Lectures for Oberwolfach Seminar on Topological K-Theory of Noncommutative Algebras

Jonathan Rosenberg
University of Maryland

May, 2005

Outline:

I. A Survey of Bivariant K-Theories

II. Algebras of Continuous Trace, Twisted K-Theory

III. Crossed Products by $\mathbb{R}$ and Connes' Thom Isomorphism

IV. Applications to Physics

I. A Survey of Bivariant K-Theories

(1) Kasparov's KK — constructed from "generalized elliptic operators"

(2) BDF-Kasparov Ext — constructed from extensions of C^*-algebras by a stable C^*-algebra, modulo split extensions. BDF one-variable version constructed from extensions of C^*-algebras by $\mathcal{K}$.

(3) Algebraic Dual K-Theory — algebraic analogue of one-variable Ext

(4) Homotopy-Theoretic KK — analogue of KK constructed using homotopy theory, with "built-in UCT"

(5) Connes-Higson E-Theory — simpler replacement for KK, designed to eliminate certain difficulties

Of these, numbers 1, 2, and 5 make sense only for C^*-algebras. #3 and #4 make sense for arbitrary Banach (and even for many Fréchet) algebras. But Kasparov's KK is by far the most important, because of the way it "fits" both with classical index theory and with "exotic" index theory like Mishchenko-Fomenko theory.

We'll start with #3 and #4 since they can be defined out of one-variable K-theory.

I.1. Algebraic Dual K-Theory Let A be a local Banach algebra, and let $DK^j(A)$ (D for dual) be the set of commutative diagrams

$$\begin{array}{ccc} K_j(A;\mathbb{Q}) & \xrightarrow{\rho_*} & K_j(A;\mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{\rho} & \mathbb{Q}/\mathbb{Z}, \end{array}$$

where $\rho : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is the quotient map. Then $DK^*(A)$ can be made into an abelian group, a subgroup of

$$\mathrm{Hom}(K_j(A;\mathbb{Q}),\mathbb{Q}) \oplus \mathrm{Hom}(K_j(A;\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}).$$

Theorem 1 *DK^* is a cohomology theory on local Banach algebras and satisfies Bott periodicity and a UCT exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_{j-1}(A),\mathbb{Z}) \to DK^j(A) \to \mathrm{Hom}(K_j(A),\mathbb{Z}) \to 0.$$

Proof. Clearly DK^* is a contravariant homotopy functor with Bott periodicity. The UCT map

$$DK^j(A) \twoheadrightarrow \text{Hom}(K_j(A), \mathbb{Z})$$

comes from chasing the diagram

$$\begin{array}{ccccccc}
K_j(A) & \longrightarrow & K_j(A;\mathbb{Q}) & \xrightarrow{\rho_*} & K_j(A;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\partial} & K_{j-1}(A) \\
\vdots & & \downarrow & & \downarrow & & \\
\mathbb{Z} & \rightarrowtail & \mathbb{Q} & \xrightarrow{\rho} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

The same diagram also gives the left side of the UCT exact sequence once we remember that $\text{Ext}^1_{\mathbb{Z}}(K_{j-1}(A), \mathbb{Z})$ is the cokernel of the map

$$\text{Hom}(K_{j-1}(A), \mathbb{Q}) \to \text{Hom}(K_{j-1}(A), \mathbb{Q}/\mathbb{Z}).$$

We need to show that DK^* comes with long exact sequences, and this uses exactness of the functors $\text{Hom}(\_\_, \mathbb{Q})$ and $\text{Hom}(\_\_, \mathbb{Q}/\mathbb{Z})$, which in turn relies on the fact that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible and thus injective as $\mathbb{Z}$-modules. $\square$

