COMPUTERIZED TOMOGRAPHY

The Idea Behind CAT Scanners

by Jerome Dancis

Computerized tomography is the technique of making pictures of a cross-section (or a slice) of an object. The pictures are produced from the results of mathematical analysis. The cat scanners, which locate tumors and blood clots in a brain, are computerized, x-ray tomography machines.

One way to make a solid, 3-dimensional "picture" of an object is to make a set of pictures of many cross-sections.

Each picture consists of a large number of gray squares. How light or dark a particular gray square will be is determined by the "numbers" calculated by the computer. The bigger the "number", the darker the square will be colored. The numbers are related to the "x-ray optical density" for the material in the square. The "x-ray optical density" is related to the percentage of x-rays absorbed by a unit thickness of material as the x-rays pass through the material. Different tissues (bones, muscle, fat, tumors,blood, blood clots) have different "x-ray optical densities". Therefore each tissue will be colored by a different shade of gray. Bones absorb about twice as many x-rays as do most other tissues. Therefore bones are easily distinguished. Unfortunately, the "x-ray density" for the other tissues vary by at most 5%. Sometimes the physicians must be able to distinguish a tissue whose "x-ray density" differs by only 1% from its surroundings. Therefore the "x-ray densities" must be calculated with great accuracy in spite of the small errors being made by both the x-ray equipment and the computer.

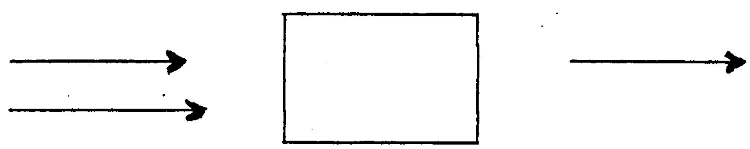
We shall now describe how the "x-ray densities" for each point may be calculated.

<u>Basic Tomography Example A.1</u>. Suppose that we wish to make a picture of some cross-section.

Suppose that we mentally divide this cross-section into 12 square pieces:

1	2	3
4	5	6
7	8	9
10	11	12

When x-rays penetrate an object, a fixed percentage of the x-rays are absorbed by the object and the remaining x-rays go through the object. Therefore, let a_i be the percentage of x-rays absorbed by square #i.

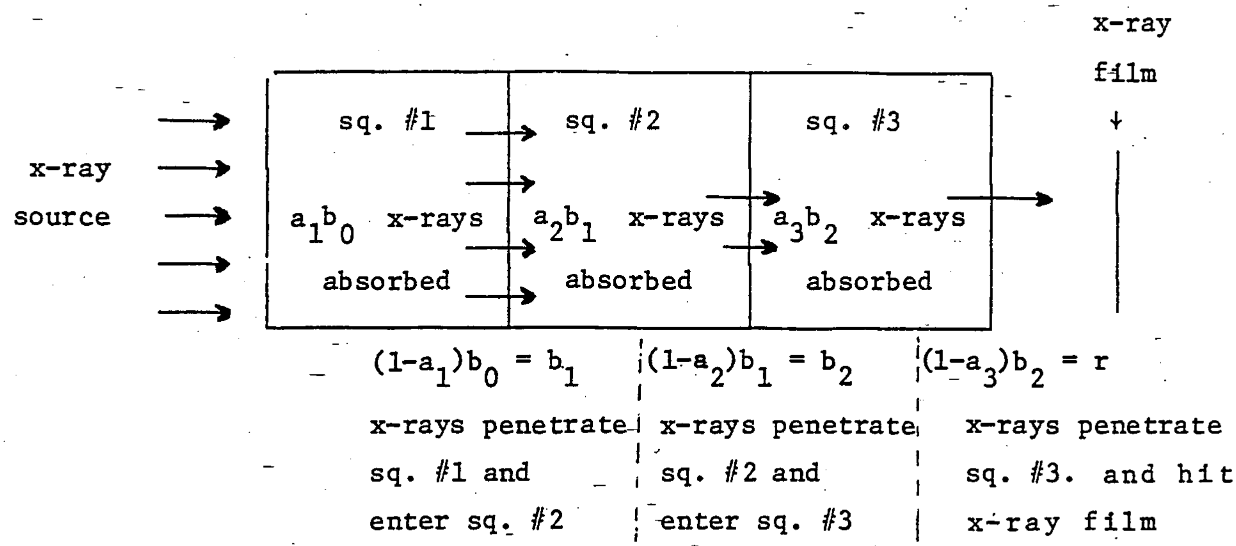


b —x-rays enter square #i

$a_i b$ x-rays absorbed by square #i

$b - a_i b = (1 - a_i) b$ x-rays leave square #i

Now let us examine what happens to x-rays which are transmitted through the top row of the section:



