THE PROJECTIVE INVARIANTS OF ORDERED POINTS ON THE LINE

BENJAMIN HOWARD, JOHN MILLSON, ANDREW SNOWDEN AND RAVI VAKIL*

ABSTRACT. The space of n (ordered) points on the projective line, modulo automorphisms of the line, is one of the most important and classical examples of an invariant theory quotient, and is one of the first examples given in any course. Generators for the ring of invariants have been known since the end of the nineteenth century, but the question of the relations has remained surprisingly open, and it was not even known that the relations have bounded degree. We show that the ideal of relations is generated in degree at most 4, and give an explicit description of the generators. The result holds for arbitrary weighting of the points. If all the weights are even (e.g. in the case of equal weight for odd n), we show that the ideal of relations is generated by quadrics. The proof is by degenerating the moduli space to a toric variety, and following an enlarged set of generators through this degeneration.

In a later work [HMSV], by different means, we will show that the projective scheme is cut out by certain natural quadrics unless $n = 6$ and each point has weight one.

CONTENTS

1. INTRODUCTION

The space of n (ordered) points on the projective line, modulo automorphisms of the line, is one of the most important and classical examples of an invariant theory quotient, and is one of the first examples given in any course (see [MS, §2], [MFK, §3], [Do, Ch. 11], [DO, Ch. I], ...).

The generators have been known for a long time: in 1894 Kempe, see [Ke], proved that for the case of n points, the lowest degree invariants generate the ring

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of invariants. This paper was said by Howe in [Howe], pg. 156, to be the "deepest result" of classical invariant theory (in the sense that it is the only result that Howe could not prove from standard constructions in representation theory). However, the question of the relations has remained surprisingly open. It was not even known that the relations have bounded degree as n varies.

In this context, it is most natural to consider more generally the space of *weighted* points on $\mathbb{CP}^1$. Let the ith point be weighted by r_i and let $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{Z}^+)^n$. Let $R_\mathbf{r}$ denote the ring of projective invariants. The weights can be interpreted as parametrizing the very ample line bundles of $(\mathbb{CP}^1)^n$. The space of weighted points (for real weights) was considered by Deligne and Mostow [DM]. In their theory the weights are the parameters for the hypergeometric function. We will see below another reason (coming from Euclidean geometry) why introducing weights is natural — the weights are the side-lengths of polygonal linkages in Euclidean three-space.

We first observe that Kempe's Theorem (that the ring of invariants is generated in "lowest degree") remains true for all integral weights (Theorem 3.7), incidentally giving a short, possibly new, proof of Kempe's Theorem. We dub these generators the *Kempe generators* of the ring. Our main result on the *relations* of the invariants is the following.

Main Theorem 1.1. *For any weights $\mathbf{r} = (r_1, \dots, r_n)$, the ideal of relations in the ring of invariants is generated by relations of degree at most four. If all the weights are even, then the ideal of relations is generated in degree two.*

The proof is completed in §8 (see Theorems 8.9 and 8.10). We give an explicit description of a generating set (suitable for example for computation) in §7, and examples at the end of §8.

The heart of the paper is §6 and §8. In these sections we enlarge the set of generators in order that the leading terms of this larger set (relative to the filtration by Lakshmibai-Gonciulea degree defined in §5) is a set of generators for the associated graded ring (the homogeneous coordinate ring of the toric fiber of our toric degeneration). The extra generators are quadratic expressions in the Kempe generators. Our main result proves that the ideal of relations among this larger set of generators is generated by quadratic relations. When these quadratic relations are rewritten in terms of the Kempe generators we obtain relations of degrees two, three and four. We do this by finding a normal form for monomials in the Kempe generators.

The polytope corresponding to the toric variety has three interpretations, which we discuss in §4. One standard interpretation is in terms of Gel'fand-Tsetlin patterns, and another is in terms of semistandard tableaux, but our proof uses a third interpretation in terms of possible lengths of diagonals emanating from an initial vertex of an n-gon with side lengths $\mathbf{r} = (r_1, \dots, r_n)$. As a consequence of this polytope interpretation we get effective bounds on the degrees of these moduli spaces, enabling us to show that the ring of projective invariants for n equally weighted points is not a complete intersection unless $n \leq 4$ or $n = 6$ (§4.5). This result seems not to be geometrically obvious.

In our next paper [HMSV], we prove that the moduli space is in fact cut out by combinatorially/geometrically obvious quadratic relations, with the single exception of $\mathbf{r} = (1, 1, 1, 1, 1, 1)$ (corresponding to the Segre cubic threefold). However, we could not prove that these quadratic relations generate the *ideal* of relations.

Thus the understanding of the ring of projective invariants of ordered points on the projective line is now quite satisfactory. This is in contrast with the equally classical (and much more complicated) question of *unordered* points. (This is equivalent to considering homogeneous degree n polynomials in two variables — the subject of "binary quantics," see [KR].) Mumford has written: "This is an amazingly difficult job, and complete success was achieved only for $n \leq 6$" [MFK, pg. 77]. Later, "by an extraordinary tour de force" (Mumford, loc. cit.) Shioda [Sh] dealt with the case $n = 8$. In the unordered case, a generating set for general n is not known (and not even known to have bounded degree as n varies).

The case of unordered points with even n essentially corresponds to the ring of hyperelliptic modular forms of genus $(n-2)/2$, and their relations. Our case of ordered points, where n is even and the the weights are even, essentially corresponds to the ring of hyperelliptic level two modular forms, and this paper completely describes generators of the ideal of relations among these forms.

Acknowledgments. Yi Hu has pointed out that our "side-splitting map" (see §3.5) is a special case of the splitting map introduced in §2.3 of [Hu]. In particular our Proposition 3.5 is a special case of his Proposition 2.11. We would like to thank Philip Foth for many helpful comments about toric degenerations. This project was inspired by reading [FH]. In [HMM] the authors will prove the conjecture of [FH] which describes the toric fiber of the toric degeneration of the moduli space of points on the line (as a topological space) as the collapsed space introduced in [KY]. The fourth author thanks Allen Knutson and Diane Maclagan for enlightening conversations. Also we thank Allen Knutson for pointing out that the first toric degenerations were given in Proposition 11.10 (page 104) of [St]. Finally we would like to thank Roger Howe for telling us about [Howe].

2. G.I.T. quotients

2.1. **Definition of G.I.T. quotient.** We refer the reader to [Do] for additional details. Suppose that G is a reductive algebraic group, V is a quasi-projective variety, and $\eta : G \times V \to V$ is regular action of G. Let $\pi : \mathcal{L} \to V$ be an ample line bundle over V. A G–linearization of $\mathcal{L}$ is a regular action $\widetilde{\eta} : G \times \mathcal{L} \to \mathcal{L}$ which is linear on fibers and makes the following diagram commute:

$$\begin{array}{ccc} G \times \mathcal{L} & \xrightarrow{\widetilde{\eta}} & \mathcal{L} \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times V & \xrightarrow{\eta} & V \end{array}$$

Given such a linearization, we automatically get linearizations on all tensor powers $\mathcal{L}^{\otimes N}$ of $\mathcal{L}$. Thus G has an action on sections s of $\mathcal{L}^{\otimes N}$ given by

$$(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x) = \widetilde{\eta}(g, s(\eta(g^{-1}, x))).$$

Let $\Gamma(V, \mathcal{L}^{\otimes N})^G$ denote the G–invariant sections of $\mathcal{L}^{\otimes N}$. The G.I.T. quotient $V/\!/_{\widetilde{\eta}}\, G$ is defined as

$$V/\!/_{\widetilde{\eta}}\, G = \mathrm{Proj}\Big(\bigoplus_{N=0}^{\infty} \Gamma(V, \mathcal{L}^{\otimes N})^G\Big).$$

If the linearization is understood, sometimes we denote $V/\!/_{\widetilde{\eta}}G$ by $V/\!/G$.

If s is a section let $\mathrm{supp}(s) = \{x \in V \mid s(x) \neq 0\}$. The set $V^{ss}_{\tilde{\eta}}$ of semistable points of V is defined as

$$V^{ss}_{\tilde{\eta}} = \bigcup_{N \geq 0} \bigcup_{s \in \Gamma(V, \mathcal{L}^{\otimes N})^G} \mathrm{supp}(s).$$

(To be more precise, we need the distinguished open subset $\mathrm{supp}(s)$ corresponding to s to be affine, but this will be true in the two cases relevant for us, when X is affine or projective.) If x is a semistable point let $\overline{G \cdot x}$ be the (Zariski) closure of the orbit $G \cdot x$ in $V^{ss}_{\tilde{\eta}}$. As a topological space $V/\!/_{\tilde{\eta}}\, G$ is the quotient space of $V^{ss}_{\tilde{\eta}}$ where points x, y are identified iff $\overline{G \cdot x}$ and $\overline{G \cdot y}$ intersect nontrivially.

2.2. **The Gel'fand-MacPherson correspondence.** The Gel'fand–MacPherson correspondence says that a G.I.T. quotient of Grassmannian space $\mathrm{Gr}_k(\mathbb{C}^n)$ by the torus $(\mathbb{C}^*)^n$ is isomorphic to a G.I.T. quotient of the product space $(\mathbb{CP}^{k-1})^n$ by the diagonal action of $\mathrm{PGL}(k, \mathbb{C})$.

Let $\mathcal{L}$ be the trivial line bundle $\mathbb{C}^{n \times k} \times \mathbb{C} \to \mathbb{C}^{n \times k}$. Given any group G acting on $\mathbb{C}^{n \times k}$, a character $\chi : G \to \mathbb{C}^*$ defines a linearization of $\mathcal{L}$ by $g \cdot (A, z) = (g \cdot A, \chi(g) z)$.

The group $\mathrm{GL}(k, \mathbb{C})$ acts on the right of $\mathbb{C}^{n \times k}$ by matrix multiplication. The group T of diagonal matrices in $\mathrm{GL}(n)$ acts on the left of $\mathbb{C}^{n \times k}$. Let $\det^a : \mathrm{GL}(k, \mathbb{C}) \to \mathbb{C}^*$ be $\det^a(g) = (\det(g))^a$ and let $\chi_{\mathbf{r}} : T \to \mathbb{C}^*$ be

$$\chi_{\mathbf{r}}(\mathrm{diag}(z_1, \ldots, z_n)) = \prod_{i=1}^n z_i^{r_i}$$

where $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{Z}^n_+$. The one–dimensional subgroup $K = \{(zI_n, z^{-1} I_k) : z \in \mathbb{C}^*\}$ of $T \times \mathrm{GL}(k, \mathbb{C})$ acts trivially on $\mathbb{C}^{n \times k}$. Let G be the quotient of $T \times \mathrm{GL}(k, \mathbb{C})$ by K. The character $\chi_{\mathbf{r}} \times \det^a$ descends to G iff $|\mathbf{r}| = \sum_i r_i = ka$, and we assume that is the case so that we have a G–linearization of the trivial line bundle.

Let $\mathcal{L}_{k,n}$ be the ample generator of the Picard group of $\mathrm{Gr}_k(\mathbb{C}^n)$; we may realize the total space of $\mathcal{L}_{k,n}$ by equivalence classes $V^{n \times k} \times \mathbb{C}/\sim$ where $V^{n \times k}$ is the open subset of $\mathbb{C}^{n \times k}$ of matrices with independent columns and $(A, z) \sim (Ag, \det(g) z)$ for $g \in \mathrm{GL}(k, \mathbb{C})$. Denote the equivalence class of (A, z) by $[A, z]$. The character $\chi_{\mathbf{r}}$ defines a T–linearization of $\mathcal{L}^a_{k,n} = \mathcal{L}^{\otimes a}_{k,n}$ by

$$t \cdot [A, z] = [tA, \chi_{\mathbf{r}}(t) z].$$

Let $\mathcal{H}$ be the ample generator of the Picard group of $\mathbb{CP}^{k-1}$, and let $\mathcal{H}^{\mathbf{r}}$ be the ample line bundle over the product $(\mathbb{CP}^{k-1})^n$ given by

$$\mathcal{H}^{\mathbf{r}} = \mathcal{H}^{\otimes r_1} \boxtimes \cdots \boxtimes \mathcal{H}^{\otimes r_n}.$$

We may identify the total space of $\mathcal{H}$ with $(\mathbb{C}^k \setminus \{0\}) \times \mathbb{C}/\sim$ where $(v, z) \sim (v\lambda, \lambda z)$ for $\lambda \in \mathrm{GL}(1)$; let $[v, z]$ denote the equivalence class. There is a unique linearization of $\mathcal{H}^{\mathbf{r}}$ for the (right) diagonal action of $\mathrm{PGL}(k, \mathbb{C})$ on $(\mathbb{CP}^{k-1})^n$.

By the First Fundamental Theorem of Invariant Theory [Do, Theorem 2.1], the homogeneous coordinate ring of the Grassmannian is generated by Plücker coordinates, hence, for any $N \geq 0$ we have

$$\Gamma(\mathrm{Gr}_k(\mathbb{C}^n), (\mathcal{L}^a_{k,n})^{\otimes N}) \cong \Gamma(\mathbb{C}^{n \times k}, \mathcal{L}^{\otimes akN})^{\mathrm{GL}(k, \mathbb{C})}$$

and consequently

$$\Gamma(\mathrm{Gr}_k(\mathbb{C}^n), (\mathcal{L}^a_{k,n})^{\otimes N})^T \cong \Gamma(\mathbb{C}^{n \times k}, \mathcal{L}^{\otimes akN})^G.$$

On the other hand the sections of the outer tensor product of hyperplane section bundles over n copies of $\mathbb{CP}^{k-1}$ are products of *homogeneous* polynomials in the homogeneous coordinates, that is,

$$\Gamma((\mathbb{CP}^{k-1})^n, (\mathcal{H}^{\mathbf{r}})^{\otimes N}) \cong \Gamma(\mathbb{C}^{n\times k}, \mathcal{L}^{\otimes|\mathbf{r}|N})^T$$

and consequently

$$\Gamma((\mathbb{CP}^{k-1})^n, (\mathcal{H}^{\mathbf{r}})^{\otimes N})^{\mathrm{PGL}(k,\mathbb{C})} \cong \Gamma(\mathbb{C}^{n\times k}, \mathcal{L}^{\otimes|\mathbf{r}|N})^G.$$

Hence we have an isomorphism of the G.I.T. quotients

$$\mathrm{Gr}_k(\mathbb{C}^n)//_{\chi_{\mathbf{r}}} T \cong \mathbb{C}^{n\times k}//_{\chi_{\mathbf{r}}\times \det^a} G \cong (\mathbb{CP}^{k-1})^n//_{\mathbf{r}}\mathrm{PGL}(k,\mathbb{C}).$$

In the Introduction we took the quotient of $(\mathbb{CP}^1)^n$ by $\mathrm{SL}(2,\mathbb{C})$ instead of $\mathrm{PGL}(2,\mathbb{C})$. We claim that there exists a $\mathrm{PGL}(2,\mathbb{C})$–linearization of $\mathcal{H}^{\mathbf{r}}$ if and only if the sum of the r_i's is even. Indeed, we have a $\mathrm{PGL}(2,\mathbb{C})$–linearization if and only if we have a $\mathrm{PSL}(2,\mathbb{C})$–linearization if and only if we have an $\mathrm{SL}(2,\mathbb{C})$ linearization in which -1 acts trivially. But -1 acts on $\mathcal{H}^{\mathbf{r}}$ by multiplication on each fiber by $(-1)^{\sum r_i}$ and the claim follows. On the other hand we have an $\mathrm{SL}(2,\mathbb{C})$–linearization and an $\mathrm{SL}(2,\mathbb{C})$ quotient for any $\mathbf{r}$. The two quotients coincide when they are both defined (i.e. the sum of the r_i's is even).

2.3. **Points on the line and Euclidean polygons.** Here we briefly explain why the G.I.T. quotient $(\mathbb{CP}^1)^n//_{\mathbf{r}}\mathrm{SL}(2,\mathbb{C})$ is homeomorphic (and symplectomorphic in the smooth case) to the space $M_{\mathbf{r}}$ of congruence classes of Euclidean n-gon linkages with side-lengths $\mathbf{r}$. We refer the reader to [KM] for details.

An n-gon $\mathbf{e}$ modulo translations in Euclidean three space $\mathbb{E}^3$ is given by an n–tuple of vectors $\mathbf{e} = (e_1, e_2, \dots, e_n)$ (the edges) such that the following *closing condition* holds

$$e_1 + e_2 + \cdots + e_n = 0.$$

It is proved in [KM] and [Kly] that the map $\mu(\mathbf{e}) = e_1 + e_2 + \cdots + e_n$ is the momentum map for the diagonal action of $\mathrm{SO}(3)$ on the product $S^2(r_1)\times\cdots\times S^2(r_n)$. Consequently the set of closed n-gon linkages is the zero level set of the momentum map and after dividing by $\mathrm{SO}(3)$ we obtain the symplectic quotient $S^2(r_1)\times\cdots\times S^2(r_n)//_{\mathbf{r}}\mathrm{SO}(3)$. It follows from a general theorem of Kirwan-Kempf-Ness that the symplectic quotient $S^2(r_1)\times\cdots\times S^2(r_n)//_{\mathbf{r}}\mathrm{SO}(3)$ is canonically homeomorphic to the G.I.T. quotient $(\mathbb{CP}^1)^n//_{\mathbf{r}}\mathrm{SL}(2,\mathbb{C})$. A direct proof using the conformal center of mass is given in [KM]. We obtain

Theorem 2.1. *There is a canonical homeomorphism of the space $M_{\mathbf{r}}$ of n-gon linkages in $\mathbb{E}^3$ with side-lengths $\mathbf{r}$ modulo orientation-preserving isometries and the moduli space $(\mathbb{CP}^1)^n//_{\mathbf{r}}\mathrm{SL}(2,\mathbb{C})$ of weighted ordered points on the line.*

3. The algebraic geometry of the moduli space of polygons

3.1. **The Plücker embedding of the Grassmannian.** In the previous section we gave a construction of $\mathrm{Gr}_k(\mathbb{C}^n)$ as a G.I.T. quotient of $\mathbb{C}^{n\times k}$. We now look at this more closely for $k = 2$.

Recall that $\mathcal{L}$ is the trivial line bundle over $\mathbb{C}^{n\times 2}$ and we have a $\mathrm{GL}(2,\mathbb{C})$–linearization of the bundle $\mathcal{L}^{\otimes N}$ via the character $\det^a$. Specifically, for $m \in \mathbb{C}^{n\times 2}$, $z\in\mathbb{C}$ and $g \in \mathrm{GL}(2,\mathbb{C})$, we have

$$g\cdot(m,z) = (mg^{-1}, \det(g)^{-aN} z).$$

Given a map $\tau : \mathbb{C}^{n\times 2} \to \mathbb{C}$ we get a section of $\mathcal{L}^{\otimes N}$ via $s_\tau(m) = (m, \tau(m))$, and all sections are of this form. For $g \in \mathrm{GL}(2, \mathbb{C})$ we have

$$(g \cdot s_\tau)(m) = g \cdot s_\tau(mg) = g \cdot (mg, \tau(mg)) = (m, \det(g)^{-aN}\tau(mg)).$$

Thus the section s_τ is invariant if and only if $\tau(mg) = \det(g)^{aN}\tau(m)$.