I.2. Homotopy-Theoretic KK-Theory

We will be brief about this since formal definitions require a lot of machinery. If A and B are local Banach algebras, the K-groups of A and B are in fact homotopy groups of spectra $\mathbb{K}(A)$ and $\mathbb{K}(B)$, in fact of $\mathbb{K}$-module spectra, where $\mathbb{K} = \mathbb{K}(\mathbb{C})$ is the spectrum of complex K-theory. Thus one can define

$$\mathbb{KK}(A, B) = \text{Hom}_{\mathbb{K}}(\mathbb{K}(A), \mathbb{K}(B)),$$

computed in a suitable category. This is itself a $\mathbb{K}$-module spectrum, so it has homotopy groups satisfying Bott periodicity which one can call the homotopy-theoretic KK-groups of A and B, $HKK_*(A, B)$. Properties of the category of $\mathbb{K}$-module spectra, studied by Bousfield, imply that one has a UCT exact sequence

$$0 \to \text{Ext}^1_{\mathbb{Z}}(K_{*-1}(A), K_*(B)) \to HKK(A, B) \to \text{Hom}(K_*(A), K_*(B)) \to 0.$$

It's fairly easy to see that all the other bivariant K-theories we are discussing have natural transformations to HKK, which in good cases are isomorphisms. This gives a way to prove a UCT in many situations.

I.3. BDF Ext-Theory Of great historical importance, because of its connection with the Weyl-von Neumann Theorem, is BDF (Brown, Douglas, Fillmore) Ext-theory. Let $\mathcal{L} = \mathcal{L}(\mathcal{H})$ be the algebra of bounded operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$, and let $\mathcal{Q} = \mathcal{L}/\mathcal{K}$ be the Calkin algebra. If A is a separable C^*-algebra, each extension of A by $\mathcal{K}$ is a pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \tau & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0. \end{array}$$

Thus we think of $*$-homomorphisms $\tau : A \to \mathcal{Q}$ as extensions, and divide out by conjugation via unitaries in $\mathcal{L}$. We can add extensions via

$$A \xrightarrow{\tau_1 \oplus \tau_2} \mathcal{Q} \oplus \mathcal{Q} \longrightarrow \mathcal{Q}(\mathcal{H} \oplus \mathcal{H}) \cong \mathcal{Q}.$$

The result is well-defined modulo unitary conjugation, and makes classes of extensions into an abelian semigroup. After dividing out by the split extensions (this is unnecessary, by a result of Voiculescu, if A is nonunital), those τ's with a lifting

$$\begin{array}{ccc} & & \mathcal{L} \\ & \nearrow & \downarrow \\ A & \xrightarrow{\tau} & \mathcal{Q}, \end{array}$$

we obtain an abelian semigroup $\mathrm{Ext}(A)$.

Theorem 2 (Arveson, Choi-Effros) *An extension $\tau : A \to \mathcal{Q}$ is invertible in* $\mathrm{Ext}(A)$ *if and only if it has a completely positive listing $A \to \mathcal{L}$. The liftable extensions form a group, and if A is nuclear, this group is all of* $\mathrm{Ext}(A)$.

Theorem 3 (O'Donovan, Salinas) Ext *is homotopy-invariant on quasidiagonal C^*-algebras.*

It is easy to construct a natural transformation Ext $\to DK^1$. In fact, given an extension

$$0 \to \mathcal{K} \to E \to A \to 0,$$

tensor the extension with nuclear C^*-algebras C with $K_1(C) = 0$ and $K_0(C) = \mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. Then use the connecting map $K_1(A \otimes C) \xrightarrow{\partial} K_0(\mathcal{K} \otimes C) \cong K_0(C)$ in the long exact K-theory sequences for the tensored extensions to define an element of DK^1. (For example, to define K-theory with coefficients in $\mathbb{Q}$, one can take C to be the "universal UHF algebra.") In "favorable circumstances," e.g., for A a type I C^*-algebra, this natural transformation Ext $\to DK^1$ is an isomorphism, and thus we obtain the UCT for Ext, originally due to Brown.

I.4. Kasparov KK-Theory Kasparov theory really deserves a course in itself, but we at least should explain what it is and why it's so important. Given a C^*-algebra B, one has the notion of a (right) Hilbert B-module $\mathcal{E}$. This is like a Hilbert space, except that the inner product takes values in B. As usual, we require $\langle v, v \rangle \geq 0$, but here $\geq$ is to be interpreted in the sense of C^*-algebras, and the norm is defined by $\|v\|_{\mathcal{E}}^2 = \|\langle v, v \rangle\|_B$.

