A K-theory perspective on T-duality in string theory

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joint work with
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Abstract

An idea which is now well established in the physics literature is that "charges" on "branes" should take values in twisted (topological) K-theory, where the twisting is given by a cohomology class that represents the field strength. It is also expected that "T-duality" should hold, meaning that the theory on one space-time (with background field) is equivalent to that on another, where tori are replaced by their duals. I will describe recent joint work with Mathai Varghese in which we show how to make this rigorous for space-times which are principal torus bundles. A surprising conclusion is that sometimes the T-dual of a torus bundle turns out to involve noncommutative tori.

Some Ideas from Physics

Physics is described by "fields" ϕ (usually sections of vector bundles or connections on vector bundles) living on a manifold ("spacetime") X. They are subject to equations of motion. In classical physics, the fields should give a critical point for the "action" $S(\phi)$. In quantum physics, the theory is described by "path integrals" like the partition function

$$Z(\beta) = \int e^{-S(\phi;\beta)}\, d\phi,$$

obtained by integrating over all states, with the classical solutions (minima of S) contributing most heavily. Here β is the inverse temperature.

The Idea of T-Duality

It has been noticed that many quantum mechanical systems, especially in supersymmetric string theory, come with a symmetry known as T-duality. This means that a theory living on a torus is formally equivalent to one living on the dual torus.

The simplest example is a free particle on a torus $\mathbb{R}^n/\Lambda$, where Λ is a lattice in $\mathbb{R}^n$. The partition function turns out to be the classical theta function

$$Z_\Lambda(\beta) = \sum_{z\in\widehat{\Lambda}} e^{-2\pi^2\beta|z|^2}$$

with $\widehat{\Lambda}$ the dual torus. By Poisson Summation, this is essentially the same as the corresponding function $Z_{\widehat{\Lambda}}$ for the dual torus $\widehat{\mathbb{R}}^n/\widehat{\Lambda}$.

H-Fluxes, K-Theory, and Twistings

In some field theories, there are "background fields" given by cohomology classes. For example, in classical electromagnetism, the field strength of the electromagnetic field defines a class in H^2. In (type II) string theory, there is a field or H-flux $\delta \in H^3$.

In addition, in some field theories there are "topological charges" living in (topological) K-theory. In string theory, these arise from the charges on D-branes, submanifolds which serve as "boundaries" for "open strings."

In the presence of a background H-flux given by $\delta \in H^3(X, \mathbb{Z})$, the picture must be twisted and the charges take their values in twisted K-theory $K^*_\delta(X)$. When H is torsion, this was defined by Karoubi and Donovan. In general, it is the K-theory of a stable continuous-trace algebra $CT(X, \delta)$ locally isomorphic to $C_0(X, \mathcal{K})$, $\mathcal{K}$ the compact operators, with twisting given by δ.

Explanation:

Every $*$-automorphism of $\mathcal{K}$ comes from conjugation by a unitary operator, so $\operatorname{Aut}\mathcal{K} = PU$. This is a $K(\mathbb{Z},2)$-space, so BPU is a $K(\mathbb{Z},3)$-space and a class $\delta \in H^3(X,\mathbb{Z})$ gives rise to a principal PU-bundle over X and an associated $\mathcal{K}$-bundle of C^*-algebras. $CT(X,\delta)$ is the algebra of sections (vanishing at infinity) of this bundle of algebras, and δ is called the Dixmier-Douady invariant.

One can in fact identify $H^3(X,\mathbb{Z}) = \operatorname{Br} X$ with the Brauer group of continuous-trace algebras over X, just as its torsion group is the Brauer group of Azumaya algebras over X with finite-dimensional fibers.

Basic Setup and the Mathematical Problem

Consider a "spacetime X compactified over a torus T," i.e., a locally compact, homotopically finite connected space X equipped with a free action of a torus T. We have a principal T-bundle

$$p\colon X \to Z$$

and an H-flux class $\delta \in H^3(X, \mathbb{Z})$. Does this situation have a T-dual, and if so, what is it?

If there is a classical T-dual, we expect to have another principal torus bundle

$$p^\#\colon X^\# \to Z$$

and an H-flux class $\delta^\# \in H^3(X^\#, \mathbb{Z})$ such that the fibers of $p^\#$ are "dual" to the fibers of p, and such that there is a K-theory isomorphism

$$K^*_\delta(X) \cong K^*_{\delta^\#}(X^\#)$$

(possibly with a degree shift).