Let i and j be integers, $1 \le i, j \le n$. Define a map $\mathbb{C}^{n\times 2} \to \mathbb{C}$ by assigning to a matrix m the determinant of the 2×2 minor of m formed by taking rows i and j. We denote this map with the 2×1 tableau

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array}\,.$$

(Note that we have the identity

$$\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array} = -\begin{array}{|c|}\hline j \\ \hline i \\ \hline\end{array}\,,$$

so it suffices to consider tableaux with $i < j$.) The product of k of these 2×1 tableaux is denoted by a $2 \times k$ tableau:

$$\begin{array}{|c|}\hline i_1 \\ \hline j_1 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline i_2 \\ \hline j_2 \\ \hline\end{array} \cdots \begin{array}{|c|}\hline i_k \\ \hline j_k \\ \hline\end{array} = \begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_k \\ \hline j_1 & j_2 & \cdots & j_k \\ \hline\end{array}\,.$$

Clearly if $\tau : \mathbb{C}^{n\times 2} \to \mathbb{C}$ is a map given by a $2 \times k$ tableau then $\tau(mg) = \det(g)^k\tau(m)$; in fact any map $\tau : \mathbb{C}^{n\times 2} \to \mathbb{C}$ satisfying $\tau(mg) = \det(g)^k\tau(m)$ is given by a linear combination of $2 \times k$ tableau. Equivalently, all invariant sections of $\mathcal{L}^{\otimes N}$ (or sections of $\mathcal{H}^{\otimes a}$, see previous section) are linear combinations of s_τ where τ is a $2 \times aN$ tableau.

The G.I.T. quotient $\mathbb{C}^{n\times 2}/\!/_{\det^a}\mathrm{GL}(2, \mathbb{C})$ (which we know to be $\mathrm{Gr}_2(\mathbb{C}^n)$) is by definition $\operatorname{Proj} S_a$ where S_a is the graded ring $\bigoplus_{N\ge 0}\Gamma(\mathbb{C}^{n\times 2}, \mathcal{L}^{\otimes N})^{\mathrm{GL}(2,\mathbb{C})}$. We have a good understanding of the generators of S_a: the ring is generated by lowest degree elements, and each graded piece is spanned by s_τ, over tableaux τ of the appropriate size. We now describe the relations satisfied by the s_τ.

A tableau is called *semistandard* if its columns are strictly increasing and its rows are weakly increasing. The following two propositions are standard facts in invariant theory, see for example [Do]. We will give additional proofs [HMSV] in terms of "Kempe graphs".

Proposition 3.1. *The relations between tableaux are generated by the* Plücker relations*:*

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} = \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array} + \begin{array}{|c|c|}\hline a & b \\ \hline d & c \\ \hline\end{array}$$

where a, b, c, d are any integers between 1 and n. If τ, σ, and γ are the above tableaux, then all relations in S_1 are generated by the corresponding relation $s_\tau = s_\sigma + s_\gamma$.

Proposition 3.2. *The $2 \times aN$ semistandard tableaux are linearly independent;*

$$\{s_\tau \mid \tau \textit{ is } 2 \times aN, \textit{ semistandard}\}$$

is a basis for the Nth graded piece of S_a.

3.2. **The homogeneous ring of** $(\mathbb{CP}^1)^n/\!/_{\mathbf{r}}\mathrm{SL}(2,\mathbb{C})$**.** Assume that $|\mathbf{r}| = \sum_i r_i = 2a$ is even. If we quotient $\mathbb{C}^{n\times 2}$ by $\mathrm{GL}(2,\mathbb{C})$ using the linearization $|\mathbf{r}|$, we obtain the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ with the line bundle $\mathcal{L}_{2,n}^{\otimes a}$. Let $\mathbf{r}$ be a weight of the torus $T \subset \mathrm{GL}(n)$ and $\chi_{\mathbf{r}}$ the corresponding character, which defines a T–linearization of $\mathcal{L}_{2,n}^{\otimes a}$. If we now quotient $\mathrm{Gr}_2(\mathbb{C}^n)$ by T we obtain the moduli space $(\mathbb{CP}^1)^n/\!/_{\mathbf{r}}\mathrm{SL}(2,\mathbb{C})$ (see the previous section on the Gel'fand-MacPherson correspondence.)

The quotient $\mathrm{Gr}_2(\mathbb{C}^n)/\!/_{\mathbf{r}}T$ is, by definition, $\mathrm{Proj}\, S_a^T$, where S_a^T is the fixed subring of S_a under the action of the torus. We now determine the invariant sections. Let s_τ be a section of $\mathcal{H}^{\otimes aN}$ corresponding to the tableau τ. An element $\mathbf{t} = (t_1, \ldots, t_n) \in T$ acts on τ via

$$\mathbf{t}\cdot\tau = \mathbf{t}^{\mathbf{c}-N\mathbf{r}}\tau$$

where c_i is the number of times i occurs in τ and we use the notation

$$\mathbf{t}^{\mathbf{c}} = \prod_{i=1}^{n} t_i^{c_i}$$

(this is most easily seen by returning to the bundle $\mathcal{L}$ over $\mathbb{C}^{n\times 2}$). Thus s_τ is invariant if and only if $\mathbf{c} = N\mathbf{r}$, i.e., $c_i = Nr_i$ for each $1 \le i \le n$. For a general section s, write $s = \sum \alpha_i s_{\tau_i}$ where the τ_i are semistandard. Clearly, s is invariant if and only if each s_{τ_i} is as well. Thus we have shown:

Proposition 3.3. *The Nth graded piece of the ring S_a^T has for a basis $\{s_\tau\}$ where τ runs over the semistandard tableaux for which i occurs precisely Nr_i times.*

From now on, we are going to be working more with the tableaux than the sections. For convenience, we put $R_{\mathbf{r}}$ to be the "ring of tableaux" which satisfy the condition of the above theorem. This is clearly no loss in generality, as the map $s : R_{\mathbf{r}} \to S_a^T$ which assigns to a tableau τ the section s_τ is an isomorphism.

We also will write $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$ for the homogeneous coordinate ring of the Grassmannian with respect to the Plücker embedding. We shall regard the elements of $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$ as tableaux with two rows and any number of columns, with entries ranging from 1 to n. The ring $R_{\mathbf{r}}$ is a subring of $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$.

3.3. **Example.** If $\mathbf{r} = (1, 1, \ldots, 1)$ (which corresponds to equilateral n-gons) a tableau is in $R_{\mathbf{r}}$ if and only if each integer between 1 and n occurs the same number of times in the tableau. For example, for $n = 6$ the tableau

1	2	3
4	5	6

is allowed. Note that the condition of being in $R_{\mathbf{r}}$ puts a restriction on the number k of columns in a tableau: for this choice of $\mathbf{r}$, if n is even then k must be a multiple of $n/2$ while if n odd then k must be a multiple of n.

3.4. **The side-splitting map on rings.** We now construct a surjection of graded rings $R_{\mathbf{r}} \to R_{\mathbf{r}'}$ for certain choices of $\mathbf{r}$ and $\mathbf{r}'$. This map will be called the *side-splitting map* for reasons which will become apparent later. In this section, $[x, y]$ will denote the integers between x and y (inclusively).

Let $\mathbf{r}$ be a side-length vector of length n and let $\mathbf{r}'$ be a side length vector of length n', where $n > n'$. Let $\phi : [1, n] \to [1, n']$ be a nondecreasing map such that

$$\mathbf{r}'_i = \sum_{\phi(j)=i} \mathbf{r}_j. \tag{1}$$

We define a map $\phi_* : R_{\mathbf{r}} \to R_{\mathbf{r}'}$ as follows: let ϕ_* be the unique graded ring homomorphism such that for any tableau $\tau \in R_{\mathbf{r}}$, $\phi_*(\tau)$ is the tableau obtained by replacing all i's in τ with $\phi(i)$. The following proposition shows that this is well-defined.

Proposition 3.4. *The map ϕ_* is a well-defined surjection of graded rings. Furthermore, it maps semistandard tableaux to semistandard tableaux (or zero).*

Proof. Define a map $\hat{\phi}_* : \mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)] \to \mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^{n'})]$ by letting $\hat{\phi}_*$ act on tableaux in the same manner that ϕ_* was specified to act on tableaux. The map $\hat{\phi}_*$ will lift the map ϕ_* to the larger ring $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$. The reason for doing this is that these larger rings are simpler than the rings $R_{\mathbf{r}}$

Recall that the rings $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)]$ and $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^{n'})]$ are generated by 2×1 tableaux with entries in $[1, n]$ ($[1, n']$ respectively) subject to the relations

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} = - \begin{array}{|c|}\hline b \\ \hline a \\ \hline\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} = \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array} + \begin{array}{|c|c|}\hline a & b \\ \hline d & c \\ \hline\end{array}$$

These relations are obviously preserved by $\hat{\phi}_*$; thus $\hat{\phi}_*$ is well-defined.

Next observe that $\hat{\phi}_*$ maps $R_{\mathbf{r}}$ into $R_{\mathbf{r}'}$. This is an immediate consequence of the condition (1) imposed on the map ϕ. By construction, the restriction of the map $\hat{\phi}_*$ to the subring $R_{\mathbf{r}}$ is equal to the map ϕ_*, that is to say, we have the following commutative diagram:

$$\begin{array}{ccc} R_{\mathbf{r}} & \hookrightarrow & \mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^n)] \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \hat{\phi}_*} \\ R_{\mathbf{r}'} & \hookrightarrow & \mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^{n'})]. \end{array}$$

This proves that the map ϕ_* is well-defined, since it is simply the restriction of a well-defined map.

The fact that ϕ_* is surjective is obvious: given any tableau τ in $\mathbb{C}[\mathrm{Gr}_2(\mathbb{C}^{n'})]$ simply change all occurrences of i to something in $\phi^{-1}(i)$ in such a way that the weights add up correctly. More precisely, if $j_1, \ldots, j_k$ are the solutions to $\phi(j) = i$ then change $\mathbf{r}_{j_1}$ of the i's in τ to be j_1's, $\mathbf{r}_{j_2}$ of the i's to j_2's, etc. By the condition (1) this is guaranteed to work. Thus ϕ_* is surjective. Note, however, that $\hat{\phi}_*$ is not necessarily surjective.

Finally, the fact that ϕ_* takes semistandard tableau in $R_{\mathbf{r}}$ to semistandard tableau in $R_{\mathbf{r}'}$ (or zero) follows immediately from the condition that ϕ is nondecreasing. This completes the proof. ∎

3.5. **Geometric interpretation of the side-splitting map.** The side-splitting map $\phi_* : R_{\mathbf{r}} \to R_{\mathbf{r}'}$ defined in the previous section induces a map $\phi^* : M_{\mathbf{r}'} \to M_{\mathbf{r}}$ on the moduli spaces of polygons. This map has a very natural geometric description, which we now give.

Let $x \in M_{\mathbf{r}'}$ be an n'-gon. Then $\phi^*(x)$ is obtained from x by splitting the ith edge of x into k pieces of lengths $r_{j_1}, \ldots, r_{j_k}$ respectively, where $j_1, \ldots, j_k$ are the solutions to $\phi(j) = i$. By "splitting" an edge we simply mean regarding a point along the edge as a vertex. Thus if ϕ is not the identity map then $\phi^*(x)$ will always have 180 degree internal angles.

Geometrically, Proposition 3.4 may be rephrased as follows:

Proposition 3.5. *The map $\phi^* : M_{\mathbf{r}'} \to M_{\mathbf{r}}$ is a closed immersion of projective varieties.*

3.6. **Example of the side-splitting map.** Let $\mathbf{r} = (1,1,1,1,1,1,1,1)$, $\mathbf{r}' = (2,2,2,2)$ and

i	1	2	3	4	5	6	7	8
$\phi(i)$	1	1	2	2	3	3	4	4

.

The semistandard tableau

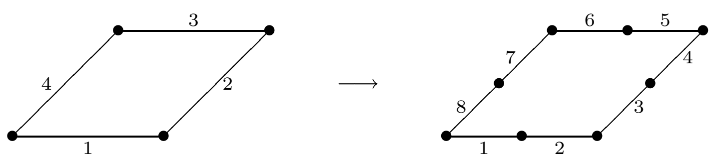
1	2	4	6
3	5	7	8

is an element of $R_{\mathbf{r}}$; its image under ϕ_* is the semistandard tableau

1	1	2	3
2	3	4	4

.

On polygons, ϕ^* takes equilateral quadrilaterals of side length 2 to equilateral octagons of side length 1. A typical example is illustrated below.



3.7. **Generators for $R_{\mathbf{r}}$.** The following theorem of Kempe gives generators of the ring $R_{\mathbf{r}}$ in the case $\mathbf{r} = (1,1,\dots,1)$ and n is even. We give a possibly new proof of this important theorem here. For Kempe's original proof, see [Howe, pg. 156]. We give another proof because we need to extend it to the case of arbitrary weights (Theorem 3.7). The graphical description of the ring used here will be key to results of [HMSV].

Theorem 3.6 (Kempe, 1894). *For $\mathbf{r} = (1,1,\dots,1)$ and n even, the ring $R_{\mathbf{r}}$ is generated by degree one tableaux; in other words, $R_{\mathbf{r}}$ is generated by $2 \times (n/2)$ semistandard tableaux for which each integer between 1 and n occurs precisely once.*

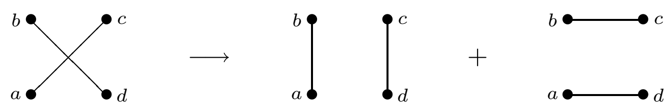
Proof. We associate a graph Γ to a tableau τ as follows: the vertices of the graph are the integers between 1 and n; for each column in the tableau containing the number i and j, place an edge between i and j. The graph is undirected (we have assumed that $i < j$ since τ is semistandard) and allowed to contain multiple edges between vertices.

A tableau τ can be written as a product $\tau_1\tau_2$ of tableaux where τ_1 is degree 1 if and only if one can pick $n/2$ edges in the graph associated to τ which contain each vertex precisely once. In fact, these edges can be used to form the columns of τ_1 by a process inverse to that of the previous paragraph. If τ can be decomposed as such we will say that it (or its graph) can be *factored*.

We now examine how Plücker relations work on graphs. On tableaux, the Plücker relation takes a tableau and two columns in that tableau and yields a sum of two tableaux. In terms of graphs, therefore, the Plücker relation should take a graph and two edges in the graph and result in two new graphs. More precisely, let τ be a tableau and Γ its associated graph. Let (a, b) and (c, d) be two edges in Γ (i.e., two columns in τ). The Plücker relations applied to these two columns in τ give

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} = \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array} + \begin{array}{|c|c|}\hline a & b \\ \hline d & c \\ \hline\end{array}.$$

Thus the graphs of the resulting tableaux are obtained from Γ by removing the edges (a, c), (b, d) and then in the first graph adding edges (a, b),(c, d) and in the second adding (a, d),(b, c). This may be represented visually by:



We first show that every degree two tableaux can be expressed algebraically in terms of degree one tableaux. Now, the graph of such a tableau is a graph on n vertices for which every vertex has valence 2. Such a graph is necessarily a disjoint union of cycles. If we take two cycles and one edge from each cycle and apply the Plücker relations, the cycles merge into one cycle in both of the resulting graphs. Thus by applying the Plücker relations repeatedly we can merge together pairs of cycles of odd length and end up with many graphs each of which consists entirely of even length cycles. An even cycle can clearly be factored by taking every other edge. Thus the original degree two tableau can be written as a sum of products of degree one tableaux.

Now suppose that τ has degree $k > 2$. Let Γ be the associated graph; note that Γ is k-regular. We form a new graph $\widetilde{\Gamma}$ by doubling the vertex set; each vertex is split into a "man" and a "woman", i.e., the vertex set of $\widetilde{\Gamma}$ is:

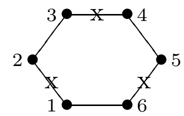
$$\mathrm{Vert}(\widetilde{\Gamma}) = \{\mathrm{man}_1, \ldots, \mathrm{man}_n, \mathrm{woman}_1, \ldots, \mathrm{woman}_n\}.$$

For each edge (i, j) of Γ let $(\mathrm{man}_i, \mathrm{woman}_j)$ and $(\mathrm{man}_j, \mathrm{woman}_i)$ be edges of $\widetilde{\Gamma}$. Note that $\widetilde{\Gamma}$ is k-regular and bi-partite between men and women. Recall Hall's Marriage Theorem for bipartite graphs: there is a matching (a 1-regular subgraph with the same vertex set) in $\widetilde{\Gamma}$ if and only if any collection of women are compatible with at least as many men. This condition is satisfied since $\widetilde{\Gamma}$ is regular. Hence there is a matching $\widetilde{\Delta}$ in $\widetilde{\Gamma}$. Let Δ be the graph on vertices $\{1, \ldots, n\}$ such that for each edge $(\mathrm{man}_i, \mathrm{woman}_j)$ in $\widetilde{\Delta}$ we place the edge (i, j) into Δ. Now Δ is 2-regular, and it is almost a subgraph of Γ. However, there could be two occurrences of an edge (i, j) in Δ but only one in Γ; this may happen if both $(\mathrm{man}_i, \mathrm{woman}_j)$ and $(\mathrm{man}_j, \mathrm{woman}_i)$ are edges of $\widetilde{\Delta}$. Nevertheless, this problematic repeated edge in Δ is a 2-cycle. Hence if we apply Plücker relations to rewrite Δ as a sum of 2-regular graphs with all even cycles, this 2-cycle (i, j) will be carried along in each term as a common factor. Finally when we pick out a 1-regular subgraph from each such term, the edge (i, j) will be chosen only once. Hence we will obtain an honest expression for Γ as a sum of terms with 1-regular subgraphs. Thus each tableau of degree k is generated by tableaux of degree one. $\square$

We illustrate the proof given above for $n = 6$. Consider the tableau

1	2	3	4	5	6
2	3	4	5	6	1

Its graph and a factorization are depicted below



This corresponds to the factorization of tableau:

1	2	3	4	5	6
2	3	4	5	6	1

=

1	3	5
2	4	6

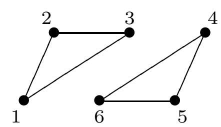
·

2	4	6
3	5	1

A more interesting example is given by the tableau

1	1	2	4	4	5
2	3	3	5	6	6

the graph of which is



Clearly, this cannot be factored. However, upon applying the Plücker relations to edges $(1, 3)$ and $(4, 6)$ we obtain the two graphs:

Each of these graphs is isomorphic to the graph in the previous example (as unlabelled graphs) and so can be factored. Such a factorization would express the original tableau as a sum with two terms, each of which is a product of two degree one tableaux.