Check that the number of x-rays which penerate all 3 squares is:

$$r = (1-a_3)b_2 = (1-a_3)(1-a_2)b_2$$

(A.1)
$$\boxed{r = (1-a_3)(1-a_2)(1-a_1)b_0\ .}$$

In this equation the unkowns are a_1, a_2 and a_3. The b_0 and r are known ; b_0 is the amount of x-rays emitted and r is the amount of x-rays which come out the other side and hit the x-ray film. The film is developed and then this number r is "counted" by a"densitometer" .

If we do this along twelve different lines(as in Figure A-1), we will obtain 12 equations in the 12 unknowns $a_1, a_2, \ldots, a_{12}$. But how can this system be solved? The equations are not linear; they are a collection of products instead of sums. Fortunately logarithms convert products into sums. Set each

$$x_i = \log(1-a_i).$$

Then equation (A.1) is converted into:

(A.2)

$$x_1 + x_2 + x_3 = \log \frac{r}{b_0} = \log \frac{\left\{\begin{array}{l}\text{no. of x-rays} \\ \text{which penetrate} \\ \text{the cross-section}\end{array}\right\}}{\left\{\begin{array}{l}\text{no. of x-rays} \\ \text{transmitted}\end{array}\right\}}$$

As we will see, this is a typical "linear density" type of equation. This equation, together with the next three propositions, is the inspiration for the following definitions.

Definitions. The total optical mass (along a line) is given by the formula

$$\log \frac{\text{no. of x-rays which penetrate}}{\text{no. of x-rays transmitted}},$$

namely the right hand side of Equation A.2. The optical density of a tissue(or of some material) is the total optical mass of the tissue along a line of length one.

Proposition A.2. Along a line with different blocks:

Optical mass :	x_1	x_2	x_3	...	x_n	x-rays →

$$\{\text{total optical mass}\} = x_1 + x_2 + \dots + x_n = \sum_i \left\{\begin{array}{l}\text{optical mass} \\ \text{for block \#i}\end{array}\right\}$$

Remark. This proposition is a corollary of the derivation of Equation (A.2).

Proposition A.3. For a "uniform" tissue:

$$\{\text{total optical mass}\} = \{\text{length}\} \times \{\text{optical density}\}$$

Remark. The proof of this proposition is presented later in this section.

Proposition A.4. Along a line, with different uniform blocks of tissues,

block :	1	2	3	...	n
optical density :	ρ_1	ρ_2	ρ_3	...	ρ_n
lengths :	ℓ_1	ℓ_2	ℓ_3	...	ℓ_n

$$\{\text{total optical mass}\} = \ell_1\rho_1 + \ell_2\rho_2 + \ldots + \ell_n\rho_n$$

$$= \sum_i \{\text{length } \#i\} \times \{\text{optical density } \#i\}$$

Remark. This proposition is a corollary of the preceding two propositions.

We now choose twelve lines and we write down the equations for the total optical mass along these lines.

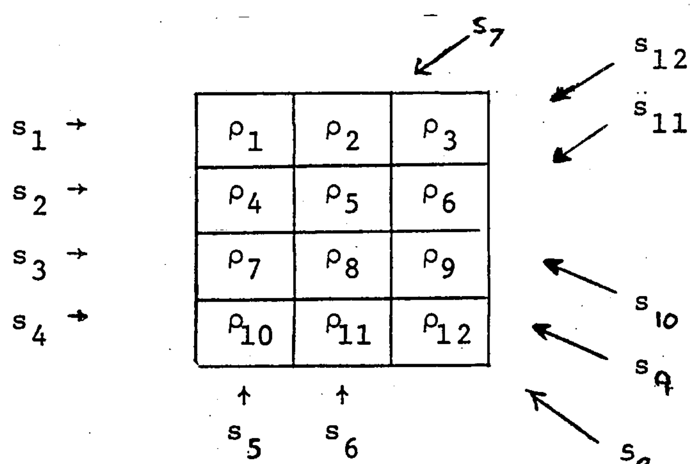


Figure A-1

Let s_j be the amount of x-rays emitted along line #j. Let r_j be the reading by the densitometer on the x-ray film situated opposite the soarce for s_j. Let the side of each square have length one and hence the diagonals have length $\sqrt{2}$.