T-Dualizability of Bundles with H-Flux

Let $p\colon X \to Z$ be a principal T-bundle as above, with T an n-torus, G its universal cover (a vector group). Also let $\delta \in H^3(X, \mathbb{Z})$. For the pair (X, δ) to be dualizable, we want the T-action on X to be in some sense compatible with δ. A natural hope is for the T-action on X to lift to an action on the principal PU-bundle defined by δ, or equivalently, to an action on $CT(X, \delta)$. Equivariant Morita equivalence classes of such liftings (with varying δ) define classes in the equivariant Brauer group. Unfortunately

$$p^*\colon \ \mathrm{Br}(Z) \xrightarrow{\cong} \mathrm{Br}_T(X)$$

and so $\mathrm{Br}_T(X)$ is not that interesting. But $\mathrm{Br}_G(X)$, constructed from local liftings, is quite a rich object.

Theorem 1 *Let*

$$T \overset{\iota}{\hookrightarrow} X \overset{p}{\twoheadrightarrow} Z$$

be a principal T-bundle as above, with T an n-torus, G its universal cover (a vector group). The following sequence is exact:

$$\mathrm{Br}_G(X) \xrightarrow{F} \mathrm{Br}(X) \cong H^3(X, \mathbb{Z}) \xrightarrow{\iota^*} H^3(T, \mathbb{Z}).$$

Here F is the "forget G-action" map.

In particular, if $n \leq 2$, every stable continuous-trace algebra on X admits a G-action compatible with the T-action on X.

When such a G-action exists, we will construct a T-dual by looking at the C^*-algebra crossed product

$$CT(X, \delta) \rtimes G.$$

The desired K-theory isomorphism will come from Connes' "Thom isomorphism" theorem.

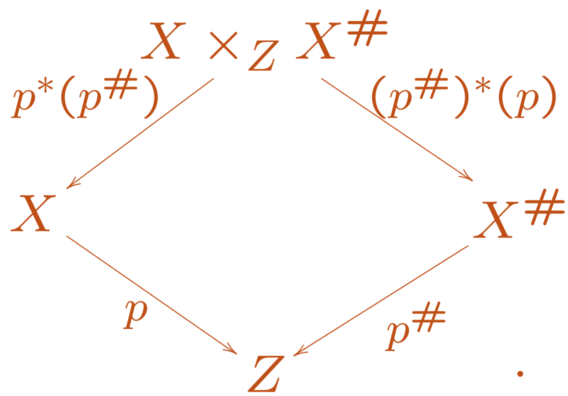
Theorem 2 *If* $n = 1$, *so* $G = \mathbb{R}$, *the forgetful map* $F\colon \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ *is an isomorphism, and thus every* $\delta \in H^3(X, \mathbb{Z})$ *is dualizable, in fact in a unique way. One has*

$$CT(X, \delta) \rtimes \mathbb{R} \cong CT(X^{\#}, \delta^{\#})$$

and

$$K^*_{\delta^{\#}}(X^{\#}) \cong K^{*+1}_{\delta}(X)$$

for a commutative diagram of $\mathbb{T}$*-bundles*



(1)

And

$$p_!(\delta) = [p^{\#}], \quad (p^{\#})_!(\delta^{\#}) = [p], \qquad (2)$$

where $p_!$ *and* $(p^{\#})_!$ *are the push-forward maps in the Gysin sequences of the two bundles.*

Results for $n = 2$

From now on, we stick to the case $n = 2$ for simplicity. H^*_M denotes cohomology with Borel cochains in the sense of C. Moore.