Examples: If $B = \mathbb{C}$, this is a Hilbert space. If $B = C_0(X)$, this is the space of sections of a continuous field of Hilbert spaces over X. B is always a Hilbert module over itself, with inner product $\langle v, w \rangle = v^* w$. $\ell^2(B)$ is the Hilbert module of sequences $\{b_k\}$ with $\sum b_k^* b_k$ convergent in B.

If $\mathcal{E}$ is a (right) Hilbert B-module, we can define C^*-algebras $\mathcal{K}(\mathcal{E})$ and $\mathcal{L}(\mathcal{E})$, consisting of bounded B-linear operators on $\mathcal{E}$ having adjoints with respect to the inner product. $\mathcal{L}(\mathcal{E})$ is the set of all such operators, while $\mathcal{K}(\mathcal{E})$ is the closed linear span of those of "rank one." If $\mathcal{E} = \ell^2(B)$, $\mathcal{K}(\mathcal{E}) \cong B \otimes \mathcal{K}$, and $\mathcal{L}(\mathcal{E}) \cong \mathcal{M}(B \otimes \mathcal{K})$.

If A and B are separable C^*-algebras, $KK(A, B)$ is the abelian group of Kasparov A-B bimodules modulo "homotopy." A Kasparov A-B bimodule is a pair $(\mathcal{E}, T)$, where $\mathcal{E} = \mathcal{E}^0 \oplus \mathcal{E}^1$ is $\mathbb{Z}/2$-graded and is both a right Hilbert B-module and a left A-module (via a map $\phi : A \to \mathcal{L}(\mathcal{E})$). $T \in \mathcal{L}(\mathcal{E})$ is required to be odd (i.e., to interchange $\mathcal{E}^0$ and $\mathcal{E}^1$), self-adjoint, and to satisfy

$$\begin{cases} \phi(a)(T^2 - 1) \in \mathcal{K}, \\ T\phi(a) - \phi(a)T \in \mathcal{K}, \end{cases}$$

for all $a \in A$.

As we mentioned, $KK(A, B)$ consists of Kasparov A-B bimodules modulo homotopy. A homotopy is the one-parameter family of Kasparov A-B bimodules induced by a single A-$C([0, 1], B)$ bimodule.

There are a number of important natural examples of Kasparov A-B bimodules. The first motivating example comes from index theory. Suppose $A = C(X)$, where X is a compact even-dimensional spin manifold, and let $B = \mathbb{C}$ and $\mathcal{E}$ be the Hilbert space of sections of the complex spinor bundle of X. This is $\mathbb{Z}/2$-graded by the splitting into half-spinor bundles. X has a Dirac operator D, which we can view as an unbounded self-adjoint operator on $\mathcal{E}$. The operator D is odd with respect to the grading, and since D is a first-order differential operator, D has bounded commutator with functions in $C^\infty(X)$.

Now define $T = D(D^2+1)^{-1/2}$ by functional calculus. This is still self-adjoint and odd with respect to the grading, but is also bounded (obvious). $T^2 - 1 \in \mathcal{K}$ by ellipticity of D, and $[T, \phi(a)] \in \mathcal{K}$ for $a \in C(M)$ since $[D, \phi(a)] \in \mathcal{L}$ for $a \in C^\infty(M)$ and $(D^2+1)^{-1/2}$ is compact (by ellipticity and pseudodifferential calculus, for example).

A $*$-homomorphism $\phi : A \to B$ can also be viewed as a special case of a Kasparov A-B bimodule; simply take $\mathcal{E}^0 = B$, $\mathcal{E}^1 = 0$ and $T = 0$. Since $\mathcal{E}$ is a rank-one B-module, $\mathcal{K}(\mathcal{E}) = B$ and $\mathcal{L}(\mathcal{E}) = \mathcal{M}(B)$, so all conditions are satisfied.

Finally, there's one other important example, that comes from extension theory. An extension of C^*-algebras

$$0 \to B \otimes \mathcal{K} \to E \to A \to 0,$$

assuming it has a completely positive splitting $s : A \to E$, gives a class not in $KK(A, B)$ but in the "shifted" group $KK^1(A, B)$.

One way to define the group $KK^1(A,B)$ is via Kasparov A-B bimodules as before, but this time without the $\mathbb{Z}/2$-grading (and of course without the requirement that the operator T be odd).