Theorem 3.7. *For any weight* $\mathbf{r}'$ *the ring* $R_{\mathbf{r}'}$ *is generated by lowest degree elements.*

Proof. This follows immediately from Theorem 3.6 and Proposition 3.4. Take $\mathbf{r}$ to be the side-length vector of length n consisting of all 1's, where $n = |\mathbf{r}'|$. There is an obvious map $\phi : [1, n] \to [1, n']$ satisfying the conditions to be a side-splitting map. The resulting surjection $\phi_* : R_{\mathbf{r}} \to R_{\mathbf{r}'}$ implies that $R_{\mathbf{r}'}$ is generated by lowest degree elements, since this is true for $R_{\mathbf{r}}$. □

Note that these results give us projective embeddings of $(\mathbb{CP}^1)^n /\!/_{\mathbf{r}} \mathrm{SL}(2, \mathbb{C})$: if $N+1$ is the number of lowest degree elements of $R_{\mathbf{r}}$ then $(\mathbb{CP}^1)^n /\!/_{\mathbf{r}} \mathrm{SL}(2, \mathbb{C})$ embeds into $\mathbb{CP}^N$. We will call this embedding the *Kempe embedding*. For example, for $n = 2m$ even and $\mathbf{r} = (1, 1, \ldots, 1)$ the hook length formula gives $N = C_m - 1$, where

$$C_m = \frac{(2m)!}{(m+1)!\, m!}$$

is the mth Catalan number; thus the moduli space of equilateral $2m$-gons naturally embeds nondegenerately into $\mathbb{CP}^{C_m - 1}$.

4. The three isomorphic polytopes

In this section, we give three quite different descriptions of the key polytope used in this paper. We will use two of them (the diagonal length polytope and the semistandard tableau polytope) in our argument, and we discuss the third (the Gel'fand-Tsetlin polytope) because it is an important description used elsewhere in the literature.

4.1. **The diagonal length polytope** $D(\mathbf{r})$**.** We define the map $F : M_{\mathbf{r}}(\mathbb{R}^2) \to \mathbb{R}^{n-3}$ by $F(\mathbf{e}) = \mathbf{d}$ where $\mathbf{d} = (d_1, d_2, \dots, d_{n-3})$ is the set of lengths of the diagonals drawn from the zeroth vertex v_0 to the remaining vertices. Precisely

$$d_i = ||e_1 + e_2 + \cdots + e_i||, \ 1 \le i \le n-1.$$

Thus the i–diagonal joins v_0 to v_i.

We note that d_1 and d_{n-1} are fixed (they are the lengths of the first and last edges), $d_1 = r_1$ and $d_{n-1} = r_n$. Also F is continuous, but is not smooth where some diagonal length d_i is equal to zero (because $\sqrt{x}$ is not a smooth function of x near $x = 0$, hence d_i is not a smooth function).

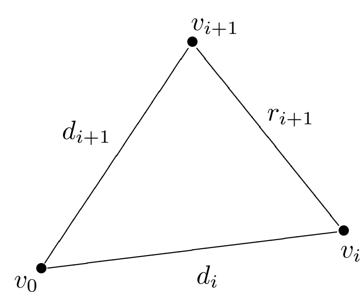
We define a subset $D(\mathbf{r})$ of $\mathbb{R}^{n-3}$ by

$$D(\mathbf{r}) = F(M_{\mathbf{r}}(\mathbb{R}^2)).$$

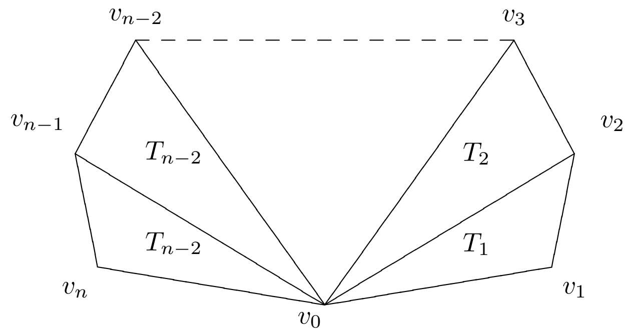
Proposition 4.1. *The set* $D(\mathbf{r}) = F(M_{\mathbf{r}}(\mathbb{R}^2))$ *is the convex polytope cut out by the* $n-2$ *triples of triangle inequalities*

(1) $d_i - d_{i+1} \le r_{i+1}$
(2) $d_i - d_{i+1} \ge -r_{i+1}$
(3) $d_i + d_{i+1} \ge r_{i+1}$

for $1 \le i \le n-2$.



Proof. Draw an abstract convex n-gon P in the plane (without specifying its side-lengths). Then the above diagonals triangulate P into $n-2$ triangles, $T_1, T_2, \dots, T_{n-2}$ where T_1 has vertices v_0, v_1 and v_2, T_2 has vertices v_0, v_2, v_3 and finally T_{n-2} has vertices v_0, v_{n-2} and v_{n-1}.



The critical observation is that we can construct a polygon with the given side-lengths if and only if we can realize each of the $n-2$ triangles $T_i,\ 1 \le i \le n-2$ as planar triangles with side-lengths d_i, r_{i+1} and d_{i+1}. For given the $n-2$ triangles we can assemble them along their common diagonals to obtain the n-gon. But the triangle T_i can be realized if and only if the three numbers d_i, r_{i+1} and d_{i+1} satisfy the triangle inequalities. □

4.2. **The Gel'fand-Tsetlin polytopes $GT(\Lambda\varpi_2, \mathbf{r})$.** To begin this section we will make a definition. (Our presentation here is borrowed from [dLMc] though we have indexed our entries differently.)

For each $n \in \mathbb{N}$, let X_n be the set of all triangular arrays $(x_{ij})_{1 \le i < j \le n}$ with $x_{ij} \in \mathbb{R}$.

Definition 4.2. *A Gel'fand-Tsetlin pattern or GT-pattern is a triangular array $\mathbf{x} = (x_{ij})_{1 \le i < j \le n} \in X_n$ such that all x_{ij} are nonnegative and satisfy the interlacing inequalities: $x_{i,j+1} \ge x_{ij} \ge x_{i,j+1}$, for $1 \le i < j \le n-1$.*

Define s_i, $1 \le i \le n$, to be the sum of the entries in the ith row,

$$\sum_{j=1}^{i} x_{ij}$$

and r_i, $1 \le i \le n$, to be the first differences of the s_i's so

$$r_1 = s_1 \text{ and } r_i = s_{i+1} - s_i, \quad 1 \le i \le n-1.$$

We then define

Definition 4.3. *The weight $wt(\mathbf{x})$ of the Gel'fand-Tsetlin pattern $\mathbf{x}$ is the n-tuple $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ where the r_i's are as immediately above.*

A GT-pattern is traditionally depicted in an inverted triangle array. See [dLMc, pg. 2]. In our case all the entries x_{ij} will be zero for $i \ge 3$, that is there are only two nonzero entries in each row of the pattern (except the first (bottom) row where there is only one nonzero entry). We will use a_i, b_i to denote the nonzero entries in the ith row. Furthermore the two nonzero entries in the top row will be assumed to be equal. Their common value will be denoted Λ. We give an example for the case of $n = 6$ (we define $a_n = b_n = a_{n-1} = \Lambda$ and $b_1 = 0$).

$$\begin{array}{ccccccccccc}
\Lambda & & \Lambda & & 0 & & 0 & & 0 & & 0 \\
& \Lambda & & b_5 & & 0 & & 0 & & 0 & \\
& & a_4 & & b_4 & & 0 & & 0 & & \\
& & & a_3 & & b_3 & & 0 & & & \\
& & & & a_2 & & b_2 & & & & \\
& & & & & a_1 & & & & &
\end{array}$$

Figure 2

We will often abbreviate such a pattern by the ordered pair $(\mathbf{a}, \mathbf{b})$ where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$.

Thus we now have $s_i = a_i + b_i,\ 1 \leq i \leq n$ and the interlacing inequalities now become

(1) $a_{i+1} \geq a_i$
(2) $a_i \geq b_{i+1}$
(3) $b_{i+1} \geq b_i$.

Here $1 \leq i \leq n$.

We will denote the the above polytope by $GT(\Lambda\varpi_2)$. Thus the polytope $GT(\Lambda\varpi_2)$ has affine hull $X(\Lambda)$, the subspace of $\mathbb{R}^{2n}$ defined by the conditions $a_n = b_n = \Lambda$ and $b_1 = 0$.

4.2.1. *The polytope $GT(\Lambda\varpi_2, \mathbf{r})$.* Suppose now that an n–tuple of positive real numbers $\mathbf{r}$ is given. We define

$$s_i = \sum_{j=1}^{i} r_j.$$

We define an affine subspace $X(\Lambda, \mathbf{r})$ of $\mathbb{R}^{2n}$ by the conditions

$$s_i = a_i + b_i, \qquad 1 \leq i \leq n.$$

Definition 4.4. *The polytope $GT(\Lambda\varpi_2, \mathbf{r})$ is the intersection of the polytope $GT(\Lambda\varpi_2)$ with the affine subspace $X(\Lambda, \mathbf{r})$.*

Thus the elements $x \in GT(\Lambda\varpi_2, \mathbf{r})$ are required to have weight $\mathbf{r}$. Recalling that $b_1 = 0$ we have $a_1 = s_1 = r_1$.

Definition 4.5. *We define a map $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{n-1}$ by $\Phi((\mathbf{a}, \mathbf{b})) = \mathbf{d}$ where*

$$d_i = a_i - b_i$$

for all i such that $1 \leq i \leq n - 1$.

The map Φ is the map of momentum polytopes underlying the Hausmann-Knutson duality between the bending integrable system on the polygon space $M_{\mathbf{r}}$ and the Gel'fand-Tsetlin integrable system on the torus quotient $\mathrm{Gr}_2(\mathbb{C}^n)//_{\mathbf{r}}T$, see [HK].

Proposition 4.6. *Suppose $\Lambda = (\sum_{i=1}^{n} r_i)/2$. Then the map Φ carries the Gel'fand-Tsetlin polytope $GT(\Lambda\varpi_2, \mathbf{r})$ bijectively onto the diagonal length polytope $D(\mathbf{r})$.*

Proof. Note first that the inequality $a_i \geq b_i$ corresponds under Φ to the inequality $d_i \geq 0$. We will prove that the interlacing inequalities relating the ith and $i+1$ rows (i.e. the rows (a_i, b_i) and (a_{i+1}, b_{i+1})) correspond under Φ to the triangle inequalities for the triangle T_{i+2} (that is the triangle with side-lengths $d_{i+1}, d_{i+2}, r_{i+3}$).

The three interlacing inequalities are

(1) $a_{i+1} \geq a_i$
(2) $a_i \geq b_{i+1}$
(3) $b_{i+1} \geq b_i$.

The reader will verify that each of the interlacing inequalities corresponds to one of the three triangle inequalities. We label the triangle inequalities so that corresponding inequalities have the same label.

(1) $d_{i+1} - d_{i+2} \leq r_{i+1}$
(2) $d_{i+1} + d_{i+2} \geq r_{i+1}$
(3) $d_{i+1} - d_{i+2} \geq -r_{i+1}$

□

We leave the analysis of the two boundary triangles to the reader.

4.3. **The polytopes $SS(\Lambda\varpi_2, \mathbf{r})$.** The following definitions are fundamental in combinatorial representation theory.

Definition 4.7. *A filling of a Young diagram by the integers between* 1 *and* n *is said to be semistandard if the columns are strictly increasing and the rows are weakly increasing.*

A Young diagram with a semistandard filling by the integers between 1 and n is called a semistandard Young tableau. Given a semistandard Young tableau τ we define its weight $wt(\tau)$ to be the sequence of nonnegative integers $\mathbf{r} = (r_1, r_2, \dots, r_n)$ where r_i is the number of i's in the tableau.

We will now restrict to the case in which the underlying Young diagram is a 2 by M rectangle.

Definition 4.8. *Let τ be a semistandard tableau obtained by a semistandard filling of a* 2 *by M rectangle filled by the integers between* 1 *and n. We let k_{1j}, resp. k_{2j} denote the number of j's in the first, resp. second row. We define the multiweight $\mathbf{wt}(\tau)$ of τ to be the* 2 *by n matrix with integer entries given by $\mathbf{wt}(\tau) = \begin{pmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \end{pmatrix}$ where*

$$\begin{pmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1,n-1} & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2,n-1} & k_{2n} \end{pmatrix}.$$

For example if $\tau = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 & 5 \\ \hline 2 & 3 & 3 & 5 & 6 & 6 \\ \hline \end{array}$ then $\mathbf{wt}(\tau) = \begin{pmatrix} 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}$.

Note also that the (ordinary) weight $wt(\tau) = \mathbf{r}$ is given by

$$r_i = k_{1i} + k_{2i}, \qquad 1 \leq i \leq n.$$

The multiplicities k_{ij} satisfy the following "stairstep inequalities" (corresponding to the condition that the columns be strictly increasing):

$$\sum_{j=1}^{i} k_{2j} \leq \sum_{j=1}^{i-1} k_{1j}, \qquad 2 \leq i \leq n-1.$$

•	•	•			
•	•	•	•		

The third stair-step inequality

They also satisfy the three equations

$$\sum_{i=1}^{n} k_{1i} = \sum_{i=1}^{n} k_{2i} \tag{2}$$

$$k_{21} = 0 \tag{3}$$

$$k_{1n} = 0. \tag{4}$$

Proposition 4.9. *The stairstep inequalities and the equations (2), (3), (4), give conditions on* $\begin{pmatrix}\mathbf{k}_1\\ \mathbf{k}_2\end{pmatrix}$ *that are necessary and sufficient in order that there exist a semistandard filling of a* 2 *by* n *rectangle such that there are* k_{1i} i*'s,* $1 \le i \le n$*, in the first row and* k_{2i} i*'s,* $1 \le i \le n$*, in the second row.*

Proof. There is only one way to fill in the rows with the given multiplicities since the rows are weakly increasing. The ith stairstep inequality gives that the string of i's in the bottom row is completed before the string of i's in the top row is completed. This guarantees that the columns are strictly increasing. □

To save space we will replace the column vector notation $\begin{pmatrix}\mathbf{k}_1\\ \mathbf{k}_2\end{pmatrix}$ by the row vector notation $(\mathbf{k}_1, \mathbf{k}_2)$

We now define the third polytope $SS(\Lambda\varpi_2, \mathbf{r})$ we will need in this paper.

Definition 4.10. *Define the polytope* $SS(\Lambda\varpi_2)$ *to be the set of* 2 *by* n *matrices with real number entries*

$$\mathbf{y} = \begin{pmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1,n-1} & y_{1n} \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2,n-1} & y_{2n} \end{pmatrix},$$

such that the y_{ij}*'s satisfy the stairstep inequalities and the three equalities above. We define the polytope* $SS(\Lambda\varpi_2, \mathbf{r})$ *to be the subset of* $SS(\Lambda\varpi_2)$ *consisting of those* $\mathbf{y}$ *which are of weight* $\mathbf{r}$ *i.e. such that*

$$y_{1j} + y_{2j} = r_j, \ 1 \le j \le n.$$

4.3.1. *The map* Ψ. We define a map Ψ by $\Psi(\mathbf{a}, \mathbf{b}) = (\mathbf{k}_1, \mathbf{k}_2)$ where

$$k_{1,i+1} = a_{i+1} - a_i \text{ and } k_{2,i+1} = b_{i+1} - b_i.$$

Proposition 4.11.

(i) *The map* Ψ *carries the Gel'fand-Tsetlin polytope* $GT(\Lambda\varpi_2))$ *bijectively onto the polytope* $SS(\Lambda\varpi_2)$.

(ii) *The map* Ψ *is weight-preserving and consequently carries the Gel'fand-Tsetlin polytope* $GT(\Lambda\varpi_2), \mathbf{r})$ *bijectively onto the polytope* $SS(\Lambda\varpi_2, \mathbf{r})$ *.*

Proof. We first prove (i). We claim that the ith stairstep inequality is equivalent under Ψ to the middle interlacing inequality $a_i \ge b_{i+1}$. Indeed the sum of the first i entries in the first row of $\Psi(x) = \begin{pmatrix}\mathbf{k}_1\\ \mathbf{k}_2\end{pmatrix}$ telescopes to a_i and the sum of the first $i+1$ entries in the second row telescopes to b_{i+1}. The other two interlacing inequalities are equivalent to the conditions that $k_{1,i+1}$ and $k_{2,i+1}$ are nonnegative.

To prove (ii) we first claim that under Ψ the number r_1 goes to k_{11} and r_i goes to $k_{1,i} + k_{2,i}$ (the number of times i appears in the tableaux T corresponding to $\Psi(x)$). Indeed the first statement is obvious and for $i \geq 2$ we have $r_i = s_i - s_{i+1} = (a_i + b_i) - (a_{i-1} + b_{i-1}) = (a_i - a_{i-1}) + (b_i + b_{i-1}) = k_{1i} + k_{2i}$.

From this it is immediate that the weight of the pattern x is equal to the weight of the tableau corresponding to $\Psi(x)$. □

Remark 4.12. *The map on integral points induced by the map Ψ is a special case of a basic map in combinatorial representation theory. The integral GT patterns and the semistandard Young tableau index weight bases for the irreducible representation with highest weight $\Lambda\varpi_2$ and the map Ψ indexes a weight-preserving change of basis (see* [dLMc, Fig. 1]*).*

4.4. **Lattice points in the diagonal polytopes $D(\mathbf{r})$.** We have defined the polytope isomorphisms

$$\Psi : GT(\Lambda\varpi_2, \mathbf{r}) \to SS(\Lambda\varpi_2, \mathbf{r}),$$

$$\Phi : GT(\Lambda\varpi_2, \mathbf{r}) \to D(\mathbf{r}).$$

Each of these takes integral integral points to integral points. Furthermore $\Psi^{-1} : SS(\Lambda\varpi_2, \mathbf{r}) \to GT(\Lambda\varpi_2, \mathbf{r})$ carries integral points to integral points. The integral points of $SS(\Lambda\varpi_2, \mathbf{r})$ (and hence $GT(\Lambda\varpi_2, \mathbf{r})$) are of fundamental importance since they correspond to semistandard tableaux.

The map Φ^{-1} does not take integral points to integral points. There are generally more integral points in $D(\mathbf{r})$ than in $GT(\Lambda\varpi_2, \mathbf{r})$. In fact, the integral points in $D(\mathbf{r})$ are in one-to-one correspondence with the integral points in $GT(2\Lambda\varpi_2, 2\mathbf{r})$. A simple example is $\mathbf{r} = (1,1,1,1)$. The Gel'fand-Tsetlin polytope $GT(2\varpi_2, \mathbf{r})$ has two integral points corresponding to the tableaux $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$, but $D(\mathbf{r})$ has three integral points, $(1,2,1)$, $(1,1,1)$, and $(1,0,1)$. The point $(1,1,1)$ is not the image of an integral point in $GT(2\varpi_2, \mathbf{r})$. Since we are primarily concerned with the semistandard tableaux which generate $R_{\mathbf{r}}$, which shall say the a point in $D(\mathbf{r})$ is a lattice point iff it maps to an integral point under Φ^{-1}.

Proposition 4.13. *Denote the lattice points in $D(\mathbf{r})$ by $D(\mathbf{r})(\mathbb{Z})$. Let $\mathbf{d} \in D(\mathbf{r})$. Then,*

$$\mathbf{d} \in D(\mathbf{r})(\mathbb{Z}) \iff d_j \equiv \left(\sum_{i=1}^{j} r_i\right) \mod 2 \textit{ for every } j \leq n-1.$$

(In particular if $\mathbf{r} = (1,1,\ldots,1)$ then $\mathbf{d} \in D(\mathbf{r})$ is a lattice point iff $d_i \equiv i \mod 2$ for each $i \leq n-1$.)

