices to show that $I = \infty$.

Since by the Lebesgue differentiation theorem almost every point of T is a point of density, it follows from (3.4) that we may choose $a \in (0, 1)$ such that a is point of density of T which satisfies either

$$(3.6) \quad 0 < |S \cap [0, a]| < a$$

or

$$(3.7) \quad 0 < |S \cap [a, 1]| < 1 - a.$$

Without loss of generality, we assume (3.6). If (3.7) holds then our arguments proceed analogously; for example in the first subcase below we would symmetrize about $x = 1$ instead of $x = 0$.

To estimate I , we shall proceed separately depending on whether $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx$ is finite or infinite.

Subcase i. Suppose $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx < \infty$. It will be convenient to work with the following set

$$\tilde{S} = (S \cup (-S)) \cap [-a, a].$$

By (3.6) we have $|\tilde{S}| = 2|S \cap [0, a]| \neq 0$. It follows from Corollary 2.4 and the definition of \tilde{S} that

$$\begin{aligned}\tilde{I} &\equiv \int_{-a}^a \int_{-\infty}^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\mathbf{1}_{\tilde{S}}(x) - \mathbf{1}_{\tilde{S}}(y)|^2}{|x-y|^2} dx dy \\ &= 4\pi^2 \|\mathbf{1}_{\tilde{S}}\|_{\dot{H}^{1/2}(\mathbb{R})}^2 = \infty.\end{aligned}$$

The symmetric definition of \tilde{S} implies that

$$(3.8) \quad \tilde{I} = 2 \int_0^a \int_{-\infty}^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx = 2(I_1 + I_2 + I_3),$$

where

$$\begin{aligned}I_1 &\equiv \int_0^a \int_{-\infty}^{-a} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx \leq \int_0^a \int_{-\infty}^{-a} \frac{1}{|y|^2} dy dx < \infty, \\ I_2 &\equiv \int_0^a \int_a^{\infty} \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx \leq \int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx < \infty, \\ I_3 &\equiv \int_0^a \int_{-a}^a \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx.\end{aligned}$$

A simple calculation for I_3 shows that

$$(3.9) \quad I_3 = \int_0^a \int_0^a \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x-y|^2} dy dx + \int_0^a \int_0^a \frac{\mathbf{1}_{\tilde{S}}(x)\mathbf{1}_{(\tilde{S})^c}(y)}{|x+y|^2} dy dx \leq 2I,$$

where the inequality for the second term in the middle of (3.9) follows from the fact that $|x-y| \leq |x+y|$ in the square $[0, a] \times [0, a]$.

It follows from (3.8) and (3.9) that

$$\infty = \tilde{I} \leq 2I_1 + 2I_2 + 4I.$$

Since I_1 and I_2 are finite, we have $I = \infty$, as desired.

Subcase ii. Suppose $\int_0^a \frac{1_S(x)}{|x-a|} dx = \infty$. Define

$$I_D = \int_0^a \int_0^a \frac{\mathbf{1}_S(x)\mathbf{1}_T(y)}{|x-y|^2} \mathbf{1}_D(x, y) dy dx \leq I,$$

where $D = \{(x, y) \in \mathbb{R}^2 : x < y\}$. To compute a lower bound for I_D first note that since a is a point of density of T , there exists a sufficiently large constant $0 < C < \infty$ such that

$$|a-x| \leq C |T \cap [x, a]|, \quad a.e. x \in [0, 1].$$

Therefore for a.e. $x \in [0, a]$

$$\begin{aligned}\frac{1}{|a-x|} &\leq \frac{C|T \cap [x, a]|}{|a-x|^2} = C |T \cap [x, a]| \cdot \min_{y \in [x, a]} \left\{ \frac{1}{|x-y|^2} \right\} \\ &\leq C \int_x^a \frac{\mathbf{1}_T(y)}{|x-y|^2} dy.\end{aligned}$$

This implies that

$$\infty = \int_0^a \frac{\mathbf{1}_S(x)}{|a-x|} dx \leq C \int_0^a \int_x^a \frac{\mathbf{1}_S(x)\mathbf{1}_T(y)}{|x-y|^2} dy dx = CI_D,$$

and it follows that $I_D = \infty$, and hence $I = \infty$, as desired.

Case II. We conclude by addressing the cases where $|S| = 0$ or $|S| = 1$. Without loss of generality we only consider $|S| = 1$, and hence assume that $S = [0, 1]$ up to a set of measure zero. It follows from (3.2) and the positivity of f that

$$A \leq f(x), \quad a.e. \ x \in [0, 1].$$

This, together with the fact that f is supported in $[-1, 1]$, implies that

$$\begin{aligned} \infty &= \int_1^\infty \int_0^1 \frac{1}{|x-y|^2} dxdy \\ &\leq \frac{1}{A^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x) - f(y)|^2}{|x-y|^2} dxdy \\ &= \frac{4\pi^2}{A^2} \|f\|_{\dot{H}^{1/2}(\mathbb{R})}^2, \end{aligned}$$

as desired. This completes the proof. \square

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We proceed by contradiction. Assume that $g \in L^2(\mathbb{R})$, that $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ with frame constants $0 < A \leq B < \infty$, and that $\Delta_1(g) < \infty$ and $\Delta_\infty(\widehat{g}) < C < \infty$, for some constant C .