The point is that the completely positive splitting and the "generalized Stinespring dilation theorem" imply the extension comes from a morphism $\phi : A \to \mathcal{M}(B \otimes \mathcal{K}) \cong \mathcal{L}(\ell^2(B))$ and a projection $p \in \mathcal{L}(\ell^2(B))$ commuting with $\phi(A)$ modulo $B \otimes \mathcal{K} \cong \mathcal{K}(\ell^2(B))$. Then we can take $T = 2p - 1$, which satisfies $T^2 = 1$.

Incidentally, it would appear that a $*$-homomorphism $\phi : A \to B$ should also define an element of $KK^1(A,B)$ (with $\mathcal{E} = B$, $T = 0$), but this element is trivial, since it is homotopic to the module with $\mathcal{E} = B$, $T = 1$, which is "degenerate" (i.e., has $T^2 - 1$ and $[T, \phi(a)]$ actually 0, not just compact, for all a).

I.5. Connes-Higson E-Theory

The last bivariant theory, E-theory, is defined using the notion of asymptotic morphism $\phi : A \dashrightarrow B$. This is not a $*$-homomorphism but a 1-parameter family of (set-theoretic) maps $\phi_t : A \to B$ which are a $*$-homomorphism "in the limit," e.g., $\phi_t(a_1)\phi_t(a_2) \to \phi_t(a_1 a_2)$ as $t \to \infty$. Any $*$-homomorphism defines an asymptotic morphism (constant in t). There is an obvious notion of homotopy for asymptotic morphisms. The notation $[[A, B]]$ denotes the homotopy classes of asymptotic morphisms $A \dashrightarrow B$.

$E(A, B)$ is defined to be $[[SA, SB \otimes \mathcal{K}]]$, where S denotes C^*-algebraic suspension (tensor product with $C_0(\mathbb{R})$). The suspension and/or stabilization are used to define a good addition operation; in some cases one doesn't need both.

Theorem 4 (Connes-Higson) *Any homotopy-invariant, half-exact, stable functor on separable C^*-algebras factors through E-theory.*

If A and B are separable C^*-algebras and A is nuclear, then $E(A, B)$ is naturally isomorphic to $KK(A, B)$.

However, the advantage of E over KK is that the former is well-behaved (has exact sequences in both variables) even when A is not nuclear.

II. Algebras of Continuous Trace, Twisted K-Theory

II.1. Algebras of Continuous Trace

Let X be a locally compact Hausdorff space. An algebra of continuous trace over X is a C^*-algebra A with spectrum X, such that for each $x_0 \in X$, there is an element $a \in A$ such that $x(a)$ is a rank-one projection for each x in a neighborhood of x_0 (Fell's condition). Such algebras were studied by Fell and Dixmier-Douady, and are algebras of sections of continuous fields of elementary C^*-algebras.

For simplicity, assume X 2nd countable ($C_0(X)$ separable) and consider only separable algebras. As far as K-theory is concerned, it is no loss of generality to stabilize, i.e., to tensor with $\mathcal{K}$. Since $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$ (for the C^*-tensor product), this is the same as restricting to algebras A with $A \cong A \otimes \mathcal{K}$.

Theorem 5 (Dixmier-Douady) *Any stable separable algebra of continuous trace over X is isomorphic to $\Gamma_0(\mathcal{A})$, the sections vanishing at infinity of a locally trivial bundle of algebras over X, with fibers $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K}) = PU = U/\mathbb{T}$. Classes of such bundles are in natural bijection with $H^3(X, \mathbb{Z})$ (Čech cohomology).*

Proof. Local triviality is mostly general topology and uses paracompactness. We just explain the last part. The point is that U (in the weak operator topology) is contractible, so PU has the homotopy type of $BS^1 = K(\mathbb{Z}, 2)$, and BPU has the homotopy type of $K(\mathbb{Z}, 3)$. Principal PU-bundles over X are thus classified by

$$[X, BPU] = [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z}).$$

□

The group $H^3(X, \mathbb{Z})$ can also be described as the Brauer group of $C_0(X)$, i.e., the group of algebras of continuous trace over X modulo Morita equivalence over X. The group operation then corresponds to tensor product. This point of view was first developed by Green (1970's, unpublished) and later used by Williams, Raeburn, et al.