Proof. We have that $\mathbf{d} \in D(\mathbf{r})(\mathbb{Z})$ if and only if $\Phi^{-1}(\mathbf{d})$ is integral by definition. This means that $a_j = ((\sum_{i=1}^{j} r_i) + d_j)/2$ and $b_j = ((\sum_{i=1}^{j} r_i) - d_j)/2$ are integers for each $j \leq n-1$. This is equivalent to $(\sum_{i=1}^{j} r_i) + d_j \in 2\mathbb{Z}$, which is equivalent to $d_j \equiv (\sum_{i=1}^{j} r_i) \mod 2$. □

4.5. **Application: $(\mathbb{CP}^1)^n /\!/_{\mathbf{r}} \mathrm{SL}(2,\mathbb{C})$ is rarely a complete intersection.** As an application of this identification of polytopes, we show that the moduli space is almost never a complete intersection. We do this by showing that its degree is too low compared to its codimension. The main point of this section is to show that the semistandard tableau polytope interpretation allows us to effectively bound the number of points in the polytope, which yields a previously inaccessible geometric consequence.

Let H_X be the Hilbert polynomial of a projective variety, X and $L(H_X)$ its leading coefficient. (We will not use this notation beyond this subsection.) If $X \subset \mathbb{CP}^N$ be a complete intersection which is nondegenerate (not contained in any hyperplane), then the degree of each defining equation of is at least 2, so the degree of X is at least $2^{\operatorname{codim} X}$. As the leading coefficient of the Hilbert polynomial is the degree divided by $\dim X!$, we have

$$\frac{1}{(\dim X)!} \leq L(H_X)\, 2^{-\operatorname{codim} X}. \tag{5}$$

We now apply this observation to show that the Kempe embedding is rarely a complete intersection. The even and odd cases must be treated differently.

First, let n be even, $\mathbf{r} = (1, 1, \ldots, 1)$ and $N = C_{n/2} - 1$. The Kempe embedding gives an embedding of $(\mathbb{CP}^1)^n /\!/_{\mathbf{r}} \mathrm{SL}(2, \mathbb{C})$ into $\mathbb{CP}^N$.

Theorem 4.14. *For $n \geq 8$ even, the image of the Kempe embedding $M_{\mathbf{r}} \to \mathbb{CP}^N$ is not a complete intersection. For $n = 2, 4, 6$ the image is a complete intersection.*

We wish to apply (5). To do so, we need to know about the leading coefficient of the Hilbert polynomial H_R of the Kempe embedding.

Lemma 4.15. *The leading coefficient $L(H_R)$of the Hilbert polynomial of the ring $R_{\mathbf{r}}$ is ≤ 1.*

Proof. The ℓth graded component of $R_{\mathbf{r}}$ has for a basis the semistandard tableaux of weight ℓ, which correspond bijectively to integral points in the polytope $\ell SS(\frac{n}{2}\varpi_2, \mathbf{r})$ (see §4), the dilate of the polytope $SS(\frac{n}{2}\varpi_2, \mathbf{r})$ by the factor ℓ. Thus the dimension of the ℓth graded component of $R_{\mathbf{r}}$ is equal to the number of integral points in $\ell SS(\frac{n}{2}\varpi_2, \mathbf{r})$. For large ℓ we have

$$H_R(\ell) = \#\left(\ell SS\left(\frac{n}{2}\varpi_2, \mathbf{r}\right)(\mathbb{Z})\right).$$

Now there is a map π from the polytope $\ell SS(\frac{n}{2}\varpi_2, \mathbf{r})$ to the $n-3$ dimensional cube with side lengths equal to ℓ given by

$$\pi(\mathbf{k}) = (k_{1,2}, k_{1,3}, \cdots, k_{1,n-2}).$$

The reader will verify that π is an injection whence

$$H_R(\ell) = \#\left(\ell SS\left(\frac{n}{2}\varpi_2, \mathbf{r}\right)(\mathbb{Z})\right) \leq (\ell + 1)^{n-3}.$$

But the degree of H_R is $n - 3$. The previous inequality (valid for large ℓ) implies that the leading coefficient of H_R is less than or equal to 1. □

We now prove the Theorem.

Proof (of Theorem 4.14). Assume the image of the Kempe embedding is a complete intersection. We now know the following: $L(H_{M_{\mathbf{r}}}) \leq 1$, $\dim M_{\mathbf{r}} = n - 3$ and $\operatorname{codim} M_{\mathbf{r}} = C_{n/2} - n + 2$. The inequality (5) would then give $2^{C_{n/2} - n + 2} \leq (n-3)!$; however, this is violated for $n \geq 8$: take $\log_2$ of both sides, together with $\log_2((n - 3)!) < (n-3)(n-2)/2$ for $n \geq 7$. Also $C_{n/2} - n + 2 > (n-3)(n-2)/2$ for $n \geq 10$. When $n = 8$ we have $2^{C_4 - 8 + 2} = 2^8 = 256 > 120 = (8-3)!$.

For $n = 2$ the ring $R_{\mathbf{r}} \cong \mathbb{C}[x]$ and the space (which is a point) is a complete intersection. For $n = 4$ the space $(\mathbb{CP}^1)^n//_{\mathbf{r}}\mathrm{SL}(2,\mathbb{C})$ is simply $\mathbb{CP}^1$ (and the Kempe embedding is surjective onto $\mathbb{CP}^1$). For $n = 6$ the image of the Kempe embedding is a cubic hypersurface in $\mathbb{CP}^4$, see [DO]. These are all complete intersections. □

Now let us examine the case n is odd. Now the image of the Kempe embedding lies within $\mathbb{CP}^{N-1}$ where N is the number of semistandard tableaux weighted by $(2, 2, \dots, 2) \in \mathbb{Z}^n$.

Theorem 4.16. *Let $n \geq 1$ be an odd integer. Then the image of the Kempe embedding $M_{\mathbf{r}} \to \mathbb{CP}^N$ is a complete intersection iff $n = 1$ or $n = 3$.*

Lemma 4.17. *Let $\mathbf{r} = (2, 2, ..., 2) \in \mathbb{Z}^n$ where n is odd. The leading coefficient of the Hilbert polynomial of the ring $R_{\mathbf{r}}$ is $\leq 2^{n-3}$.*

Proof. The ℓth graded component of $R_{\mathbf{r}}$ has for a basis the semistandard tableaux of weight 2ℓ, which correspond bijectively to integral points in the polytope $\ell SS(n\varpi_2, \mathbf{r})$, the dilation of the polytope $SS(n\varpi_2, \mathbf{r})$ by a factor of ℓ. Thus the dimension of the ℓth graded component of $R_{\mathbf{r}}$ is equal to the number of integral points in $\ell SS(n\varpi_2, \mathbf{r})$. We repeat the previous argument replacing the cube of dimension $n-3$ with side-lengths ℓ by the cube of dimension $n-3$ with side-lengths 2ℓ. We obtain

$$H_R(\ell) = \#(\ell SS(n\varpi_2, \mathbf{r})(\mathbb{Z})) \leq (2\ell + 1)^{n-3}.$$

□

We now prove the theorem for the case n odd.

Proof (of Theorem 4.16). Assume the image of the Kempe embedding is a complete intersection. Let $R(n) = N$ be the number of integral points in $SS(n\varpi_2, \mathbf{r})$ (the notation $R(n)$ is used because $R(n)$ is the nth Riordan number). We now know the following: $L(H_{M_{\mathbf{r}}}) \leq 2^{n-3}$, $\dim M_{\mathbf{r}} = n - 3$ and $\operatorname{codim} M_{\mathbf{r}} = R(n) - n + 2$. The inequality (5) would then give $2^{R(n)-n+2} \leq 2^{n-3}(n-3)!$. We claim this inequality is violated for $n \geq 7$. It is simple to check that $R(n) > 2^{n-3}$ for $n \geq 7$. But for $n \geq 7$,

$$2^{R(n)-n+2} > 2^{2^{n-3}-n+2} > 2^{n-3}(n-3)!.$$

We then deal with the small cases by hand. If $n = 5$ then $R(5) = 6$ and $2^{R(n)-n+2} = 8$ and $2^{n-3}(n-3)! = 8$ so the inequality is not violated. However, the ring $R_{\mathbf{r}}$ is still not a complete intersection; there are six generators and five essential quadratic relations (this is classical, and may also be seen in §6). But the space is two dimensional, so to be a complete intersection its ideal should have just three generators. Therefore $R_{\mathbf{r}}$ is not a complete intersection for $n \geq 5$.

If $n = 1$ then $R_{\mathbf{r}} \cong \mathbb{C}$ (the space is empty) and if $n = 3$ then $R_{\mathbf{r}} \cong \mathbb{C}[x]$ (the space is a point). These are both complete intersections. □

5. A toric degeneration of $M_{\mathbf{r}}$

We construct a toric degeneration of the moduli space of polygons by descending a toric degeneration of the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ to its torus quotients. The degeneration of the Grassmannian is essentially the same as that given in [GL] and our toric fiber coincides with that of [St], page 104. Foth and Hu [FH] first observed that the toric degenerations of flag varieties constructed by Alexeev and Brion [AB] descend to give toric degenerations of their torus quotients. The momentum polytopes of the corresponding toric varieties will be $SS(\Lambda\varpi_2)$ and $SS(\Lambda\varpi_2, \mathbf{r})$ respectively. In what follows we consider problems concerning lattice points in these polytopes. In all cases the underlying lattice will be the standard integer lattice.

5.1. **A toric degeneration of** $\mathrm{Gr}_2(\mathbb{C}^n)$**.** Our toric degeneration is essentially the same as that given in [GL]. Recall that $2a = \sum_i r_i = |\mathbf{r}|$ and $\mathcal{L}^a_{2,n}$ is the very ample line bundle of $\mathrm{Gr}_2(\mathbb{C}^n)$ corresponding to the character $\det^a$ of $\mathrm{GL}(2,\mathbb{C})$. Let

$$R = \bigoplus_{N=0}^{\infty} \Gamma(\mathrm{Gr}_2(\mathbb{C}^n), (\mathcal{L}^a_{2,n})^{\otimes N}).$$

The degree one ($N = 1$) sections s_τ generate R as a ring, where τ is a semistandard 2 by Λ tableau filled with indices 1 through n. For ease of notation, we shall identify s_τ with τ.

We have that R is an infinite-dimensional $\mathbb{C}$-vector space, with basis consisting of semistandard 2 by Na tableaux as N ranges from 0 to infinity. (When $N = 0$ the section $s_\emptyset$ of the empty tableau is taken to be the constant section 1 of the trivial line bundle.) The multi-weights $\mathbf{wt}(\tau)$ defined in the previous section make R into a multi-graded vector space. For each multi-weight $\mathbf{k}$, let $\pi_\mathbf{k}$ denote the projection of R onto the one-dimensional subspace of weight $\mathbf{k}$. Let $\mathrm{gr}_1(R)$ denote the associated multi-graded ring, defined as follows. As a $\mathbb{C}$-vector space $\mathrm{gr}_1(R)$ is the same as R; only the ring structures will differ. We define multiplication on $\mathrm{gr}_1(R)$ by

$$\bar{\tau} \cdot_{\mathrm{gr}_1(R)} \bar{\sigma} = \pi_\mathbf{k}(\tau \cdot_R \sigma),$$

where $\mathbf{k} = \mathbf{wt}(\tau) + \mathbf{wt}(\sigma)$ and τ,σ are semistandard tableaux. Here we have used $\bar{\tau}$ and $\bar{\sigma}$ to denote the elements in $\mathrm{gr}_1(R)$ corresponding to the (multihomogeneous) elements τ and σ.

Definition 5.1. *Given semistandard tableaux τ and σ, let $\tau * \sigma$ be the unique semistandard tableau such that $\mathbf{wt}(\tau * \sigma) = \mathbf{wt}(\tau) + \mathbf{wt}(\sigma)$.*

Remark 5.2. *The tableau $\tau * \sigma$ can be described alternatively as the tableau obtained by concatenating τ and σ, then rearranging the top row and bottom row indices so that both rows are nondecreasing.*

Lemma 5.3. *The product of $\bar{\tau}$ and $\bar{\sigma}$ in $\mathrm{gr}_1(R)$ is given by:*

$$\tau \cdot_{\mathrm{gr}_1(R)} \sigma = \overline{\tau * \sigma}.$$

Proof. If basic straightening relations are applied to the product $\tau\sigma$, all the monomials have the same set of indices as τ and σ together. Furthermore, there is exactly one monomial in the sum which has the same set of indices in its top row as the set of indices in the top row of the concatenated $\tau\sigma$. Finally when enough applications of basic straightening relations are applied so that each term in the sum is semistandard, the unique term with the same indices in the top row as $\tau\sigma$ must be equal to $\tau * \sigma$. □

Let $\widetilde{\mathbb{T}}$ be the complex torus $\mathbb{C}^{2\times n}/\mathbb{Z}^{2\times n}$ of dimension $2n$. Identify the characters of $\widetilde{\mathbb{T}}$ with the multi-weights $(\mathbf{k}_1, \mathbf{k}_2) \in \mathbb{Z}^{2n}$ by

$$(\mathbf{k}_1, \mathbf{k}_2)((t_{1,1}, \ldots, t_{1,n}, t_{2,1}, \ldots, t_{2,n})) = \prod_{i,j} t_{i,j}^{k_{i,j}}.$$

We define an action of $\widetilde{\mathbb{T}}$ on semistandard tableaux. If $\mathbf{wt}(\tau) = (\mathbf{k}_1, \mathbf{k}_2)$ and $t \in \widetilde{\mathbb{T}}$, let

$$t \cdot \tau = \mathbf{wt}(\tau)(t)\,\tau.$$

Extend the action to be linear on R.

The torus $\widetilde{\mathbb{T}}$ does not act effectively on R. For $k = 1, 2$ and $1 \leq \ell \leq n$ let $T_{k,\ell}$ denote the one dimensional subgroup of $\widetilde{\mathbb{T}}$ where all components $t_{i,j}$ except $t_{k,\ell}$ are equal to 1. The subgroups $T_{2,1}$ and $T_{1,n}$ act trivially since in any semistandard tableau the index 1 does not appear in the second row nor does the index n appear in the first row. Furthermore, since the tableaux also have the same number of entries in the first and second rows, the one-dimensional subgroup $(t, t, \ldots, t, t^{-1}, t^{-1}, \ldots, t^{-1})$ acts trivially. Let $\mathbb{T}$ denote the quotient of $\widetilde{\mathbb{T}}$ by these three one-dimensional subgroups. Now the character lattice $\chi^*(\mathbb{T})$ is canonically a sub-lattice of $\chi^*(\widetilde{\mathbb{T}})$ of dimension $2n-3$, given by the equations

$$k_{2,1} = k_{1,n} = 0 \quad \text{and} \quad \sum_\ell k_{1,\ell} = \sum_\ell k_{2,\ell}.$$

Now $\mathbb{T}$ acts effectively on R (and $\mathrm{gr}_1(R)$) as a collection $\mathbb{C}$ module homomorphisms. Furthermore, $\mathbb{T}$ acts by ring homomorphisms on $\mathrm{gr}_1(R)$, since

$$(t \cdot \tau)(t \cdot \sigma) = \mathbf{wt}(\tau)(t)\mathbf{wt}(\sigma)(t)\tau\sigma = \mathbf{wt}(\tau * \sigma)(t)(\tau * \sigma) = t \cdot (\tau * \sigma).$$

Let $\mathrm{Gr}_2(\mathbb{C}^n)_0$ denote the associated toric variety $\mathrm{Proj}(\mathrm{gr}_1(R))$, which contains the projective quotient $\mathbb{T}/\Delta(\mathbb{T})$ of $\mathbb{T}$ as an open subset. Here $\Delta(\mathbb{T})$ denotes the image in $\mathbb{T}$ of the diagonal subgroup of $\widetilde{\mathbb{T}}$.

Now we shall construct $\mathrm{Gr}_2(\mathbb{C}^n)_0$ as the special fiber of a degeneration of the ring R by choosing a $\mathbb{T}$ stable $\mathbb{N}$-filtration as in [AB].

Definition 5.4. *Let $C \geq 2$ be an integer. For each semistandard 2 by Na tableau* $\tau = \begin{array}{|c|c|}\hline \tau_{1,1} & \tau_{1,2} \\ \hline \tau_{2,1} & \tau_{2,2} \\ \hline \end{array} \cdots \begin{array}{|c|}\hline \tau_{1,Na} \\ \hline \tau_{2,Na} \\ \hline \end{array}$ *define*

$$C_\tau = \sum_{\ell=1}^{m} \tau_{1,\ell} + C\tau_{2,\ell}.$$

For each non-negative integer m let

$$F_m(R) = \bigoplus_{C_\tau \leq m} \mathbb{C}[\tau] \subset R.$$

Let $\mathrm{gr}_2(R)$ be the associated graded ring, with graded components $F_m(R)/F_{m-1}(R)$. We shall say that the tableau τ has LG-degree C_τ (since this is the construction of Lakshmibai-Gonciulea).

Now R is both filtered *and* graded. The filtration is given by the LG-degrees of tableaux and the grading comes from the decomposition of R into sections of tensor powers of $\mathcal{L}^a_{2,n}$. The filtration and grading are not compatible, since the set of elements of R of degree k and less is not a union of LG–filtration levels.

Now $\mathrm{gr}_2(R)$ is graded in two ways; we have the standard degree $\deg(\tau) = N$ where τ is 2 by Na and there is also the LG-degree C_τ. The reason why the standard grading (by the standard degree) is well-defined on $\mathrm{gr}_2(R)$ is that the standard degree of a tableau may be computed from its multi-weight.

We will use the following notation for the rest of the paper. We have an identification of vector spaces

$$R = \bigoplus \mathbb{C}\tau_i$$

where τ_i runs over the set of semistandard tableau with two rows. If the LG-degree of τ_i is m then we let $\bar{\tau}_i$ denote the corresponding (basis) element of $\mathrm{gr}_2(R)^{(m)} = F_m(R)/F_{m-1}(R)$. Thus as vector spaces we have

$$\mathrm{gr}_2(R) = \bigoplus \mathbb{C}\bar{\tau}_i.$$

Lemma 5.5. *Multiplication within* $\mathrm{gr}_2(R)$ *is given by* $\bar{\tau} \cdot_{\mathrm{gr}_2(R)} \bar{\sigma} = \overline{\tau * \sigma}$.