By Theorem 2.1,

$$\sqrt{A} \leq |Zg(x, w)| \leq \sqrt{B} \quad a.e. \text{ on } [0, 1]^2.$$

Since $Z\widehat{g}(x, w) = e^{2\pi i x w} Zg(-w, x)$ we have

$$\sqrt{A} \leq |Z\widehat{g}(x, w)| \leq \sqrt{B} \quad a.e. \text{ on } [0, 1]^2.$$

Next, the assumption $\int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma < C$ for all $N > 0$ implies that

$$\text{supp } \widehat{g} \subseteq [-1, 1].$$

Thus, for $(x, w) \in [0, 1]^2$, we have

$$Z\widehat{g}(x, w) = \sum_{n \in \mathbb{Z}} \widehat{g}(x-n) e^{2\pi i n w} = \widehat{g}(x) + \widehat{g}(x-1) e^{2\pi i w},$$

so that we have

$$(3.10) \quad \sqrt{A} \leq |\widehat{g}(x) + \widehat{g}(x-1) e^{2\pi i w}| \leq \sqrt{B} \quad \text{for a.e. } (x, w) \in [0, 1]^2.$$

In particular, it follows that

$$\sqrt{A} \leq | |\widehat{g}(x)| \pm |\widehat{g}(x-1)| | \leq \sqrt{B}, \quad \text{for a.e. } x \in [0, 1].$$

It now follows from Theorem 3.1 that $|\widehat{g}| \notin \dot{H}^{1/2}(\mathbb{R})$, which implies that $\widehat{g} \notin \dot{H}^{1/2}(\mathbb{R})$. In other words, $\Delta_1(g) = \|\widehat{g}\|_{\dot{H}^{1/2}(\mathbb{R})}^2 = \infty$. This contradiction completes the proof. \square

Since orthonormal bases are Riesz bases with frame constants $A = B = 1$, Corollary 1.5 follows from Theorem 1.4.

4. FURTHER COMMENTS

1. Theorem 1.5 is sharp in the sense investigated in Theorem 1.2, see [5]. In fact, Theorem 1.5 no longer holds if one weakens the Δ_1 decay hypotheses by a certain logarithmic amount. For example, if $d > 1$ and $\widehat{g}(\gamma) = \mathbf{1}_{[0,1]}(\gamma)$ then $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$, and

$$\int \frac{|t|}{\log^d(|t| + 2)} |g(t)|^2 dt < \infty \quad \text{and} \quad \sup_{N > 0} \int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma < \infty.$$

2. There are two noteworthy cases in which the proof of Theorem 1.4 can be significantly simplified. If one assumes that $\mathcal{G}(g, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ then the frame constants satisfy $A = B = 1$ and it follows from (3.10) that $|\widehat{g}(x)| = \mathbf{1}_R(x)$ for some set $R \subset \mathbb{R}$ of positive and finite measure. Corollary 2.4 completes the proof in this case. Likewise, if $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ whose frame bounds A and B are sufficiently close to one another, e.g., $\sqrt{B} < 3\sqrt{A}$, then a direct argument involving Theorem 2.2 and Theorem 2.3 completes the proof. The main difficulty in Theorem 1.4 and Theorem 3.1 arises when the frame constants A and B are far apart.

3. We conclude by noting that if one weakens the hypotheses in Theorem 1.4 to $\Delta_\infty(\widehat{g}) < \infty$ and $\Delta_{1+\epsilon}(g) < \infty$, for some $\epsilon > 0$, then the result is a simple consequence of the *Amalgam Balian-Low Theorem*. The Amalgam Balian-Low Theorem, e.g., [6], states that if $\mathcal{G}(g, 1, 1)$ is a Riesz basis for $L^2(\mathbb{R})$ then

$$g \notin W(C_0, l^1) \quad \text{and} \quad \widehat{g} \notin W(C_0, l^1),$$

where

$$W(C_0, l^1) = \{f : f \text{ is continuous and } \sum_{k \in \mathbb{Z}} \|f \mathbf{1}_{[k, k+1]}\|_{L^\infty(\mathbb{R})} < \infty\}.$$

The assumptions $\Delta_{1+\epsilon}(g) < \infty$ and $\Delta_\infty(\widehat{g}) < \infty$ imply that \widehat{g} is continuous and supported in $[-1, 1]$, which, in turn, implies that $\widehat{g} \in W(C_0, l^1)$.

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