For X a finite CW complex, Serre and Grothendieck had earlier studied the Brauer group of $C(X)$ in the purely algebraic sense, i.e., the group of algebras of sections of bundles of matrix algebras over X, modulo algebraic Morita equivalence over X. Translated into our language, their result is:

Theorem 6 *Let X be a finite CW complex. Then an element of the Brauer group $H^3(X, \mathbb{Z})$ of continuous-trace algebras over X is represented by a bundle of finite-dimensional matrix algebras if and only if the class is torsion.*

II.2. Twisted K-Theory Now we can define the twisted K-theory of a (locally compact) space X with respect to a cohomology class $\delta \in H^3(X, \mathbb{Z})$ as the K-theory of the corresponding continuous-trace algebra $CT(X, \delta)$ (which is locally isomorphic to $C_0(X, \mathcal{K})$. This is somewhat analogous to the twisted cohomology (or cohomology with local coefficients) attached to a flat line bundle.

Example: Let $X = S^3$, so that $H^3(X) \cong \mathbb{Z}$. Thus we have a stable CT algebra over X for each integer n. It can be obtained by clutching together two copies of $C(D^3, \mathcal{K})$ via a map $S^2 \to \mathrm{Aut}(\mathcal{K}) = PU$ of degree n. One finds that if $n \neq 0$,

$$K_*(CT(S^3, \delta_n)) = \begin{cases} 0, & * \text{ even}, \\ \mathbb{Z}/n, & * \text{ odd}. \end{cases}$$

III. Crossed Products by $\mathbb{R}$ and Connes' Thom Isomorphism

For what we will do later we will need a few facts about crossed products by $\mathbb{R}$, closely related to the Pimsner-Voiculescu sequence for crossed products by $\mathbb{Z}$. First let's mention the Takai Duality Theorem.

Theorem 7 (Takai) *Let A be a C^*-algebra and let α be an action of a locally compact abelian group on A. Let $A \rtimes_\alpha G$ be the C^* crossed product. Recall this is the completion of $C_c(G, A)$ in the universal C^* norm, with convolution multiplication determined by the formal relation $g \cdot a \cdot g^{-1} = \alpha_g(a)$. Define the dual action $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes_\alpha G$ by multiplication by $\widehat{G}$ on functions on G. (The formal relations are $\gamma \cdot a = a \cdot \gamma$, $\gamma \cdot g \cdot \gamma^{-1} = \langle \gamma, g \rangle g$.) Then*

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}.$$

Proof. Proof is just like that of the Stone-von Neumann-Mackey Theorem, which is the special case $A = \mathbb{C}$. □

Theorem 8 (Connes) *Let A be a C^*-algebra and let α be an action of $\mathbb{R}$ on A. Then there is a natural isomorphism*

$$\phi : K_*(A) \to K_{*+1}(A \rtimes_\alpha \mathbb{R}).$$

Put another way, the K-theory of $A \rtimes \mathbb{R}$ is in some sense independent of the action α.
Proof. We will sketch two proofs, Connes' original one and a modification of one due to Rieffel. In both cases there are two steps, construction of ϕ and the proof that it's an isomorphism.

The original proof of Connes relies on the "2×2 matrix trick."

Lemma 9 (Connes) *Let α be an action of a locally compact group G on a C^*-algebra A, and let u be a unitary cocycle for G. (Thus u is a strictly continuous map $G \to U(\mathcal{M}(A))$ and $u_{gh} = u_g \alpha_g(u_h)$.) Then there is an action of G on $M_2(A)$ restricting to α on one corner and to α' on the other corner. Here $\alpha'_g = \mathrm{Ad}\, u_g \circ \alpha_g$.*

Proof. The cocycle condition guarantees that α' is an action. Simply define β on $M_2(A)$ by the formula:

$$\beta_g \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_t(a) & \alpha_t(b)u_t^* \\ u_t\alpha_t(c) & u_t\alpha_t(d)u_t^* \end{pmatrix}$$

and check that it works. □

Actions α and α' related as in Lemma 9 are called exterior equivalent.

In many ways, the most satisfying proof of Theorem 8 is the original one by Connes. This depends on the following lemma.