The equations for these x-ray emissions are:

line #	total optical mass	=	$\log \dfrac{\left\{\text{reading on densitometer}\right\}}{\left\{\text{amount of x-rays emitted at soarce}\right\}}$
1	$\rho_1 + \rho_2 + \rho_3$	=	$\log r_1/s_1$
2	$\rho_4 + \rho_5 + \rho_6$	=	$\log r_2/s_2$
3	$\rho_7 + \rho_8 + \rho_9$	=	$\log r_3/s_3$
4	$\rho_{10} + \rho_{11} + \rho_{12}$	=	$\log r_4/s_4$
5	$\rho_{10} + \rho_7 + \rho_4 + \rho_1$	=	$\log r_5/s_5$
6	$\rho_{11} + \rho_8 + \rho_5 + \rho_2$	=	$\log r_6/s_6$
7	$\sqrt{2}\,\rho_2 + \sqrt{2}\,\rho_4$	=	$\log r_7/s_7$
8	$\sqrt{2}\,\rho_{12} + \sqrt{2}\,\rho_8 + \sqrt{2}\,\rho_4$	=	$\log r_8/s_8$
9	$\sqrt{2}\,\rho_9 + \sqrt{2}\,\rho_5 + \sqrt{2}\,\rho_1$	=	$\log r_9/s_9$
10	$\sqrt{2}\,\rho_6 + \sqrt{2}\,\rho_2$	=	$\log r_{10}/s_{10}$
11	$\sqrt{2}\,\rho_6 + \sqrt{2}\,\rho_8 + \sqrt{2}\,\rho_{10}$	=	$\log r_{11}/s_{11}$
12	$\sqrt{2}\,\rho_3 + \sqrt{2}\,\rho_5 + \sqrt{2}\,\rho_7$	=	$\log r_{12}/s_{12}$

We now have twelve <u>linear</u> equations in twelve unknowns $(\rho_1, \rho_2, \ldots, \rho_{12})$. These equations may be solved (on a computer) and the ρ's will be found.

The computer can now print a picture of the cross-section in which the squares with higher ρ's will be colored darker shades of gray.

The proof of Proposition A.3 is explained by the following lemmas.

<u>Lemma A.5</u>. Suppose we have two pieces of the some tissue but one piece is n times the length of the second piece, where n is

an <u>integer</u>. Then:

$$\left\{\begin{matrix}\text{optical mass}\\ \text{for big piece}\\ \text{(n times as long)}\end{matrix}\right\} = n \times \left\{\begin{matrix}\text{optical mass}\\ \text{for small piece}\end{matrix}\right\}$$

<u>Proof</u>. We mentally divide the big piece into n equal parts, where each part is identical to the small piece. Let x be the optical mass for the small piece. Then the picture is:

big piece :

x	x	...	x

small piece:

x

Proposition A.4 says that:

$$\left\{\begin{matrix}\text{optical mass}\\ \text{for big piece}\end{matrix}\right\} = \underbrace{x + x + \ldots + x}_{n \text{ times}} = nx. \quad \checkmark$$

<u>Lemma A.6</u>. A piece of tissue, with optical density ρ and length n/m, has total optical mass, $(n/m)\rho$, when n and m are integers.

<u>Proof</u>. Consider three pieces of this tissue with optical density ρ:

optical mass:	x_1	x_n	x_m
length:	1	n	n/m

The preceding lemma says that

$$x_n = nx_1 \quad \text{and} \quad x_n = mx_m .$$

Therefore

$$x_m = \frac{n}{m} x_1 = \frac{n}{m} \rho \qquad \checkmark$$

Outline of proof of Proposition A.3. When the length is $\sqrt{2}$.