Proof. Suppose that τ,σ are semistandard tableaux. Let $\tau\sigma = \tau * \sigma + \sum_i c_i\gamma_i$ be the product of τ and σ computed in R, where the γ_i are semistandard. We shall show that $C_{\gamma_i} < C_{\tau*\sigma}$ for each i. Consider the basic straightening relations

$$\boxed{\begin{array}{c} i_1 \\ \hline i_4 \end{array}} \cdot \boxed{\begin{array}{c} i_2 \\ \hline i_3 \end{array}} = \boxed{\begin{array}{c} i_1 \\ \hline i_3 \end{array}} \cdot \boxed{\begin{array}{c} i_2 \\ \hline i_4 \end{array}} - \boxed{\begin{array}{c} i_1 \\ \hline i_2 \end{array}} \cdot \boxed{\begin{array}{c} i_3 \\ \hline i_4 \end{array}}$$

where $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$. (We are considering the above 2 by 2 tableaux as factors of larger tableaux which actually lie within R.) Let

$$\alpha = \boxed{\begin{array}{c} i_1 \\ \hline i_4 \end{array}} \cdot \boxed{\begin{array}{c} i_2 \\ \hline i_3 \end{array}}, \quad \alpha_1 = \boxed{\begin{array}{c} i_1 \\ \hline i_3 \end{array}} \cdot \boxed{\begin{array}{c} i_2 \\ \hline i_4 \end{array}}, \quad \alpha_2 = \boxed{\begin{array}{c} i_1 \\ \hline i_2 \end{array}} \cdot \boxed{\begin{array}{c} i_3 \\ \hline i_4 \end{array}}.$$

We have

$$\begin{aligned} C_\alpha &= i_1 + i_2 + C(i_3 + i_4), \\ C_{\alpha_1} &= i_1 + i_2 + C(i_3 + i_4), \\ C_{\alpha_2} &= i_1 + i_3 + C(i_2 + i_4). \end{aligned}$$

Now $C_\alpha = C_{\alpha_1}$. Since $i_3 > i_2$ and $C \geq 2$,

$$C_\alpha - C_{\alpha_2} = (i_2 - i_3) + C(i_3 - i_2) = (C-1)(i_3 - i_2) > 0.$$

Since the equation $\tau\sigma = \tau * \sigma + \sum_i c_i\gamma_i$ is gotten from a sequence of basic straightening relations, and $\tau * \sigma$ is the final leftmost term, it is clear from the above that each $C_{\gamma_i} < C_{\tau*\sigma}$. Hence the product of $\bar{\tau}$ and $\bar{\sigma}$ in $\mathrm{gr}_2(R)$ is $\overline{\tau * \sigma}$. ☐

Corollary 5.6. *The* $\mathbb{C}$*–algebras* $\mathrm{gr}_1(R)$ *and* $\mathrm{gr}_2(R)$ *are isomorphic.*

Remark 5.7. *Of course* $\mathrm{gr}_1(R)$ *and* $\mathrm{gr}_2(R)$ *are not isomorphic as graded* $\mathbb{C}$*–algebras: the grading of* $\mathrm{gr}_1(R)$ *is by a cone in the character lattice of* $\mathbb{T}$*, whereas the grading of* $\mathrm{gr}_2(R)$ *is a grading by the natural numbers.*

It is well-known that if R is a filtered ring then there is a one-parameter flat degeneration with special fiber the associated graded ring of R. We sketch one way to do this, borrowed from [AB], using the Rees algebra. Let z be an indeterminate and let $\mathcal{R}$ be the Rees algebra

$$\mathcal{R} = \bigoplus_{m=0}^{\infty} F_m(R)z^m \subset R[z].$$

Theorem 5.8. *(Alexeev–Brion* [AB]*)*

- $\mathcal{R}$ *is flat over* $\mathbb{C}[z]$.
- $\mathcal{R} \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}] \cong R[z, z^{-1}]$.
- $\mathcal{R} \otimes_{\mathbb{C}[z]} \mathbb{C}[z]/(z) \cong \mathrm{gr}_2(R)$.

5.2. **Restriction of the degeneration to torus invariants.** The small torus $T \cong (\mathbb{C}^\times)^n$ acts on R by ring homomorphisms, by

$$t \cdot \tau = \chi^a_{\mathbf{r}}(t)\chi_{\mathbf{s}}(t^{-1})\tau,$$

where τ is a 2 by Na semistandard tableau and $\mathbf{s} = (s_1, s_2, \dots, s_n)$ where s_i is the number of occurrences of i in τ. We extend the action of the small torus T to $\mathcal{R}$ by making the indeterminate z an invariant. Let $\mathcal{R}^T$ denote the torus invariants.

Theorem 5.9.

- *$\mathcal{R}^T$ is flat over $\mathbb{C}[z]$.*
- *$\mathcal{R}^T \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}] \cong R^T[z, z^{-1}]$.*
- *$\mathcal{R}^T \otimes_{\mathbb{C}[z]} \mathbb{C}[z]/(z) \cong \mathrm{gr}_2(R)^T$.*

Proof. Flatness follows from the fact that restriction to T-invariants is an exact functor of $\mathbb{C}[z]$-modules. The other properties are immediate. □

Recall that the T-invariant sub-module $R_{\mathbf{r}} = R^T$ of R is generated by symbols $\bar{\tau}$ for those tableaux τ, with weight a multiple of $\mathbf{r}$. We still say that the LG-degree of $\bar{\tau}$ in R^T is C_τ as above. However, the standard degrees of τ and $\bar{\tau}$ are no longer the number of columns of τ, but rather the number N whenever τ has shape 2 by Na (where $a = |\mathbf{r}|/2$), since then τ corresponds to a section s_τ of the Nth tensor power of the line bundle $\mathcal{L}_{2,n}^{\otimes a}$ over $\mathrm{Gr}_2(\mathbb{C}^n)$, see §3.

Definition 5.10. *We let $(R_{\mathbf{r}})_0$ denote the projective coordinate ring of the toric fiber of the above induced toric degeneration or $R_{\mathbf{r}} = R^T$. Thus*

$$(R_{\mathbf{r}})_0 \cong \mathrm{gr}_2(R)^T.$$

The main point of what follows is that the graded $\mathbb{C}$–algebra $(R_{\mathbf{r}})_0$ can be identified with the semigroup algebra of the graded semigroup $S_{\mathbf{r}}$ of lattice points in the family of integral dilates $D(N\mathbf{r}), N \geq 0$, of the diagonal polytope $D(\mathbf{r})$. Here we define the degree of a lattice point $\mathbf{d} \in D(N\mathbf{r})$ to be N. This definition of degree makes the semi-group algebra $\mathbb{C}[S_{\mathbf{r}}]$ into a graded $\mathbb{C}$-algebra with relations generated by binomial relations $x_{\mathbf{d}_1}x_{\mathbf{d}_2} - x_{\mathbf{d}'_1}x_{\mathbf{d}'_2}$ where $\mathbf{d}_1 + \mathbf{d}_2 = \mathbf{d}'_1 + \mathbf{d}'_2$ in $S_{\mathbf{r}}$. Here we use the symbol $x_{\mathbf{d}}$ to be the element of $\mathbb{C}[S_{\mathbf{r}}]$ corresponding to $\mathbf{d} \in D(N\mathbf{r})$.

Proposition 5.11. *We have an isomorphism of graded $\mathbb{C}$–algebras*

$$(R_{\mathbf{r}})_0 \cong \mathbb{C}[S_{\mathbf{r}}].$$

Proof. First of all $\mathrm{gr}_2(R^T)$ and R^T are isomorphic as $\mathbb{C}$-modules. A basis for the Nth graded summand is given by the set $\bar{\tau}$ for τ a semistandard 2 by Na tableaux of weight $N\mathbf{r}$. These correspond to the integral points occurring in $SS(Na\varpi_2, N\mathbf{r})$ for $N \geq 0$. The map $\Psi \circ \Phi^{-1}$ is a degree–preserving bijection between $S_{\mathbf{r}}$ and this basis. We claim this map induces a graded ring homomorphism. Multiplication in $\mathrm{gr}_2(R^T)$ is given by $\bar{\tau}\bar{\sigma} = \overline{\tau * \sigma}$. But the multi-weight of $\tau * \sigma$ is the sum of the multi-weights of τ and σ by definition. Hence multiplication in $\mathrm{gr}_2(R^T)$ corresponds to addition of multi-weights, which is the (commutative) semigroup structure of the integral points in the various $SS(Na\varpi_2, N\mathbf{r})$. The map $\Psi \circ \Phi^{-1}$ is additive and so it preserves the semigroup structure. Therefore it induces to a ring homomorphism. □

We will identify the rings $(R_{\mathbf{r}})_0$ and $\mathbb{C}[S_{\mathbf{r}}]$ using the above isomorphism henceforth.

6. The projective coordinate ring of the toric fiber $(M_{\mathbf{r}})_0$

Fix $\mathbf{r} \in (\mathbb{Z}^+)^n$. We shall show in this section that the degree one and degree two elements of the toric ring $\mathbb{C}[S_{\mathbf{r}}]$ generate $\mathbb{C}[S_{\mathbf{r}}]$. Furthermore we will show that the ideal of $\mathbb{C}[S_{\mathbf{r}}]$ (and hence $(R_{\mathbf{r}})_0$) is generated by relations of degrees two, three, and four. It follows immediately from the proofs given that $\mathbb{C}[S_{2\mathbf{r}}] \cong (R_{2\mathbf{r}})_0$ has a presentation by degree one generators and quadratic relations.

6.1. Generators for $\mathbb{C}[S_{\mathbf{r}}]$.

Definition 6.1. *If A, B are subsets of a vector space, their Minkowski sum $A+B$ is*

$$A + B = \{a + b \mid a \in A,\, b \in B\}.$$

Additionally when the vector space contains a given lattice let $A(\mathbb{Z})$ denote the lattice points which are contained in A.

Definition 6.2. *For each positive integer k, let $D(k) = D(k\mathbf{r})$.*

Lemma 6.3. *For each positive integer m,*

$$D(2m + 1)(\mathbb{Z}) = D(1)(\mathbb{Z}) + D(2m)(\mathbb{Z}).$$

Proof. Given $\mathbf{d} \in D(2m+1)(\mathbb{Z})$ we shall construct $\mathbf{d}' \in D(1)(\mathbb{Z})$ in the proximity of $\mathbf{d}/(2m+1)$ such that $\mathbf{d} - \mathbf{d}' = \mathbf{d}'' \in D(2m)(\mathbb{Z})$. An element $\mathbf{d}' = (d'_1, \ldots, d'_{n-1})$ in $D(1)$ is a lattice point iff each d'_i is an integer with the same parity as $(r_1 + \cdots + r_i)$; that is $d'_i \equiv (r_1 + \cdots + r_i) \mod 2$. On the other hand $\mathbf{d}'' = (d''_1, \ldots, d''_{n-1})$ in $D(2m)$ is a lattice point iff each d''_i is an even integer. Define

$$d'_i = k, \text{ such that } k \in \mathbb{Z},\ k \equiv (r_1 + \cdots + r_i) \mod 2, \text{ and } \left|k - \frac{d_i}{2m+1}\right| < 1.$$

For this to be well-defined we need to show that k exists and is unique. Uniqueness follows immediately since there can be only one integer of a given parity in an open interval of length 2. For existence we must check that $d_i/(2m+1)$ does not have opposite parity to $(r_1 + \cdots + r_i)$, since in this case there is no integer of the correct parity less than one unit from $d_i/(2m+1)$. But $d_i \equiv (r_1 + \cdots + r_i) \mod 2$ since $\mathbf{d} \in D(2m+1)(\mathbb{Z})$ is a lattice point and $(2m+1)$ is odd. Hence if $d_i/(2m+1)$ is an integer then $d_i/(2m+1) \equiv (r_1 + \cdots + r_i) \mod 2$ as well since $2m+1$ is odd. Therefore the parity condition for $\mathbf{d}'$ is satisfied.

Let $\mathbf{d}'' = \mathbf{d} - \mathbf{d}'$. Note that each d''_i is even since d_i and d'_i have the same parity. We have that in fact d''_i is the nearest even integer to $2md_i/(2m+1)$. Thus the lattice point conditions are satisfied. Now we only need to show that $\mathbf{d}' \in D(1)$ and $\mathbf{d}'' \in D(2m)$.

Since $d_1 = (2m+1)r_1$ and $d_{n-1} = (2m+1)r_{n-1}$ we have that $d'_1 = r_1$, $d''_1 = 2mr_1$, $d'_{n-1} = r_{n-1}$, and $d''_{n-1} = 2mr_{n-1}$. It remains to show the triangle inequalities,

1. $d'_{i-1} \leq d'_i + r_i$, $d''_{i-1} \leq d''_i + 2mr_i$,
2. $d'_i \leq d'_{i-1} + r_i$, $d''_i \leq d''_{i-1} + 2mr_i$,
3. $r_i \leq d'_{i-1} + d'_i$, $2mr_i \leq d''_{i-1} + d''_i$.

Let $\ell = 2m + 1$. We have that $|d_{i-1} - d_i| \leq \ell r_i$, hence $|d_{i-1}/\ell - d_i/\ell| \leq r_i$. Recall that d'_{i-1} is the nearest integer to d_{i-1}/ℓ with parity $(r_1 + \cdots + r_{i-1}) \mod 2$ and d'_i is the nearest integer to d_i/ℓ with parity $(r_1 + \cdots + r_i) \mod 2$. The distance between d_{i-1}/ℓ and d_i/ℓ is at most r_i and also $d'_{i-1} - d'_i \equiv r_i \mod 2$. We also have that $|d'_{i-1} - d_{i-1}/\ell| < 1$ and $|d'_i - d_i/\ell| < 1$. Therefore $|d'_{i-1} - d'_i| < r_i + 2$ and consequently

$|d'_{i-1}-d'_i| \le r_i+1$. But $|d'_{i-1}-d'_i| \ne r_i+1$ because $d'_{i-1}-d'_i \equiv r_i \mod 2$. It follows that $|d'_{i-1}-d'_i| \le r_i$. Also $|d''_{i-1} - \frac{\ell-1}{\ell}d_{i-1}| < 1$ and $|d''_i - \frac{\ell-1}{\ell}d_i| < 1$. Therefore $|d''_{i-1} - d''_i| < (\ell-1)r_i + 2$, so $|d''_{i-1} - d''_i| \le (\ell-1)r_i + 1$. But $d''_{i-1} - d''_i$ is even and $(\ell-1)r_i + 1$ is odd since ℓ is odd. Therefore $|d''_{i-1} - d''_i| \le (\ell-1)r_i = 2mr_i$. Therefore both (1) and (2) hold.

We have $d'_{i-1} + d'_i > d_{i-1}/\ell + d_i/\ell - 2 \ge r_i - 2$. Thus $d'_{i-1} + d'_i \ge r_i - 1$. But $d'_{i-1} + d'_i \equiv r_i \mod 2$ so $d'_{i-1} + d'_i \ne r_i - 1$. Thus $d'_{i-1} + d'_i \ge r_i$. We have $d''_{i-1} + d''_i > \frac{\ell-1}{\ell}(d_{i-1} + d_i) - 2 \ge (\ell-1)r_i - 2$. Thus $d''_{i-1} + d''_i \ge (\ell-1)r_i - 1$. But $d''_{i-1} + d''_i$ is even and ℓ is odd so $d''_{i-1} + d''_i \ne (\ell-1)r_i - 1$. Thus $d'_{i-1} + d'_i \ge (\ell-1)r_i = 2mr_i$. Therefore (3) holds. ☐

Lemma 6.4. *For each positive integer m,*

$$D(2m)(\mathbb{Z}) = \underbrace{D(2)(\mathbb{Z}) + \cdots + D(2)(\mathbb{Z})}_{m}.$$

Proof. We show the lemma by induction on $m \ge 1$. The case $m = 1$ is a tautology. Suppose that $m \ge 2$. We show that

$$D(2m)(\mathbb{Z}) = D(2)(\mathbb{Z}) + D(2m-2)(\mathbb{Z}).$$

Suppose $\mathbf{d} = (d_1, \ldots, d_{n-1}) \in D(2m)(\mathbb{Z})$. We construct

$$\mathbf{d}' \in D(2)(\mathbb{Z}), \quad \mathbf{d}'' \in D(2m-2)(\mathbb{Z})$$

where $\mathbf{d}'+\mathbf{d}'' = \mathbf{d}$, by placing $\mathbf{d}'$ in the proximity of $\mathbf{d}/m$. Recall that the integrality condition is that the components of $\mathbf{d}$, $\mathbf{d}'$, and $\mathbf{d}''$ are even integers.

Let $e^- : \mathbb{R} \to 2\mathbb{Z}$ be the function which assigns the nearest even integer, where odd integers $2t+1$ are mapped to $2t$. To be concise,

$$e^-(x) = \min\{k \in 2\mathbb{Z} : k+1 \ge x\}.$$

Similarly let $e^+ : \mathbb{R} \to 2\mathbb{Z}$ assign the nearest even integer where odd integers $2t+1$ are mapped to $2t+2$,

$$e^+(x) = \max\{k \in 2\mathbb{Z} : k-1 \le x\}.$$

We will often use the following properties of e^- and e^+:

- each of e^- and e^+ is weakly increasing.
- if $k \in 2\mathbb{Z}$, then $e^\pm(x+k) = e^\pm(x) + k$.
- $e^+(-x) = -e^-(x)$.
- if $x + y \in 2\mathbb{Z}$, then $e^+(x) + e^-(y) = x + y$.
- if $x + y \ge k \in 2\mathbb{Z}$, then $e^+(x) + e^-(y) \ge k$.

Let $\mathbf{d} = (d_1, \ldots, d_{n-1}) \in D(2m)(\mathbb{Z})$. Let

$$\mathcal{J}^0_{\mathbf{d}} = \{i \mid d_{i-1}/m \text{ and } d_i/m \text{ are odd integers}, d_{i-1} + d_i = 2mr_i,\ 2 \le i \le n-1\},$$

$$\mathcal{J}^1_{\mathbf{d}} = \{i \mid d_{i-1} \le 2mr_i,\ d_i \le 2mr_i,\ 2 \le i \le n-1\}.$$

Clearly $\mathcal{J}^0_{\mathbf{d}} \subset \mathcal{J}^1_{\mathbf{d}}$. Let $\mathcal{J}_{\mathbf{d}}$ be such that $\mathcal{J}^0_{\mathbf{d}} \subset \mathcal{J}_{\mathbf{d}} \subset \mathcal{J}^1_{\mathbf{d}}$. Let $\{i_1, \ldots, i_s\} = \mathcal{J}_{\mathbf{d}}$ such that $i_t < i_{t+1}$ for all t, and set $i_0 = 1$ and $i_{s+1} = n$.
Let $\mathbf{d}' = (d'_1, \ldots, d'_{n-1}) \in (2\mathbb{Z})^{n-1}$ be

$$d'_i = \begin{cases} e^-(d_i/m) & \text{for } i_{2t} \le i < i_{2t+1}, 2t \le s, \\ e^+(d_i/m) & \text{for } i_{2t+1} \le i < i_{2t+2}, 2t+1 \le s. \end{cases}$$

Let $\mathbf{d}'' = (d''_1, \dots, d''_{n-3}) \in (2\mathbb{Z})^{n-1}$ be

$$d''_i = \begin{cases} e^+((m-1)d_i/m) & \text{for } i_{2t} \le i < i_{2t+1}, 2t \le s, \\ e^-((m-1)d_i/m) & \text{for } i_{2t+1} \le i < i_{2t+2}, 2t+1 \le s. \end{cases}$$

We will show that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$, $\mathbf{d}' \in D(2)(\mathbb{Z})$, and $\mathbf{d}'' \in D(2m-2)(\mathbb{Z})$. Note that $e^\pm(x) + e^\mp(y) = k$ whenever $x + y = k$ and $k \in 2\mathbb{Z}$. We have that $d_i/m + (m-1)d_i/m = d_i \in 2\mathbb{Z}$ for all i, so

$$d'_i + d''_i = e^\pm(d_i/m) + e^\mp((m-1)d_i/m) = d_i.$$

Thus $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$.