Lemma 10 (Connes) *Let α be an action of a $\mathbb{R}$ on a C^*-algebra A, and let e be a projection in A which is a smooth vector for α. Then there is an exterior equivalent action α' of $\mathbb{R}$ on A which fixes e.*

Proof. The fact that e is α-smooth means that it lies in the domain of the derivation δ which is the infinitesimal generator of α. Write δ formally as $i \operatorname{ad} H$, where H is an unbounded self-adjoint multiplier of A. Then replace H by

$$H' = eHe + (1-e)H(1-e) = H + i[\delta(e), e],$$

which commutes with e. Define α' by $\operatorname{Ad}\left(e^{itH'}\right)$, defined by expanding the series, and check that it works. □

Proof of Theorem 8 *from Lemma* 10. If ϕ is to be natural and compatible with suspension, it's enough to define it on classes of projections $e \in A$. Since we can perturb a projection to a smooth projection, and close projections are equivalent in K_0, we may assume e is smooth. Apply Lemmas 10 and 9. We obtain an action β on $M_2(A)$ with α in one corner and α' in the other corner, where α' fixes e. The inclusions $A \hookrightarrow M_2(A)$ into the two corners are both isomorphisms on K-theory, and are equivariant for α, resp., α'. So we can reduce to the case where e is fixed. Then $1 \mapsto e$ is an equivariant map $\mathbb{C} \to A$, so $\phi([e])$ is defined by naturality from the trivial case $A = \mathbb{C}$, $A \rtimes \mathbb{R} \cong C_0(\mathbb{R})$, where there is an obvious isomorphism $K_0(\mathbb{C}) \to K_1(C_0(\mathbb{R}))$. The fact that ϕ is an isomorphism follows from naturality and Takai duality. $\square$

Another proof of Theorem 8. We give another proof based on the Pimsner-Voiculescu sequence. This is based on ideas from a different proof by Rieffel. An advantage of this proof is that it might work for local Banach algebras. Start by defining an action of $\mathbb{R}$ on $C_0([0,1), A)$ by

$$(\widetilde{\alpha}_t f)(s) = \alpha_{ts}(f(s)).$$

Note that we have an exact sequence

$$0 \to S(A \rtimes_\alpha \mathbb{R}) \to C_0([0,1), A) \rtimes_{\widetilde{\alpha}} \mathbb{R} \xrightarrow{(e_0)_*} SA \to 0.$$

The map ϕ will just be the index map for the corresponding K-theory exact sequence. Since $C_0([0,1), A)$ is contractible, "all" we need to show is that $K_*(B) = 0$ implies $K_*(B \rtimes \mathbb{R})$. (Here $B = C_0([0,1), A)$ and the action is $\widetilde{\alpha}$.)

Since we want to use Pimsner-Voiculescu, we want to relate crossed products by $\mathbb{R}$ to crossed products by $\mathbb{Z}$. Now we need:

Theorem 11 (Packer-Raeburn) *If α is an action of a locally compact group G on a C^*-algebra, and if N is a closed normal subgroup, then after stabilizing, $B \rtimes G$ is an iterated crossed product $((B \otimes \mathcal{K}) \rtimes N) \rtimes (G/N)$.*

To finish the proof, use the Packer-Raeburn trick to write

$$(B \otimes \mathcal{K}) \rtimes \mathbb{R} \cong D \rtimes_\beta (\mathbb{R}/\mathbb{Z}), \quad D = (B \otimes \mathcal{K}) \rtimes \mathbb{Z}.$$

By P-V, $K_*(B) = 0$ implies $K_*(D) = 0$. So we need to show this implies $K_*(D \rtimes_\beta (\mathbb{R}/\mathbb{Z})) = 0$. Use Takai Duality (possibly with additional stabilization) to write

$$D \cong (D \times_\beta \mathbb{R}/\mathbb{Z}) \rtimes_{\widehat{\beta}} \mathbb{Z}.$$

Reiterating,

$$D \cong (D \times_\beta \mathbb{R}/\mathbb{Z}) \rtimes_{\widehat{\beta}} \mathbb{Z}.$$