Theorem 3 *If $n = 2$, there is a commutative diagram of exact sequences*:

$$
\begin{array}{ccccccc}
 & & H^0(Z,\mathbb{Z}) & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
H^2(X,\mathbb{Z}) & \longrightarrow & H^2_M(G, C(X,\mathbb{T})) & \xrightarrow{\xi} & \ker F & \xrightarrow{\eta} & 0 \\
 & & \downarrow a & & \downarrow & & \\
 & & C(Z, H^2_M(\mathbb{Z}^2,\mathbb{T})) & \xleftarrow{M} & \mathrm{Br}_G(X) & & \\
 & & \downarrow h & & \downarrow F & & \\
 & & H^1(Z,\mathbb{Z}) & \xleftarrow{p_!} & H^3(X,\mathbb{Z}) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

$M\colon \mathrm{Br}_G(X) \to C(Z, H^2_M(\mathbb{Z}^2,\mathbb{T})) \cong C(Z,\mathbb{T})$ *is the Mackey obstruction map*, $h\colon C(Z,\mathbb{T}) \to H^1(X,\mathbb{Z})$ *sends a continuous function $Z \to S^1$ to its homotopy class*, $F\colon \mathrm{Br}_G(X) \to \mathrm{Br}(X)$ *is the forgetful map*.

Applications to T-duality

Theorem 4 *Let $p\colon X \to Z$ be a principal $\mathbb{T}^2$-bundle as above. Let $\delta \in H^3(X, \mathbb{Z})$ be an "H-flux" on X. Then:*

1. If $p_!\delta = 0 \in H^1(Z, \mathbb{Z})$, there is a (uniquely determined) classical T-dual to (p, δ), consisting of $p^\#\colon X^\# \to Z$, which is a another principal $\mathbb{T}^2$-bundle over Z, and $\delta^\# \in H^3(X^\#, \mathbb{Z})$, the "T-dual H-flux" on $X^\#$. One has a natural isomorphism

$$K^*_{\delta^\#}(X^\#) \cong K^*_\delta(X).$$

2. If $p_!\delta \neq 0 \in H^1(Z, \mathbb{Z})$, then a classical T-dual as above does not *exist. However, there is a "nonclassical" T-dual bundle of noncommutative tori over Z. It is not unique, but the non-uniqueness does not affect its K-theory, which is naturally $\cong K^*_\delta(X)$.*

An example

Let $X = T^3$, $p\colon X \to S^1$ the trivial $\mathbb{T}^2$-bundle. If $\delta \in H^3(X, \mathbb{Z}) \neq 0$, $p_!(\delta) \neq 0$ in $H^1(S^1)$. By Theorem 4, there is no classical T-dual to (p, δ). (The problem is that non-triviality of δ would have to give rise to non-triviality of $p^\#$.)

One can realize $CT(X, \delta)$ in this case as follows. Let $\mathcal{H} = L^2(\mathbb{T})$. Define the projective unitary representation $\rho_\theta : \mathbb{Z}^2 \to PU(\mathcal{H})$ by letting the first $\mathbb{Z}$ factor act by multiplication by z^k, the second $\mathbb{Z}$ factor act by translation by $\theta \in \mathbb{T}$. Then the Mackey obstruction of ρ_θ is $\theta \in \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{T})$. Let $\mathbb{Z}^2$ act on $C(\mathbb{T}, \mathcal{K}(\mathcal{H}))$ by α, which is given at the point θ by ρ_θ. Define the C^*-algebra

$$
\begin{aligned}
B &= \mathrm{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C(\mathbb{T}, \mathcal{K}(\mathcal{H})), \alpha) \\
&= \Big\{ f : \mathbb{R}^2 \to C(\mathbb{T}, \mathcal{K}(\mathcal{H})) : \\
&\qquad f(t+g) = \alpha(g)(f(t)),\ \ t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \Big\}.
\end{aligned}
$$

Then B is a continuous-trace C^*-algebra having spectrum T^3, having an action of $\mathbb{R}^2$ whose induced action on the spectrum of B is the trivial bundle $\mathbb{T}^3 \to \mathbb{T}$. The crossed product algebra $B \rtimes \mathbb{R}^2 \cong C(\mathbb{T}, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^2$ has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta \otimes \mathcal{K}(\mathcal{H})$, where A_θ is the noncommutative 2-torus. In fact, the crossed product $B \rtimes \mathbb{R}^2$ is isomorphic to $C^*(H_k) \otimes \mathcal{K}$, where H_k is the integer Heisenberg-type group,

$$H_k = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\},$$

a lattice in the usual Heisenberg group $H_{\mathbb{R}}$ (consisting of matrices of the same form, but with x, y, $z \in \mathbb{R}$).

The Dixmier-Douady invariant δ of B is k times a generator of H^3. We see that the group C^*-algebra of H_k serves as a non-classical T-dual.