We show that $\mathbf{d}' \in D(2)$. The proof that $\mathbf{d}'' \in D(2m-2)$ is similar. Since $d_1 = 2mr_1$ and $d_{n-1} = 2mr_{n-1}$, we have $d'_1 = 2r_1$ and $d'_{n-1} = 2r_{n-1}$. Now suppose $2 \le i \le n-1$. We must show the three triangle inequalities that define $D(2)$:

(1) $d'_i \le d'_{i-1} + 2r_i$,
(2) $d'_{i-1} \le d'_i + 2r_i$,
(3) $2r_i \le d'_{i-1} + d'_i$.

Suppose that $i \notin \mathcal{J}_\mathbf{d}$. We have $d'_{i-1} = e^\pm(d_{i-1}/m)$ and $d'_{i-1} = e^\pm(d_i/m)$ (the same function is applied to each). The functions e^- and e^+ are weakly increasing, and since $d_i/m \le d_{i-1}/m + 2r_i$ we have that

$$d'_i = e^\pm(d_i/m) \le e^\pm(d_{i-1}/m + 2r_i) = e^\pm(d_{i-1}/m) + 2r_i = d'_{i-1} + 2r_i$$

so (1) holds. Similarly inequality (2) holds. Since $i \notin \mathcal{J}^0_\mathbf{d}$ we know that either $d_{i-1} + d_i > 2mr_i$ or one of d_{i-1}/m or d_i/m is not an odd integer. Suppose $d_{i-1} + d_i > 2mr_i$. We have that $d'_{i-1} + d'_i \ge d_{i-1}/m + d_i/m - 2 > 2r_i - 2$. But $d'_{i-1} + d'_i$ is even so $d'_{i-1} + d'_i \ge 2r_i$ and (3) holds. Suppose that one of d_{i-1}/m or d_i/m is not an odd integer and $d_{i-1} + d_i = 2mr_i$. Without loss of generality suppose that d_{i-1}/m is not odd. Then $e^+(d_{i-1}/m) = e^-(d_{i-1}/m)$. Since the sum $d_{i-1}/m + d_i/m = 2r_i$ is even, we have that $e^+(d_{i-1}/m) + e^-(d_i/m) = 2r_i$. Now

$$d'_{i-1} + d'_i \ge e^-(d_{i-1}/m) + e^-(d_i/m) = e^+(d_{i-1}/m) + e^-(d_i/m) = 2r_i,$$

and again (3) holds.

Suppose that $i \in \mathcal{J}_\mathbf{d}$. Hence $i \in \mathcal{J}^1_\mathbf{d}$ and so $d_{i-1}/m \le 2r_i$ and $d_i/m \le 2r_i$. Therefore each of $d'_{i-1} \le 2r_i$ and $d'_i \le 2r_i$. We have $d'_{i-1} = e^\pm(d_{i-1}/m)$ and $d'_i = e^\mp(d_i/m)$. Whenever two numbers x, y satisfy $x + y \ge k \in 2\mathbb{Z}$, then $e^\pm(x) + e^\mp(y) \ge k$, thus (3) holds since $d_{i-1}/m + d_i/m \ge 2r_i$. Suppose that $d'_{i-1} = e^+(d_{i-1}/m)$ and so $d'_i = e^-(d_i/m)$. We show that (1) holds. We have that

$$2r_i \ge d_{i-1}/m - d_i/m \ge d'_{i-1} - d'_i - 2,$$

so if (1) fails then $d'_{i-1} - d'_i = 2r_i + 2$. But $d'_{i-1} + d'_i \ge 2r_i$ since we have shown (3) already, and hence we get that $d'_{i-1} \ge 2r_i + 1$, a contradiction with $d'_{i-1} \le 2r_i$. The other cases are similar. □

Theorem 6.5. *The toric ring $\mathbb{C}[S_\mathbf{r}]$ is generated by elements of degrees one and two; furthermore, $(R_{2\mathbf{r}})_0$ is generated by elements of degree one.*

Proof. The first statement is a direct consequence of Lemmas 6.3 and 6.4. The second statement follows from Lemma 6.4. □

6.2. The word problem for $S_{\mathbf{r}}$ and the relations for $\mathbb{C}[S_{\mathbf{r}}]$.

We will actually solve the presentation problem for $\mathbb{C}[S_{\mathbf{r}}]$ by solving the seemingly more difficult *word problem* for the graded semigroup $S_{\mathbf{r}} = \cup_{N\geq 0} D(N)(\mathbb{Z})$. Our technique is to define a normal form for words in $S_{\mathbf{r}}$ expressed in terms of degree one and degree two elements, then show that any word can be brought into normal form by a sequence of quadric relations.

Definition 6.6. *Let* $\xi_{2m+1} : D(2m+1)(\mathbb{Z}) \to D(1)(\mathbb{Z})$ *be given by* $\xi_{2m+1}(\mathbf{d}) = \mathbf{d}'$ *where* $\mathbf{d}'$ *is as in the proof of Lemma 6.3.*

Definition 6.7. *Let A be an integer matrix. Let the jth column of A be denoted $c_j(A)$. If each column of A belongs to either $D(1)(\mathbb{Z})$ or $D(2)(\mathbb{Z})$ then we say that A is a D-matrix. (D-matrices represent monomials in $\mathbb{C}[S_{\mathbf{r}}]$ which are products of degree one and degree two generators.) The elements of $D(1)(\mathbb{Z})$ (resp. $D(2)(\mathbb{Z})$) are said to have degree one (resp. two). We define* $\deg(A)$ *to be the sum of the degrees of the columns of A whenever A is a D-matrix.*

Definition 6.8. *(normal form) Suppose that A is a D-matrix. Let*

$$\mathbf{a} = (a_1, \dots, a_{n-1}) = \sum_j c_j(A) \in D(\deg(A))(\mathbb{Z}).$$

Suppose that $\deg(A) = 2m$ *is even. Let $\mathcal{J}_{\mathbf{a}} = \{i_1, \dots, i_k\}$ be the set of all i, $2 \leq i \leq n-1$, such that $a_{i-1} \leq 2mr_i$ and $a_i \leq 2mr_i$, where $i_t < i_{t+1}$ for all t, $1 \leq t < k$. Let $i_0 = 1$ and let $i_{k+1} = n$. (Note that $\mathcal{J}_{\mathbf{a}} = \mathcal{J}^1_{\mathbf{a}}$ as in the proof of Lemma 6.4.) Suppose A has the following properties:*

- (N0) *Each column of A has degree two.*
- (N1) *For each i the row entries $a_{i,j}$ satisfy $|a_{i,j} - a_i/m| < 2$.*
- (N2) *For $i_{2t} \leq i < i_{2t+1}$, row i is weakly increasing. For $i_{2t+1} \leq i < i_{2t+2}$, row i is weakly decreasing.*

Then we say that A is in normal form.

Now suppose that $\deg(A) = 2m+1$ *is odd. Then we say that A is in normal form if the first column of A is equal to $\xi_{2m+1}(\mathbf{a})$ and if the matrix A' obtained from A by removing the first column is in normal form.*

Example 6.9. *Here is an example of two D-matrices A and B where B is the normal form representative of A. This for the case $\mathbf{r} = (1, 1, 1, 1, 1, 1, 1, 1)$.*

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 4 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

The lattice point given by A and B is $\mathbf{d} = (5, 4, 3, 8, 7, 6, 5) \in D(5)(\mathbb{Z})$; this is the sum of the columns. According to the definition of normal form, the first column should be equal to $\xi_5(\mathbf{d}) = (1, 0, 1, 2, 1, 2, 1) \in D(1)(\mathbb{Z})$; this is indeed the first column of B. Now $\mathbf{d}' = \mathbf{d} - \xi_5(\mathbf{d}) = (4, 4, 2, 6, 6, 4, 4) \in D(4)(\mathbb{Z})$. Note that $\mathcal{J}_{\mathbf{d}'} = \{2, 3, 7\}$. So the two remaining degree two columns of B should have the property that rows $1, 3, 4, 5, 6$ weakly increase and rows $2, 7$ weakly decrease. Furthermore, the entries *in a given row shouldn't differ by more than 2. This is true for the last two columns of B, and so B is in normal form. Hence B the normal representative among the set of all tuples of degree one and degree two lattice points whose components sum to* $\mathbf{d} = (5, 4, 3, 8, 7, 6, 5)$.

Lemma 6.10. *(uniqueness) For any* $\mathbf{a} = (a_1, \ldots, a_{n-1}) \in D(\ell)(\mathbb{Z})$ *there is at most one matrix A in normal form such that the columns of A sum to* $\mathbf{a}$.

Proof. Suppose $\ell = 2m = \deg(A)$ is even and A is in normal form. Then each column of A is degree two so all the matrix entries are even integers and there are m columns. For each i let k_i be an even integer such that $k_i \leq a_i/m \leq k_i + 2$. By condition (N1) we know that each $a_{i,j}$ is either k_i or $k_i + 2$. Let t_i be the number of $a_{i,j}$ equal to k_i. Then, $t_i k_i + (m - t_i)(k_i + 2) = a_i$, so $2t_i = m(k_i + 2) - a_i$, and thus t_i is determined by the value of a_i. Finally the monotonicity condition (N2) determines each $a_{i,j}$.

Suppose $\ell = 2m + 1 = \deg(A)$ is odd and A is in normal form. The first column of A must be equal to $\xi_{2m+1}(\mathbf{a})$ so it is determined. Now the matrix A' which is A with the first column removed is degree $2m$ and is in normal form, so its entries are determined by the argument given above for matrices of even degree. □

Definition 6.11. *We say that two D-matrices A and B are equivalent if the sum of the columns of A is equal to the sum of the columns of B. (Note that each equivalence class contains at most one representative in normal form by the above lemma — later we will see that a normal form representative always exists.)*

Definition 6.12. *Let A be a D-matrix and let $\mathbf{d}_1$ and $\mathbf{d}_2$ be two different columns of A. We define operations of types (F2), (F3), and (F4) as follows.*

- (F2) *If* $\deg(\mathbf{d}_1) = \deg(\mathbf{d}_2) = 1$ *then remove columns* $\mathbf{d}_1$ *and* $\mathbf{d}_2$ *and place* $\mathbf{d}_1 + \mathbf{d}_2$ *as the last column.*
- (F3) *If* $\deg(\mathbf{d}_1) = 1$ *and* $\deg(\mathbf{d}_2) = 2$, *let* $\mathbf{d} = \mathbf{d}_1 + \mathbf{d}_2$. *If* $\mathbf{d}_1$ *precedes* $\mathbf{d}_2$ *then replace* $\mathbf{d}_1$ *with* $\xi_3(\mathbf{d})$ *and replace* $\mathbf{d}_2$ *with* $\mathbf{d} - \xi_3(\mathbf{d})$. *If* $\mathbf{d}_2$ *precedes* $\mathbf{d}_1$ *then replace* $\mathbf{d}_2$ *with* $\xi_3(\mathbf{d})$ *and replace* $\mathbf{d}_1$ *with* $\mathbf{d} - \xi_3(\mathbf{d})$.
- (F4) *If* $\deg(\mathbf{d}_1) = \deg(\mathbf{d}_2) = 2$ *then choose* $\mathbf{d}'_1$ *and* $\mathbf{d}'_2$ *each of degree two such that* $\mathbf{d}'_1 + \mathbf{d}'_2 = \mathbf{d}_1 + \mathbf{d}_2$. *Replace* $\mathbf{d}_1$ *with* $\mathbf{d}'_1$ *and replace* $\mathbf{d}_2$ *with* $\mathbf{d}'_2$.

Lemma 6.13. *Suppose that A is a D-matrix. Then there is a finite sequence $(A_0, A_1, \ldots, A_p)$ of equivalent D-matrices where $A_0 = A$, the final matrix A_p is in normal form, and for each i the matrix A_{i+1} is obtained from A_i by a single operation of type (F2), (F3), or (F4). (In the special case that A is even degree and all columns are degree two then all these operations are of type (F4).)*

Proof. First note that (F2) operations can be applied to any pair of degree one columns until either every column is degree two (when $\deg(A)$ is even) or there is only one column of degree one (when $\deg(A)$ is odd.) Assume now that A has at most one column of degree one.

Suppose that $\deg(A) = 2m$ is even, so that each column of A is degree two and there are m columns. For $\mathbf{d} = (d_1, \ldots, d_{n-1}) \in D(4)(\mathbb{Z})$, let $\mathcal{J}_{\mathbf{d}} = \mathcal{J}^0_{\mathbf{d}}$ where $\mathcal{J}^0_{\mathbf{d}}$ is as in the proof of Lemma 6.4, and let $f^-, f^+ : D(4)(\mathbb{Z}) \to D(2)(\mathbb{Z})$ be given by $f^-(\mathbf{d}) = \mathbf{d}'$ and $f^+(\mathbf{d}) = \mathbf{d}''$ where $\mathbf{d}'$ and $\mathbf{d}''$ are again as in the proof of Lemma 6.4. Let an f^-, f^+ operation be the following. Choose j, j' such that $1 \leq j < j' \leq m$. Replace columns $c_j(A), c_{j'}(A)$ with $f^-(c_j(A) + c_{j'}(A)), f^+(c_j(A) + c_{j'}(A))$. Then,

$$c_j(A) + c_{j'}(A) = f^-(c_j(A) + c_{j'}(A)) + f^+(c_j(A) + c_{j'}(A)),$$

so an f^-, f^+ operation is of type (F4). Let $\mathbf{a} = (a_1, \ldots, a_{n-1}) = \sum_j c_j(A)$. We claim that after a finite number of f^-, f^+ operations, each entry $a_{i,j}$ of row i satisfies $|a_{i,j} - a_i/m| < 2$. The ith row (which has even integer entries that sum to a_i) is of minimal distance (using the standard Euclidean metric) from the constant vector $(a_i/m, \ldots, a_i/m)$ iff $|a_{i,j} - a_i/m| < 2$ for each j. Suppose x and y are even integers. It is easy to check that

$$(x - a_i/m)^2 + (y - a_i/m)^2 \geq \left(e^+\left(\frac{x+y}{2}\right) - a_i/m\right)^2 + \left(e^-\left(\frac{x+y}{2}\right) - a_i/m\right)^2$$

and the inequality is strict iff $|x - y| \geq 4$. Hence f^-, f^+ operations cannot take the ith row further from the constant vector $(a_i/m, \ldots, a_i/m)$. Now suppose that the ith row is as close as possible to $(a_i/m, \ldots, a_i/m)$ by applying f^-, f^+ operations. Suppose there is some $a_{i,j}$ such that $|a_{i,j} - a_i/m| \geq 2$. Then there is some j' such that $|a_{i,j'} - a_{i,j}| > 2$ since $\sum_j a_{i,j} = a_i$. Since $a_{i,j'} - a_{i,j}$ is even we have $|a_{i,j'} - a_{i,j}| \geq 4$. But now an f^-, f^+ operation on columns j, j' places row i strictly closer to $(a_i/m, \ldots, a_i/m)$, a contradiction. Therefore after sufficiently many f^-, f^+ operations the resulting matrix satisfies (N1). Assume now that A satisfies (N1). We shall now switch to a different kind of (F4) operation (which does not disrupt (N1)) which will eventually give us a matrix that also satisfies (N2). Recall the definition of $\mathcal{J}_\mathbf{a}$. We have $\mathcal{J}_\mathbf{a} = \{i_1, \ldots, i_k\}$ is the set of all i, $2 \leq i \leq n-1$, such that $a_{i-1} \leq 2mr_i$ and $a_i \leq 2mr_i$, where $i_t < i_{t+1}$ for all t, $1 \leq t < k$. Let $i_0 = 1$ and let $i_{k+1} = n$. Let

$$D(4)(\mathbb{Z})_\mathbf{a} = \{\mathbf{d} \in D(4)(\mathbb{Z}) \mid \mathcal{J}^0_\mathbf{d} \subset \mathcal{J}_\mathbf{a} \subset \mathcal{J}^1_\mathbf{d}\}.$$

Let

$$g^- : D(4)(\mathbb{Z})_\mathbf{a} \to D(2)(\mathbb{Z}),$$

$$g^-(\mathbf{d})_i = \begin{cases} e^-(d_i/2) & \text{for } i_{2t} \leq i < i_{2t+1}, 2t \leq k, \\ e^+(d_i/2) & \text{for } i_{2t+1} \leq i < i_{2t+2}, 2t+1 \leq k. \end{cases}$$

Let

$$g^+ : D(4)(\mathbb{Z})_\mathbf{a} \to D(2)(\mathbb{Z}),$$

$$g^+(\mathbf{d})_i = \begin{cases} e^+(d_i/2) & \text{for } i_{2t} \leq i < i_{2t+1}, 2t \leq k, \\ e^-(d_i/2) & \text{for } i_{2t+1} \leq i < i_{2t+2}, 2t+1. \leq k \end{cases}$$

We claim that for any two columns $c_j(A)$, $c_{j'}(A)$ the sum $\mathbf{d} = (d_1, \ldots, d_{n-1}) = c_j(A) + c_{j'}(A)$ is a member of $D(4)(\mathbb{Z})_\mathbf{a}$. First we show that $\mathcal{J}_\mathbf{a} \subset \mathcal{J}^1_\mathbf{d}$. Suppose that $i \in \mathcal{J}_\mathbf{a}$. Then $a_{i-1}/m \leq 2r_i$ and $a_i/m \leq 2r_i$, so the entries in rows $i-1$ and i are at most $2r_i$ since $|a_{i-1,j} - a_{i-1}/m| < 2$ and $|a_{i,j} - a_i/m| < 2$ for all j. Hence the sum of any two entries in row $i-1$ is at most $4r_i$ and the sum of any two entries in row i is at most $4r_i$. Therefore each of d_{i-1} and d_i is at most $4r_i$ so $i \in \mathcal{J}^1_\mathbf{d}$. Next we show that $\mathcal{J}^0_\mathbf{d} \subset \mathcal{J}_\mathbf{a}$. Suppose $i \in \mathcal{J}^0_\mathbf{d}$. This means that $d_{i-1} + d_i = 4r_i$ and each of $d_{i-1}/2$ and $d_i/2$ is an odd integer. Since $a_{i-1,j} + a_{i,j} \geq 2r_i$ and $a_{i-1,j'} + a_{i,j'} \geq 2r_i$ and

$$a_{i-1,j} + a_{i-1,j'} + a_{i,j} + a_{i,j'} = d_{i-1} + d_i = 4r_i,$$

we have that $a_{i-1,j} + a_{i,j} = 2r_i$ and $a_{i-1,j'} + a_{i,j'} = 2r_i$. Suppose by way of contradiction that $a_{i-1} > 2mr_i$. Then each entry of row $i-1$ is at least $2r_i$. Then $a_{i-1,j} = a_{i-1,j'} = 2r_i$ and $a_{i,j} = a_{i,j'} = 0$. But now $d_i = a_{i,j} + a_{i,j'} = 0$ which contradicts that $d_i/2$ is an odd integer. Therefore $a_{i-1} \leq 2mr_i$. Similarly we can show $a_i \leq 2mr_i$. Hence $i \in \mathcal{J}_\mathbf{a}$.