By P-V again and the fact that $K_*(D) = 0$, we see $1-(\widehat{\beta})_*$ is an isomorphism of $K_*(D \times_\beta \mathbb{R}/\mathbb{Z})$. But $\widehat{\beta}$ is the restriction of the $\widehat{\mathbb{R}}$-action $\widehat{\alpha}$, so it acts trivially on K-theory. Thus $1-(\widehat{\beta})_*$ is both 0 and bijective. So $K_*(D \rtimes_\beta (\mathbb{R}/\mathbb{Z})) = 0$, and $K_*(B \rtimes \mathbb{R}) = 0$. So we've seen that $K_*(B) = 0$ implies $K_*(B \rtimes \mathbb{R}) = 0$, which completes the proof of Connes' theorem. $\square$

IV. Applications to Physics

K-theory, including twisted K-theory, is starting to appear in the physics literature quite frequently. Good first places to look are

E. Witten, D-Branes and K-Theory, J. High Energy Physics 12 (1998) 019.
D. Freed, K-Theory in Quantum Field Theory, math-ph/0206031.

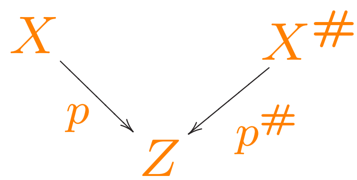
The idea, to quote Witten, is that "D-brane charge takes values in the K-theory of space-time." (In string theory, a D-brane is a submanifold of space-time on which strings can begin and end. The "D" stands for Dirichlet and has to do with the boundary conditions on open strings.) Twisting of K-theory comes in because of a background field, called the H-flux, given by a 3-dimensional cohomology class.

I will not try to explain the physics involved in all this but instead will discuss some mathematics related to it.

Another interesting feature of string theory is the notion of T-duality (T for torus), which postulates an equivalence of theories on two different space-times X and $X^\#$, which are related by exchange of tori in X by their dual tori in $X^\#$. Let's try to make this precise in the case where the tori involved are 1-dimensional. The duality in this case should exchange Type IIA and Type IIB theories (for those who know what this means).

We consider two principal $\mathbb{T}$-bundles X and $X^\#$ over a common base Z. Each is supposed to be equipped with an H-flux, so there are associated cohomology classes δ and $\delta^\#$ in $H^3(X)$ and $H^3(X^\#)$, respectively.

From the diagram



and the classes $\delta \in H^3(X)$, $\delta^{\#} \in H^3(X^{\#})$, we have continuous-trace algebras $CT(X, \delta)$ and $CT(X^{\#}, \delta^{\#})$. The circle group $\mathbb{T}$ acts freely on X and $X^{\#}$, but not necessarily on $CT(X, \delta)$ and $CT(X^{\#}, \delta^{\#})$. In fact, given an action of a group G on a space X and a class $\delta \in H^3(X)$, the action lifts to an action on $CT(X, \delta)$ if and only if

(a) G fixes δ in H^3, and

(b) the G-action on X lifts to an action on the principal PU-bundle associated to δ.

In our situation, (a) is obvious since the G involved is connected, but (b) is unclear. In fact, one can check:

Lemma 12 (Raeburn-Williams-Rosenberg) *The $\mathbb{T}$-action on X lifts to an action on the principal bundle associated to δ if and only if $\delta \in p^*(H^3(Z))$. But if we view $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, the action always lifts to $\mathbb{R}$.*

Proof. One can do the first part purely topologically, but also one can observe that since $\mathbb{T}$ acts transitively on fibers of p, if there were an action α on $CT(X,\delta)$, then $CT(X,\delta) \rtimes_\alpha \mathbb{T}$ would be a continuous-trace algebra over Z, say with class $c \in H^3(Z)$, and by Takai duality, we'd have

$$CT(X,\delta) \cong CT(Z,c) \rtimes_{\widehat{\alpha}} \mathbb{Z} \cong p^*CT(Z,c).$$

For the second part, say X and Z are manifolds and everything is smooth. (One can reduce to this case.) Then one can construct a lifting using a connection on the bundle. $\square$

Now let's come back to T-duality. $X \xrightarrow{p} Z$ and $X^{\#} \xrightarrow{p^{\#}} Z$ should be T-dual if the fibers of $p^{\#}$ are dual to the fibers of p, if there is a well-defined procedure for creating $(X^{\#}, \delta^{\#})$ from (X, δ), if doing this process twice gets us back where we started, and if there is a natural isomorphism of twisted K-theories

$$K^*(X, \delta) \cong K^{*+1}(X^{\#}, \delta^{\#}).$$

(The last condition is forced by equivalence of the IIA string theory on X and the IIB theory on $X^{\#}$.) In fact we can achieve all of these.

