

An example of finding critical points

$$f := 5x^4 - 7x^2 + 8xy^3 + y^4 - 20xy$$

$$5x^4 - 7x^2 + 8xy^3 - 20xy + y^4$$

The first thing we do is to take the partial derivatives

$$f_x := \text{diff}(f, x)$$

$$20x^3 - 14x + 8y^3 - 20y$$

$$f_y := \text{diff}(f, y)$$

$$4y^3 + 24xy^2 - 20x$$

Now we solve the simultaneous equations $f_x=f_y=0$.

```
crits = solve([fx, fy], [x,y], Real)
Warning: possibly spurious solutions [solvelib::checkVectorSolutions]
Warning: possibly spurious solutions [solvelib::checkVectorSolutions]
```

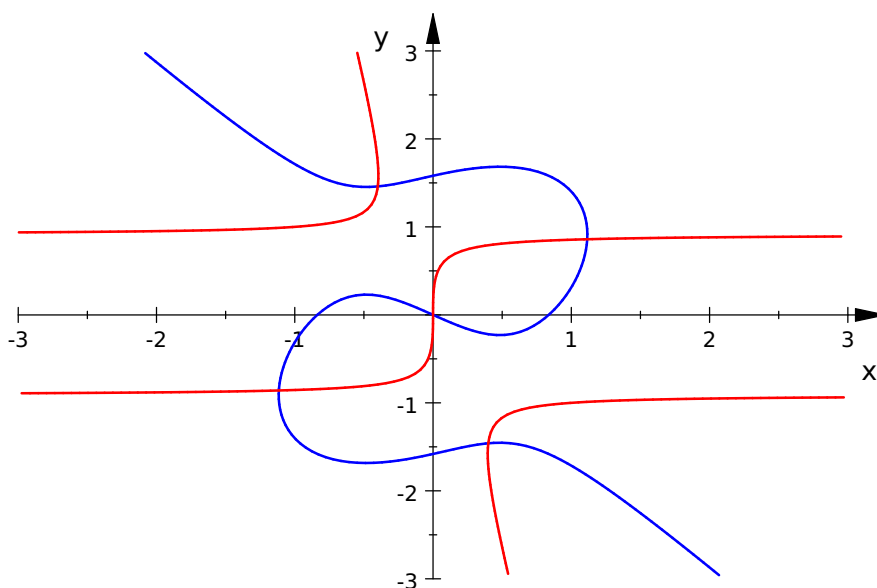
$$\text{crits} = \left(\begin{pmatrix} x \\ y \end{pmatrix} \in \left(\left\{ \left(-\frac{15372y^7}{625} + \frac{60414y^5}{625u} - \frac{14339y^3}{125} + \frac{216y}{5} \right) \mid u \in \mathbb{R} \right\} \cap \left\{ -\frac{15372u^7}{625} + \right.$$

where

$$\sigma_1 = \text{RootOf}\left(z_1^4 - \frac{2034z_1^3}{427} + \frac{3390z_1^2}{427} - \frac{4825z_1}{854} + \frac{625}{427}, z_1\right)$$

This is frankly a mess, so graphical methods look as if they will be better.

```
a:=plot(plot::Implicit2d(fx, x = -3..3, y = -3..3),
plot::Implicit2d(fy, x = -3..3, y = -3..3, LineColor = RGB::Red))
```



The plot $f_x=0$ is shown in blue, the plot $f_y=0$ is shown in red. The critical points of f are the intersection points. There is clearly one at the origin, and then there are two more above the x -axis and two more below the x -axis. We can see that the critical points besides $(0,0)$ are roughly at $(-0.4, 1.5)$, $(1.2, .8)$, $(.4, -1.5)$, and $(-1.2, -.8)$. We can

solve numerically to get good approximations.

```
crits:=numeric::solve([fx, fy], [x,y])
```

```
{[x = -0.5513368177 - 0.4890600055 i, y = 0.9777571534 - 0.09273909436 i], [x =
```

We have found the critical points, but some of them are complex. Let's print things in a table, to make them easier to read.

```
output::tableForm(crits)
```

```
[x = - 0.5513368177 + 0.4890600055*I, y = 0.9777571534 + 0.09273909436*I  
[x = - 0.5513368177 - 0.4890600055*I, y = 0.9777571534 - 0.09273909436*I  
[x = -0.3995623922, y = 1.459513254]  
[x = -1.114936411, y = -0.8593428195]  
[x = 0, y = 0]  
[x = 0.3995623922, y = -1.459513254]  
[x = 0.5513368177 + 0.4890600055*I, y = - 0.9777571534 + 0.09273909436*I  
[x = 0.5513368177 - 0.4890600055*I, y = - 0.9777571534 - 0.09273909436*I  
[x = 1.114936411, y = 0.8593428195]
```

Indeed there are 5 real critical points, roughly where we expected. There are also 4 complex critical points that we don't care about. Now let's classify the critical points by type. For this we need f_{xx} and the discriminant.

```
fxx:=diff(fx,x);  
disc:=fxx*diff(fy,y)-diff(fx,y)^2
```

$$60x^2 - 14$$

$$(60x^2 - 14)(12y^2 + 48xy) - (24y^2 - 20)^2$$

We need to evaluate these at the critical points. (Note: some critical points are complex and don't count.)

```
for j from 1 to 9 do  
  print([x,y,fxx,disc] | crits[j]);  
end_for
```

```
[-0.5513368177 - 0.4890600055 i, 0.9777571534 - 0.09273909436 i, -10.11244415
```

```
[0.5513368177 + 0.4890600055 i, -0.9777571534 + 0.09273909436 i, -10.1124441
```

```
[1.114936411, 0.8593428195, 60.58499204, 3317.966724]
```

```
[0.5513368177 - 0.4890600055 i, -0.9777571534 - 0.09273909436 i, -10.11244415
```

```
[-0.3995623922, 1.459513254, -4.420993683, -957.979358]
```

```
[0, 0, -14, -400]
```

```
[0.3995623922, -1.459513254, -4.420993683, -957.979358]
```

```
[-0.5513368177 + 0.4890600055 i, 0.9777571534 + 0.09273909436 i, -10.1124441
```

[-1.114936411, -0.8593428195, 60.58499204, 3317.966724]

Note that the ordering is different from before. The complex critical points are the ones numbered 1, 2, 4, and 8. The remaining ones are real. The ones near (1.11, 0.85) and its negative have f_{xx} and the discriminant positive, so those are local minima. The ones at (0,0) and near (-0.40, 1.45) and its negative have negative discriminant, so those are saddle points.