We define a g^-, g^+ operation to be the following. Let $j < j'$ and replace columns $c_j(A)$, $c_{j'}(A)$ with $g^-(c_j(A) + c_{j'}(A))$, $g^+(c_j(A) + c_{j'}(A))$ in that order. Clearly any such g^-, g^+ operation is of type (F4) and it preserves the inequalities $|a_{i,j} - a_i/m| < 2$. We claim that a finite number of such operations results in a matrix in normal form. First notice that g^-, g^+ operations don't change the multi-set of entries in any given row since they preserve the (N1) condition. We determine how g^-, g^+ operations affect the order of the row entries. The output of g^- and g^+ is determined by the type of interval i belongs to; either $i_{2t} \leq i < i_{2t+1}$ for some t or $i_{2t+1} \leq i < i_{2t+2}$ for some t. Let us examine the case $i_{2t} \leq i < i_{2t+1}$. Here g^- applies the e^- rule and g^+ applies the e^+ rule. Hence the result of a g^-, g^+ operation to columns j, j' with $j < j'$ is to put entries $a_{i,j}, a_{i,j'}$ into (weakly) increasing order. After applying these operations to all pairs j, j', the resulting ith row is weakly increasing. The case $i_{2t+1} \leq i < i_{2t+2}$ is similar; this row will be weakly decreasing after g^-, g^+ operations are performed on all pairs of columns.

Suppose $\deg(A) = 2m + 1$ is odd, so there is one column of degree one and m columns of degree two. Apply a single (F3) operation so that the first column is the degree one column and columns 2 through $m+1$ are degree two. Always let A' denote A without the first column. We will show after enough operations of types (F3) and (F4) that the first column is $\xi_{2m+1}(\mathbf{a})$ and that A' satisfies conditions (N0) and (N1) for normality. Then g^-, g^+ operations can be performed on A' so that A' will eventually satisfy (N2).

The ith row must satisfy that $a_{i,1} \equiv (r_1 + \cdots + r_i) \mod 2$, each $a_{i,j}$ is even for $j \geq 2$, and the sum $\sum_j a_{i,j} = a_i$. Clearly row i is closest to the vector

$$v_i = \left(\frac{a_i}{2m+1}, \frac{2a_i}{2m+1}, \frac{2a_i}{2m+1}, \ldots, \frac{2a_i}{2m+1} \right) \in \mathbb{Z}^{m+1}$$

iff

$$(*) \quad |a_{i,1} - a_i/(2m+1)| < 1 \text{ and } |a_{i,j} - 2a_i/(2m+1)| < 2 \text{ for all } j \geq 2.$$

These inequalities are necessary for the first column $c_1(A)$ to be $\xi_{2m+1}(\mathbf{a})$ and for A' to satisfy (N1). If each row satisfies $(*)$ then in fact $c_1(A) = \xi_{2m+1}(\mathbf{a})$ and A' satisfies (N1).

Suppose that $(*)$ holds for row i. We claim that operations of types (F3) and (F4) preserve $(*)$. We have that $|2a_{i,1} - a_{i,j}| < 4$ for each $j \geq 2$. But $2a_{i,1} - a_{i,j}$ is even so in fact $|2a_{i,1} - a_{i,j}| \leq 2 < 3$. Therefore, $|(a_{i,1} + a_{i,j})/3 - a_{i,1}| < 1$. But this implies that $\xi_3(c_1(A) + c_j(A))_i = a_{i,1}$ since $a_{i,1}$ has parity $(r_1 + \cdots + r_i) \mod 2$ and is less than one unit from $(a_{i,1} + a_{i,j})/3$. Therefore row i is fixed by any (F3) operation. On the other hand if an (F4) operation is applied to columns j and j' then it either fixes $a_{i,j}$ and $a_{i,j'}$ or swaps their order since $|a_{i,j} - a_{i,j'}| \leq 2$.

Suppose that row i is as close as possible to v_i by applying (F3) and (F4) operations. Suppose by way of contradiction that $|a_{i,1} - a_i/(2m+1)| \geq 1$. Then there is some $j_0 \geq 2$ such that $a_i/(2m+1)$ is strictly between $a_{i,j_0}/2$ and $a_{i,1}$ since $a_i/(2m+1)$ is the weighted average of the entries in row i, where $a_{i,1}$ is weighted by 1 and $a_{i,j}$ is weighted by 2 for each $j \geq 2$. Therefore $|2a_{i,1} - a_{i,j_0}| > 2$. But $2a_{i,1} - a_{i,j_0}$ is even so in fact $|2a_{i,1} - a_{i,j_0}| \geq 4$. So $|(a_{i,1} + a_{i,j_0})/3 - a_{i,1}| \geq 4/3$. Without loss of generality suppose that $a_{i,1} < a_{i,j_0}/2$. Then we have

$$a_{i,1} < (a_{i,1} + a_{i,j_0})/3 < a_{i,j_0}/2.$$

Let k be the nearest integer of parity $(r_1+\cdots+r_i) \mod 2$ to $(a_{i,1}-a_{i,j_0})/3$. Then we have $a_{i,1} < k \le a_{i,j_0}/2$. Let δ_i be the change in the distance between row i and v_i after applying an (F3) operation to columns 1 and j_0. Let $a = a_i/(2m+1)$ and let $t = k - a_{i,1}$. Then

$$\delta_i = (a_{i,1}+t-a)^2+(a_{i,j_0}-t-2a)^2-(a_{i,1}-a)^2-(a_{i,j_0}-2a)^2 = 2t(t-(a_{i,j_0}-a_{i,1}-a)).$$

But we know that $0 < t \le a_{i,j_0}/2 - a_{i,1} < a_{i,j_0} - a_{i,1} - a$. The first inequality follows from the fact that $a_{i,1} < k$ and the last inequality follows from that fact that $a_{i,j_0} > 2a = 2a_i/(2m+1)$. Hence δ_i is negative which means an (F3) operation takes row i strictly closer to v_i, a contradiction. Hence $|a_{i,1}-a_i/(2m+1)| < 1$. Now by our argument above for even degree matrices, we must have that the remaining entries $a_{i,j}$ differ by at most 2 from one another ((F4) operations can accomplish this) and consequently we also have that $|a_{i,j} - 2a_i/(2m+1)| < 2$ for each $j \ge 2$. Therefore, working row by row, we end up with a matrix A such that $c_1(A) = \xi_{2m+1}(\mathbf{a})$ and A' satisfies (N1). Now apply g^-, g^+ operations to A' so that finally A' satisfies (N2) as well. □

Corollary 6.14. *For any D-matrix A, there is a unique matrix $\mathcal{N}(A)$ in normal form which is equivalent to A.*

Theorem 6.15. *The ideal of $\mathbb{C}[S_\mathbf{r}] \cong (R_\mathbf{r})_0$ is generated by quadratic relations of degrees two, three, and four. Furthermore, the ideal of $(R_{2\mathbf{r}})_0$ is generated by quadratic relations.*

Proof. One only needs to determine if two monomials in degree one and two variables are equal. This corresponds to deciding if two D-matrices are equivalent, which is true iff they have the same normal form. Operations of types (F2),(F3), and (F4) correspond to degree two, degree three, and degree four relations in the ideal of $\mathbb{C}[S_\mathbf{r}]$. By Lemma 6.13 these operations are enough to place any D-matrix A into its normal form $\mathcal{N}(A)$. Hence relations up to degree four must generate the ideal of $\mathbb{C}[S_\mathbf{r}]$. For the case of $(R_{2\mathbf{r}})_0$ a D-matrix is of even degree and we only need type (F4) operations to place it into normal form. In this case an (F4) operation corresponds to a quadratic relation. □

7. Lifting generators and relations from the toric fiber

We wish to get a presentation for our ring $R_\mathbf{r}$ by lifting the presentation for the associated graded ring $\mathrm{gr}(R_\mathbf{r})$ that we found in §6. It is a basic fact that generators of an associated graded *algebra* or a *module* may be lifted to generators of the original object (whether it be an algebra or module.) Here we will use both facts, that is, we will lift generators of the algebra $\mathrm{gr}(R_\mathbf{r})$ to get generators of $R_\mathbf{r}$, and we will lift generators of the ideal of $\mathrm{gr}(R_\mathbf{r})$ (a module) to get generators of the ideal of $R_\mathbf{r}$.

We begin with the following lemma which will be basic in what follows. We leave the proof to the reader. For background on filtrations and gradings we refer to [Bou]. Throughout this section all filtrations will be increasing and indexed by the nonnegative integers $\mathbb{Z}^+$.

Lemma 7.1. *Suppose that M is a filtered module over a filtered ring R and that their filtrations are compatible in the sense that*

$$F_i(R) \otimes F_j(M) \subset F_{i+j}(M).$$

Suppose that $x_1, x_2, \ldots, x_n$ are elements of M such that their images $\overline{x}_i, 1 \leq i \leq n$, under the leading term map generate $\mathrm{gr}(M)$ *as a* $\mathrm{gr}(R)$ *module. Then $x_i, 1 \leq i \leq n$ generate M.*

Remark 7.2. *An analogous argument shows that if the images in* $\mathrm{gr}(R)$ *of a finite set of elements $r_1, r_2, \ldots, r_n$ of R generate* $\mathrm{gr}(R)$ *then the elements r_1. r_2, $\ldots$, r_n generate R.*

Our goal in this section is to prove the statement for *relations* that is the analogue of the statement in the remark for generators.

Definition 7.3. *Let M be a filtered module and $x \in M$. We define the filtration level (or order) $v(x) \in \mathbb{Z}^+$ of x to be the smallest n such that $x \in F_n(M)$.*

Assume that R is graded as a vector space and that we have chosen homogeneous generators $f_1, f_2, \ldots, f_n$ for R such that the images $\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_n$ of these generators in $\mathrm{gr}(R)$ generate $\mathrm{gr}(R)$. We assume the degree of f_i is $e_i, 1 \leq i \leq n$.

We obtain two exact sequences.

$$I \xrightarrow{\iota} \mathbb{C}[x_1, x_2, \cdots, x_n] \xrightarrow{\pi} R$$

and

$$J \longrightarrow \mathrm{gr}(\mathbb{C}[x_1, x_2, \cdots, x_n]) \xrightarrow{\pi} \mathrm{gr}(R).$$

Here π sends x_i to $f_i, 1 \leq i \leq n$. In the above the polynomial ring $\mathbb{C}[x_1, x_2, \cdots, x_n]$ is a weighted polynomial ring, the variable x_i has weight e_i. We define a filtration on R by defining the filtration level of r to be the minimum of the degrees of the polynomials in $\pi^{-1}(r)$. The reader will verify that this filtration coincides with the quotient filtration of the standard filtration on $\mathbb{C}[x_1, x_2, \cdots, x_n]$. We remind the reader that the quotient filtration is characterized by the fact that the induced map on each filtration level is a surjection, see [Bou, pg. 164].

We note that $\mathrm{gr}(\mathbb{C}[x_1, x_2, \cdots, x_n]))$ is the polynomial ring $\mathbb{C}[\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n]$. We leave the reader the task of proving (by induction on the filtration level):

Lemma 7.4. *Suppose R is a filtered $\mathbb{C}$–algebra which is graded as a vector space and $f_1, \ldots, f_n$ have the property that their images $\overline{f}_1, \ldots, \overline{f}_n$ generate* $\mathrm{gr}(R)$. *Then the given filtration on R coincides with the quotient filtration associated to the surjection $\pi : \mathbb{C}[x_1, x_2, \cdots, x_n] \to R$ given by $\pi(x_i) = f_i$.*

Example 7.5. *We give the following example to show what can go wrong if $\overline{f}_1, \ldots, \overline{f}_n$ do not generate* $\mathrm{gr}(R)$. *We give this example because a similar phenomenon occurs for the case of equilateral hexagons. Consider the affine coordinate ring R of the saddle surface $z = xy$. Then R is generated by x and y. We give R the filtration that is the quotient of the filtration on $\mathbb{C}[x, y, z]$, so the images $\overline{x}, \overline{y}$ and $\overline{z}$ have degree one in R. We also have a surjection $\pi : \mathbb{C}[u, v] \to R$ given by $\pi(u) = x$ and $\pi(v) = y$. Clearly π is not onto at filtration level one. There is no contradiction with the lemma above because the images $\overline{x}$ and $\overline{y}$ do not generate* $\mathrm{gr}(R)$.

Since we give I the filtration induced as a submodule of the polynomial ring both I and R have the filtrations needed to apply [Bou, pg. 169, Prop. 2] to deduce that we have an exact sequence

$$\mathrm{gr}(I) \xrightarrow{\mathrm{gr}(\iota)} \mathrm{gr}(\mathbb{C}[x_1, x_2, \cdots, x_n]) \xrightarrow{\mathrm{gr}(\pi)} \mathrm{gr}(R).$$

and consequently $\mathrm{gr}(\iota) : \mathrm{gr}(I) \to J$ is an isomorphism.

We are now ready to state and prove the result we want on lifting relations from $\mathrm{gr}(R)$ to R. We emphasize that we are assuming that the generators for R map to generators for $\mathrm{gr}(R)$ under the leading term map.

Proposition 7.6. *Suppose $p_1, p_2, \dots, p_k \in \mathrm{gr}(\mathbb{C}[x_1, x_2, \dots, x_n])$ generate the ideal of relations in the given generators for $\mathrm{gr}(R)$. Then*

(1) *There exist lifts $\tilde{p}_i, 1 \le i \le k$, to $C[x_1, x_2, \dots, x_n]$ such that for all i the polynomial $\tilde{p}_i$ is a relation for R.*
(2) *For any choice of such lifts $\tilde{p}_i, 1 \le i \le k$ the lifts generate the ideal of relations of R.*

Proof. Since we have shown that $J \equiv \mathrm{gr}(I)$ the first statement in the proposition is obvious (since the leading term map is onto by definition of $\mathrm{gr}(I)$). However the lift of a homogeneous element will usually not be homogeneous (the ideal I may not contain any nonzero homogeneous elements). The second statement follows from Lemma 7.1 — the images of the lifts generate the ideal $\mathrm{gr}(I)$ so the lifts generate I. □

8. The projective coordinate ring of $M_{\mathbf{r}}$

Now we apply the lifting results of the previous section to get a presentation of the ring $R_{\mathbf{r}}$. We begin by carrying over some definitions from the toric ring $(R_{\mathbf{r}})_0$ to $R_{\mathbf{r}}$. Recall the definition of normal form for D-matrices.

Definition 8.1. *For any $\mathbf{d} \in D(m\mathbf{r})(\mathbb{Z})$ let $\tau_{\mathbf{d}} \in R_{\mathbf{r}}$ be the unique tableau with multi-weight $\Psi(\Phi^{-1}(\mathbf{d})) \in SS(ma\varpi_2, m\mathbf{r})$. If a D-matrix is in normal form with columns $\mathbf{d}_1, \dots, \mathbf{d}_s$ we shall say the product $\tau_{\mathbf{d}_1}\tau_{\mathbf{d}_2}\cdots\tau_{\mathbf{d}_s} \in R_{\mathbf{r}}$ is a normal monomial. Also we shall say that $\bar{\tau}_{\mathbf{d}_1}\bar{\tau}_{\mathbf{d}_2}\cdots\bar{\tau}_{\mathbf{d}_s} \in (R_{\mathbf{r}})_0$ is a normal monomial.*

Example 8.2. *We return to Example 6.9, and see what are the related monomials in tableaux. We had:*

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 4 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

The associated monomials in tableaux are:

$$m_A = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & 7 & 8 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 & 6 & 6 & 7 \\ \hline 2 & 3 & 3 & 5 & 5 & 7 & 8 & 8 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 4 & 4 & 5 & 7 \\ \hline 2 & 2 & 5 & 6 & 6 & 7 & 8 & 8 \\ \hline \end{array},$$

$$m_B = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & 7 & 8 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 & 5 & 6 & 7 \\ \hline 2 & 3 & 3 & 5 & 6 & 7 & 8 & 8 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 \\ \hline 2 & 3 & 5 & 6 & 6 & 7 & 8 & 8 \\ \hline \end{array}.$$

Here m_B is a normal monomial since B is in normal form, whereas m_A is not a normal monomial because A is not in normal form. However, m_B and m_A have the same LG-filtration level.

Definition 8.3. *Suppose $k = 2m$ is even. Choose tableaux $\tau_{i,j}$ such that $1 \leq i \leq M_k$, $1 \leq j \leq m$, where M_k is the number of lattice points in $D(k\mathbf{r})(\mathbb{Z})$, and such that*

$$\mathcal{N}_{k\mathbf{r}} = \left(\prod_{j=1}^{m} \tau_{1,j}, \prod_{j=1}^{m} \tau_{2,j}, \ldots, \prod_{j=1}^{m} \tau_{M_k,j} \right) \in (R_{\mathbf{r}}^{(k)})^{M_k}$$

is the set of normal monomials of degree k,

$$\mathcal{SS}_{k\mathbf{r}} = (\sigma_1, \ldots, \sigma_{M_k}) \in (R_{\mathbf{r}}^{(k)})^{M_k}, \text{ where}$$

$$\sigma_i = \tau_{i,1} * \tau_{i,2} * \cdots * \tau_{i,m},$$

such that LG-deg(σ_i) $\leq$ LG-deg(σ_{i+1}) for each i, $1 \leq i < M_k$.

Suppose $k = 2m + 1$ is odd. Choose tableaux $\tau_{i,j}$ such that $1 \leq i \leq M_k$, $1 \leq j \leq m + 1$, where M_k is the number of lattice points in $D(k\mathbf{r})(\mathbb{Z})$, and such that

$$\mathcal{N}_{k\mathbf{r}} = \left(\prod_{j=1}^{m+1} \tau_{1,j}, \prod_{j=1}^{m+1} \tau_{2,j}, \ldots, \prod_{j=1}^{m+1} \tau_{M_k,j} \right) \in (R_{\mathbf{r}}^{(k)})^{M_k}$$

is the set of normal monomials of degree k,

$$\mathcal{SS}_{k\mathbf{r}} = (\sigma_1, \ldots, \sigma_{M_k}) \in (R_{\mathbf{r}}^{(k)})^{M_k}, \text{ where}$$

$$\sigma_i = \tau_{i,1} * \tau_{i,2} * \cdots * \tau_{i,m+1},$$

such that LG-deg(σ_i) $\leq$ LG-deg(σ_{i+1}) for each i, $1 \leq i < M_k$.

Note that M_k is the number of semistandard tableaux of weight $k\mathbf{r}$, which is the dimension of $(R_{\mathbf{r}})^{(k)}$.