Theorem 13 (Raeburn-Rosenberg) *Lift the $\mathbb{T}$-action on X to an $\mathbb{R}$-action α on $CT(X, \delta)$. It turns out all such choices are exterior equivalent. Then*

$$CT(X, \delta) \rtimes_\alpha \mathbb{R} \cong CT(X^{\#}, \delta^{\#}),$$
$$K^*(X, \delta) \cong K^{*+1}(X^{\#}, \delta^{\#}).$$

Here $X^{\#} \xrightarrow{p^{\#}} Z$ is a principal $\mathbb{T}$-bundle over Z whose fibers are naturally dual to the fibers of p. Doing this twice gets us back to (X, δ).

In fact, we have formulas from which $p^\#$ *and* $\delta^\#$ *can be computed. Recall that a principal* $\mathbb{T}$*-bundle over* Z *is determined by a characteristic class* $[p] \in H^2(Z)$*, and that for any circle bundle, we have a Gysin sequence*

$$\cdots \to H^1(Z) \xrightarrow{\cup [p]} H^3(Z) \xrightarrow{p^*} H^3(X) \xrightarrow{p_!} H^2(Z) \to \cdots .$$

Then

$$p_!(\delta) = [p^\#], \quad (p^\#)_!(\delta^\#) = [p].$$

Proof. We don't have room for all the details, but it's easy to see that $CT(X, \delta) \rtimes_\alpha \mathbb{R}$ must be a continuous trace algebra with spectrum a circle bundle over Z. Furthermore, Takai duality shows X and $X^\#$ play symmetrical roles. The isomorphism of twisted K-theories follows from Connes' Theorem in the previous lecture. The characteristic class formula is proved by checking certain examples and using functoriality.

Certainly, if δ is in the image of p^*, then α can be chosen trivial on $\mathbb{Z} = \ker(\mathbb{R} \to \mathbb{T})$. Then $CT(X, \delta) \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z} \cong CT(X, \delta) \otimes C(S^1)$. So by Packer-Raeburn (since things are stable)

$$CT(X^{\#}, \delta^{\#}) \cong CT(X \times S^1, \delta \times 1) \rtimes \mathbb{T},$$

with $\mathbb{T}$ acting freely on X and trivially on S^1, so $X^{\#} = Z \times S^1$ and $p^{\#}$ is a trivial bundle. But if p is trivial (so $X = S^1 \times Z$) and $\delta = a \times b$, where a is the generator of $H^1(S^1)$ and $b \in H^2(Z)$, then $p_!(\delta) = b$. Furthermore, it is known there is an action θ of $\mathbb{Z}$ on $C_0(Z, \mathcal{K})$ with $C_0(Z, \mathcal{K}) \rtimes_\theta \mathbb{Z}$ having spectrum T, where $T \to Z$ is the principal bundle with characteristic class b. If one forms $\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}} C_0(Z, \mathcal{K})$, one can check that this is isomorphic to $CT(X, \delta)$. Thus we can assume $\alpha = \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}} \theta$, so

$$\begin{aligned} CT(X^{\#}, \delta^{\#}) &\cong \left(\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}} C_0(Z, \mathcal{K})\right) \rtimes_{\mathrm{Ind}\,\theta} \mathbb{R} \\ &\cong_{\text{Morita}} C_0(Z, \mathcal{K}) \rtimes_\theta \mathbb{Z}, \end{aligned}$$

which has spectrum T. So $[p^{\#}] = b = p_!(\delta)$. The general cases are reduced to these. $\square$

The conclusion of this analysis is that use of crossed products of continuous-trace algebras, twisted K-theory, and the Connes Thom isomorphism enables us to put on a firm mathematical basis a phenomenon suggested empirically by physicists!