Remark 8.4. *The choice of the ordering of normal monomials is not unique, since it is possible for two normal monomials to have the same total LG-degree, which is $\sum_j$ LG-deg($\tau_{i,j}$).*

Proposition 8.5. *The components of $\mathcal{SS}_{k\mathbf{r}}$ are exactly the semistandard tableaux of weight $k\mathbf{r}$.*

Proof. Let $\mathcal{N}_{k\mathbf{r}}$ be as above. The normal monomials of degree k are in bijection with the integral points of the polytope $SS(ka\varpi_2, k\mathbf{r})$, by

$$\prod_j \tau_{i,j} \mapsto \sum_j \mathbf{wt}(\tau_{i,j}) = \mathbf{wt}(\sigma_i).$$

But the semistandard tableaux of weight $k\mathbf{r}$ are in bijection with the integral points of $SS(ka\varpi_2, k\mathbf{r})$ by $\sigma \mapsto \mathbf{wt}(\sigma)$. □

Theorem 8.6. *The tuple $\mathcal{N}_{k\mathbf{r}}$ is a basis for $(R_{\mathbf{r}})^{(k)}$.*

Proof. Let $c_{i,j} \in \mathbb{Z}$ for $1 \leq i, j \leq M_k$ be given by

$$\prod_\ell \tau_{j,\ell} = \sum_{i=1}^{M_k} c_{i,j}\ \sigma_i.$$

We claim the matrix $[c_{i,j}]$ is unipotent (or upper-triangular). Since the product of any two semistandard tableaux $\tau\sigma$ is $\tau * \sigma + \sum_i c_i \rho_i$ where each $c_i \in \mathbb{Z}$, each ρ_i is semistandard, and LG-deg(ρ_i) < LG-degree($\tau * \sigma$), one can argue by induction on $s \geq 2$ that

$$\prod_{\ell=1}^{s} \tau_\ell = (\tau_1 * \tau_2 * \cdots * \tau_s) + \sum_i c_i \rho_i$$

where each $c_i \in \mathbb{Z}$, each ρ_i is semistandard with

$$\text{LG-deg}(\rho_i) < \text{LG-deg}(\tau_1 * \tau_2 * \cdots * \tau_s).$$

Since the tuple $\mathcal{SS}_{k\mathbf{r}} = (\sigma_1, \ldots, \sigma_{M_k})$ satisfies $\text{LG-deg}(\sigma_i) \leq \text{LG-deg}(\sigma_{i+1})$, the matrix $[c_{i,j}]$ is upper-triangular, and since the first term of the sum is the concatenated product, we get 1's on the diagonal of $[c_{i,j}]$. Since $\mathcal{SS}_{k\mathbf{r}}$ is a basis of $R_{\mathbf{r}}^{(k)}$, the theorem follows. □

Definition 8.7. *Let $C_{k\mathbf{r}}$ be the (unipotent) basis exchange matrix from $\mathcal{N}_{k\mathbf{r}}$ to $\mathcal{SS}_{k\mathbf{r}}$.*

Proposition 8.8. *For any two tableaux ρ_1, ρ_2 with $\rho_1\rho_2 \in R_{\mathbf{r}}^{(k)}$, there exists $i_0 \leq M_k$ such that*

$$\rho_1\rho_2 = \prod_{j=1}^{s} \tau_{i_0,j} + \sum_{i<i_0} c_i \left(\prod_{j=1}^{s} \tau_{i,j} \right), \text{where}$$

$$\mathcal{N}_{k\mathbf{r}} = \left(\prod_{j=1}^{s} \tau_{1,j}, \prod_{j=1}^{s} \tau_{2,j}, \ldots, \prod_{j=1}^{s} \tau_{M_k,j} \right)$$

and $\bar{\rho}_1\bar{\rho}_2 = \prod_{j=1}^{s} \bar{\tau}_{i_0,j}$ is the leading term of the right hand side with respect to LG-degree.

Proof. We have already established the analogous statement in terms of semistandard tableaux. There exists $i_0 \leq M_k$ such that

$$\rho_1\rho_2 = \sigma_{i_0} + \sum_{i<i_0} c'_i \sigma_i, \text{where}$$

$$\mathcal{SS}_{k\mathbf{r}} = (\sigma_1, \ldots, \sigma_{M_k})$$

and $\bar{\rho}_1\bar{\rho}_2 = \bar{\sigma}_{i_0} = \prod_{j=1}^{s} \bar{\tau}_{i_0,j}$ is the leading term of the right hand side with respect to LG-degree. □

Theorem 8.9.

(1) *The ring $R_{\mathbf{r}}$ is generated by semistandard tableaux in degrees one and two. The relations in these tableaux are generated by quadratic relations of degrees two, three, and four.*
(2) *The ring $R_{2\mathbf{r}}$ is generated by semistandard tableaux of degree one. The relations in these tableaux are generated by quadratic relations.*

Proof. The tableaux $\bar{\tau}$ of degrees one and two generate $(R_{\mathbf{r}})_0$ so their lifts τ must generate $R_{\mathbf{r}}$.

Now we choose lifts of the relations in $(R_{\mathbf{r}})_0$ of types (F2), (F3), and (F4) to get degree two, three, four relations in the tableaux of degree one and two.

First consider a relation of type (F2) which is $\bar{\sigma}_1\bar{\sigma}_2 = \bar{\tau}$ where $\sigma_1 * \sigma_2 = \tau$ and σ_1, σ_2 are degree one. Write

$$\sigma_1\sigma_2 = \sum_{i=1}^{M_2} c_i\tau_i \qquad (F2')$$

where $M_2 = \dim R_{\mathbf{r}}^{(2)}$, $(\tau_1, \ldots, \tau_{M_2}) = \mathcal{SS}_{2\mathbf{r}} = \mathcal{N}_{2\mathbf{r}}$. There is some k such that $\tau = \tau_k$, and furthermore $\tau = \tau_k$ is the leading term of the right hand side with respect to LG-degree. Hence this relation is a lift of the (F2) relation $\bar{\sigma}_1\bar{\sigma}_2 = \bar{\tau}$.

Now consider a relation of type (F3), which is $\bar{\sigma}\bar{\tau} = \bar{\sigma}'\bar{\tau}'$ where the right hand side is a normal monomial, and $\bar{\sigma}, \bar{\sigma}'$ are degree one and $\bar{\tau}, \bar{\tau}'$ are degree two. Write

$$\sigma\tau = \sum_{i=1}^{M_3} c_i \sigma_i \tau_i \qquad (F3')$$

where $M_3 = \dim R_{\mathbf{r}}^{(3)}$ and $\mathcal{N}_{3\mathbf{r}} = (\sigma_1\tau_1, \ldots, \sigma_{M_3}\tau_{M_3})$. The normal monomial $\bar{\sigma}'\bar{\tau}'$ is the leading term of the right hand side with respect to LG-degree, so this relation is a lift of the (F3) relation $\bar{\sigma}\bar{\tau} = \bar{\sigma}'\bar{\tau}'$.

Finally consider a relation of type (F4), which is $\bar{\tau}_1\bar{\tau}_2 = \bar{\tau}_1'\bar{\tau}_2'$ where the right hand side is a normal monomial, and each of $\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_1', \bar{\tau}_2'$ is degree two. Write

$$\tau_1\tau_2 = \sum_{i=1}^{M_3} c_i \tau_{i,1}\tau_{i,2} \qquad (F4')$$

where $M_4 = \dim R_{\mathbf{r}}^{(4)}$ and $\mathcal{N}_{4\mathbf{r}} = (\tau_{1,1}\tau_{1,2}, \ldots, \tau_{M_4,1}\tau_{M_4,2})$. The normal monomial $\bar{\tau}_1'\bar{\tau}_2'$ is the leading term of the right hand side with respect to LG-degree, so this relation is a lift of the (F4) relation $\bar{\tau}_1\bar{\tau}_2 = \bar{\tau}_1'\bar{\tau}_2'$.

It follows from Proposition 7.6 and Theorem 6.15 that the relations in the degree one and degree two tableaux are generated by the lifted relations above,

$$\rho_1\rho_2 = \sum_{i=1}^{M_j} c_i \mathbf{n}_i \qquad (Fj')$$

where $\mathcal{N}_{j\mathbf{r}} = (\mathbf{n}_1, \ldots, \mathbf{n}_{M_j})$ and $\rho_1\rho_2 \in R_{\mathbf{r}}^{(j)}$, $j = 2, 3, 4$.

Similarly the semistandard Young tableaux of weight $2\mathbf{r}$ generate $R_{2\mathbf{r}}$ and the relations in these tableaux are generated by quadratic relations, which are lifts of relations of type (F4) for the toric ring $(R_{\mathbf{r}})_0$. □

Now recall Theorem 3.7 which states that in fact $R_{\mathbf{r}}$ is generated by elements of degree one provided that $|\mathbf{r}| = \sum_i r_i$ is even. We shall now improve this theorem to include a statement about the relations.

Theorem 8.10. *Assume that $|\mathbf{r}| = \sum_{i=1}^n r_i$ is even. Then $R_{\mathbf{r}}$ is generated by the semistandard tableaux of weight $\mathbf{r}$ and the relations amongst these tableaux are generated by relations of degree four and less. If furthermore each r_i is even then the relations among the tableaux of weight $\mathbf{r}$ are generated by quadratic relations.*

Proof. We have that the degree tableaux of weight $\mathbf{r}$ generate $R_{\mathbf{r}}$ by Theorem 3.7.

Each tableaux of weight $2\mathbf{r}$ is a quadratic function of tableaux of weight $\mathbf{r}$. For each tableaux τ of weight $2\mathbf{r}$, write τ in terms of tableaux of weight $\mathbf{r}$:

$$\tau = \sum_i a_i(\tau)\sigma_{i,1}\sigma_{i,2}.$$

Let $f(\tau) = \sum_i a_i(\tau)\sigma_{i,1}\sigma_{i,2}$. The relations of type (F2'), (F3'), and (F4') in Theorem 8.9 above may be replaced with relations in tableaux of weight $\mathbf{r}$ by substituting each degree two tableau τ with $f(\tau)$. □

8.1. **An explicit algorithm for listing the relations.** First write each degree two tableau τ in terms of degree one tableaux (see Theorem 3.7):

$$\tau = f(\tau) = \sum_i a_i(\tau)\sigma_{i,1}\sigma_{i,2}.$$

For each $j = 2, 3, 4$, do the following:

(1) List the normal monomials $\mathcal{N}_{j\mathbf{r}} = (\mathbf{n}_1, \ldots, \mathbf{n}_{M_j})$ and semistandard tableaux $\mathcal{SS}_{j\mathbf{r}} = (\sigma_1, \ldots, \sigma_{M_j})$ weakly increasing with respect to LG-degree.
(2) Compute the change of basis matrix $C_{j\mathbf{r}}$ using Plücker relations.
(3) Compute $C_{j\mathbf{r}}^{-1}$ (recall $C_{j\mathbf{r}}$ is unipotent with integer entries).
(4) List all non-normal monomials $\rho_1\rho_2$ of degree j.
(5) For each such $\rho_1\rho_2$ above, use Plücker relations to compute the coefficients $c'_{i,\rho_1\rho_2}$ where

$$\rho_1\rho_2 = \sum_{i=1}^{M_j} c'_{i,\rho_1\rho_2}\sigma_i.$$

(6) Compute $[c_{i,\rho_1\rho_2}]_{i=1}^{M_j} = C_{j\mathbf{r}}^{-1}\,[c'_{i,\rho_1\rho_2}]_{i=1}^{M_j}$. Then,

$$\rho_1\rho_2 = \sum_{i=1}^{M_j} c_{i,\rho_1\rho_2}\mathbf{n}_i.$$

(7) Replace each degree two tableau τ occuring in the relation above with $f(\tau)$.

8.2. **A minimal set of generating relations for $R_{2\mathbf{r}}$.** The relations given for $R_\mathbf{r}$ above may not be minimal. The prime example is that of $n = 6$ with each $r_i = 1$, where a single cubic relation generates the ideal of relations. However, the presentation given for $R_\mathbf{s}$ when each s_i is even (say $\mathbf{s} = 2\mathbf{r}$) turns out to be a minimal presentation.

Theorem 8.11. *A minimal set of generating relations for $R_{2\mathbf{r}}$ is given by*

$$\rho_1\rho_2 = \sum_{i=1}^{M_4} c_{i,\rho_1\rho_2}\sigma_i\tau_i$$

where $(\sigma_1\tau_1, \ldots, \sigma_{M_4}\tau_{M_4}) = \mathcal{N}_{4\mathbf{r}}$ and the products $\rho_1\rho_2$ range over all non-normal products of degree two tableaux in $R_\mathbf{r}$ (they are degree one in $R_{2\mathbf{r}}$).

Proof. Let I be the ideal generated by the above relations, and let J be the ideal generated by all but one of them, say $p(\rho_1, \rho_2) = \tau_1\tau_2 - \sum_{i=1}^{M_4} c_{i,\rho_1\rho_2}\sigma_i\tau_i$. We show that J does not contain $p(\rho_1, \rho_2)$. Suppose $p(\rho_1, \rho_2) \in J$. Then,

$$p(\rho_1, \rho_2) = \sum_\ell a_\ell p(\alpha_\ell, \beta_\ell)$$

for some coefficients a_ℓ, where the products $\alpha_\ell\beta_\ell$ are not equal to $\rho_1\rho_2$ in the polynomial ring $\mathbb{C}[\gamma_1, \ldots, \gamma_{M_2}]$ where the γ_i range over the tableaux of weight $2\mathbf{r}$. But the monomial $\rho_1\rho_2$ does not even occur on the right hand side of the equation since it is not one of the $\alpha_\ell\beta_\ell$'s nor is it a normal monomial. Hence we have a contradiction. □

8.3. **Some examples.** Here we will concentrate on the case of equal weights. When the r_i's are equal there is a natural action of the symmetric group on the moduli space, so that the resulting quotient is the space of *unordered* points on the line.

8.3.1. *Four points.* Let $\mathbf{r} = (1,1,1,1)$. The moduli space $M_{\mathbf{r}}$ is simply the projective line $\mathbb{CP}^1$. The invariant semistandard 2 by 2 tableaux are

$$X = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad Y = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

These give the embedding into $\mathbb{CP}^1$. The moduli space of the square is one dimensional over $\mathbb{C}$, and hence there can be no relations between X and Y. Hence the embedding surjects onto $\mathbb{CP}^1$. Identify $\mathbb{CP}^1$ with $\mathbb{C} \cup \{\infty\}$ where $[z,1] \mapsto z \in \mathbb{C}$, and $[1,0] \mapsto \infty$. Now let $z_1, z_2, z_3, z_4 \in \mathbb{C}$. The condition for semistability is that no three of these points coincide, and the well known cross ratio function is given by

$$f(z_1,z_2,z_3,z_4) = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)} = (X/Y)([z_1,1],[z_2,1],[z_3,1],[z_4,1]).$$

8.3.2. *Five points.* Let $\mathbf{r} = (2,2,2,2,2)$. The moduli space $M_{\mathbf{r}}$ (the pentagon space) is more complicated, it is embedded in $\mathbb{CP}^5$ and satisfies five quadratic equations. It can be shown that $M_{\mathbf{r}}$ is $\mathbb{CP}^2$ with four points blown up. The generators are

$$A = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 3 & 4 & 4 & 5 & 5 \\ \hline\end{array} \quad B = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 4 \\ \hline 3 & 3 & 4 & 5 & 5 \\ \hline\end{array} \quad C = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 4 & 4 & 5 & 5 \\ \hline\end{array}$$

$$D = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 & 5 \\ \hline\end{array} \quad E = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 2 & 3 & 3 & 5 & 5 \\ \hline\end{array} \quad F = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 4 \\ \hline 2 & 2 & 4 & 5 & 5 \\ \hline\end{array}$$

These give an embedding of the moduli space of the pentagon into $\mathbb{CP}^5$. There are five non-normal quadratic products, BC, AE, AF, BF, and CE. The generating set of relations is

$$BC = AD, \qquad AE = BD - D^2 + DE, \qquad AF = CD - D^2 + DF$$
$$BF = D^2 - DE, \qquad CE = D^2 - DF$$

8.3.3. *Six points.* The space of equilateral hexagons is a cubic hypersurface in $\mathbb{CP}^4$. We leave this as an exercise for the reader; otherwise see [DO, pg. 17] for more details.

8.3.4. *Eight points.* By straightforward computer check, one may verify that the quadratic relations generate the cubic and quartic relations when $n = 8$. By our main theorem on relations, we know that *all* relations are generated by the quadratic relations. There are fourteen generators and fourteen relations. We will use this calculation in [HMSV].

Generators for the ring of octagons:

$$A = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline\end{array} \quad B = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 7 & 8 \\ \hline\end{array} \quad C = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & 7 & 8 \\ \hline\end{array}$$

$$D = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 5 & 6 & 8 \\ \hline\end{array} \quad E = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 6 & 7 & 8 \\ \hline\end{array} \quad F = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 6 \\ \hline 3 & 5 & 7 & 8 \\ \hline\end{array}$$

$$G = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & 8 \\ \hline \end{array} \quad H = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & 8 \\ \hline \end{array} \quad I = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & 6 & 8 \\ \hline \end{array}$$

$$J = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 & 7 & 8 \\ \hline \end{array} \quad K = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & 7 & 8 \\ \hline \end{array} \quad L = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & 6 & 8 \\ \hline \end{array}$$

$$M = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & 7 & 8 \\ \hline \end{array} \quad N = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 6 & 8 \\ \hline \end{array}$$

Relations for the ring of octagons:

$$\begin{aligned}
0 &= -AH + AI + AM - AN + BF - BG - BK + BL \\
0 &= -BF + CE \\
0 &= +CL - CN - DK + DM \\
0 &= -FL + GK \\
0 &= +CG - CI - DF + DH + FI - FN - GH + GM \\
0 &= -BG + CG - CI + DE - DF + DI + FI - FN - GI + GN \\
0 &= -AH + BF - DF + DH + FI - FJ + FL + FM \\
&\quad -2FN - GH + HJ - HK + HN \\
0 &= +EL - EN - GJ + IJ \\
0 &= +EK - EM - FJ + FM - FN + HJ - HK + IK \\
0 &= -AH + BF - DF + DH + FI - FJ + FL + FM \\
&\quad -2FN - GH + HJ - HK + IM \\
0 &= -AI + DE - DF + DI + EK - EL - EM + EN \\
&\quad +FI - FJ + FL + FM - 2FN - GI + HJ - HK + IN \\
0 &= -BK + CJ + EK - EM - FJ + FM \\
&\quad -FN + JM - KM + KN \\
0 &= -AM + CG - CI + CJ - CL + CN - DF + DH \\
&\quad +FI - FJ + FL + FM - FN - GH + JM - KM + MN \\
0 &= -AN - BG - BK + CG - CI + CJ + DE - DF \\
&\quad +DI + DJ - DK + DN + EK - EM + FI - FJ \\
&\quad +FL + FM - 2FN - GI - GJ + JM + JN - KM + NN
\end{aligned}$$

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Benjamin Howard: Department of Mathematics, University of Maryland, College Park, MD 20742, USA, bhoward@math.umd.edu

John Millson: Department of Mathematics, University of Maryland, College Park, MD 20742, USA, jjm@math.umd.edu

Andrew Snowden: Department of Mathematics, Princeton University, Princeton, NJ 08544, USA, asnowden@math.princeton.edu

Ravi Vakil: Department of Mathematics, Stanford University, Stanford, CA 94305-2125, USA, vakil@math.stanford.edu