Consider three pieces of this tissue with optical density ρ:

optical mass :	x_1	x	x_2
length :	1.4141	$\sqrt{2}$	1.4142

Since $\sqrt{2}$ is between 1.4141 and 1.4142, the optical mass x will be between x_1 and x_2. Therefore

$$1.4141\rho \;=\; x_1 > x > x_2 \;=\; 1.4142\,\rho$$

(the inequalities are backwards because the optical densities are negative).

Therefore if we take a limit as the left tissue expands to length $\sqrt{2}$ and as the right tissue is shrunk to length $\sqrt{2}$, the result will be

$$x = \sqrt{2}\,\rho \qquad \checkmark$$

Now, we will discuss the situation involved in making a picture of a cross section of a head. A cross-section of a head, together with a small region around the head, is divided into squares of size 1 mm to 1½ mm. This results in dividing the cross-section into about 25,000 tiny squares. This results in 25,000 unknowns and therefore 25,000 equations are needed. A linear system consisting of 25,000 "sparse" equations in 25,000 unknowns, can be solved, with difficulty, on a computer.

In the future, many scientists, engineers and technicians will be using new (not yet invented) computerized tomography machines to "look inside" all sorts of objects. Today, computerized tomography machines are being developed which make a motion picture of a heart beating. This will enable physicians to determine the amount of damage cause by a heart attack.

The Basic Tomography Example presents the basic idea behind tomography. This is the example that physicicans and engineers should have in mind when they look at tomography pictures.

The Basic Tomography Example A.1 is an oversimplified idealized view of computerized tomography. The problem with the example is that it ignores all sorts of complications. The complications come from mathematics, computers, x-ray machines, x-ray films, etc. In order to overcome these complications, a much more sophisticated and complicated method is used. A master degree in applied mathematics is needed in order to fully understand the current methods. But the Basic Tomography Example A.1 explains the basic idea behind computerized tomography and it is the example that was in the back of the minds of the researchers who successful developed computerized tomography.

To find out more about computerized tomography, you may <u>browse</u> the following articles. Look at the pictures,

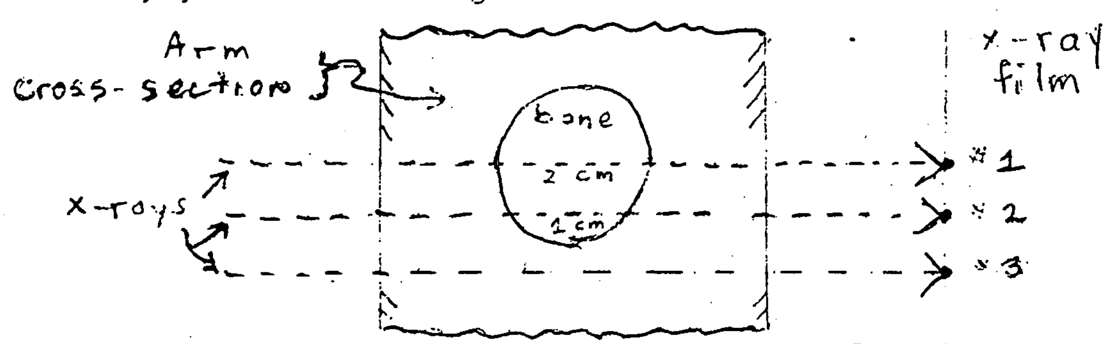
BIBLIOGRAPHY

(iv) W. Swindell and H. Barrett, Physics Today (December 1977)

The mathematical discussion on how x-rays penetrate tissue (Propositions A.2, A.3 and A.4) is also valid in many different contexts. In the exercises, these propositions will be used to examine such diverse applications as light penetrating glass and gamma rays penetrating ground beef (in order to determine the fat content).

EXERCISES

Exercise A.7. Suppose an ordinary x-ray picture of a round bone in an arm is being taken (no tomography). Suppose that 10 units of x-rays are shot. Find the amount of x-rays which reach points #1,2,3 in this diagram:

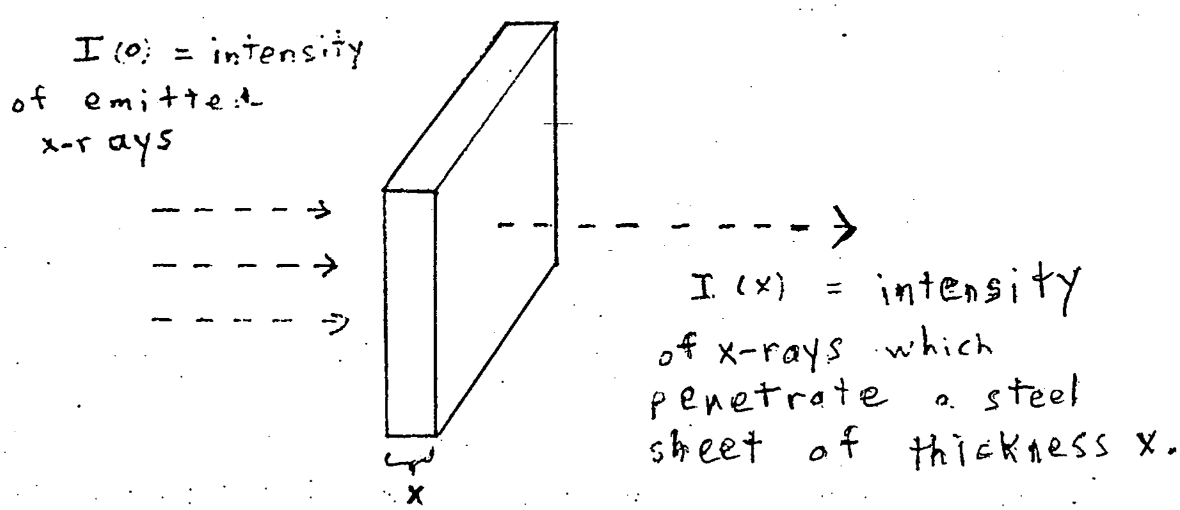


Suppose that the x-ray optical density of the bone is $\rho_b = -2$ and of the flesh $\rho_f = -1$.

Your answers should help you understand why the edges of a bone appear very "cloud-like" on the film of an ordinary x-ray negative.

Exercise A.8. Let ρ be the x-ray optical density for steel.

(a) Suppose that some parallel x-rays penetrate a sheet of steel



Show that

$$I(x) = I(0)e^{\rho x}.$$

(b) Suppose the sheet of steel has a crack. Explain why the crack will be displayed on an ordinary x-ray picture (no tomography) of the sheet.

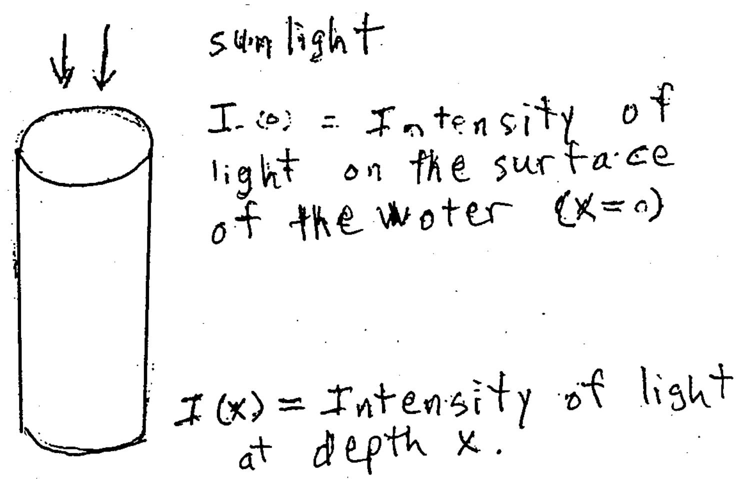
(c) Let us consider an oil pipeline made of steel and filled with oil. Suppose there is a crack on the inside of the pipe. (No leakage yet!) Suppose that the x-ray optical density of steel is 3 times the density of oil. Explain why the crack will be displayed on an ordinary x-ray picture (no tomography) of the cracked section of the oil pipeline.

Exercise A.9. Suppose that a one cm. thick piece of tinted glass absorbs 15% of the light which passes through it. How thick must the glass be in order to absorb 90% of the light?

Use the fact that light is absorbed by glass in the same mathematical manner as x-rays are absorbed by tissue. Therefore Proposition A.3 is also valid for light passing through tinted glass.

Exercise A.10. The optical density for light rays passing through sea-water is -1.4 (when distance is measured in meters.)

(a) Suppose there is a light above a column of sea-water



Show that

$$I(x) = I(0)e^{-1.4x}.$$

(b) Calculate the percentage of light which goes through 10 meters of water. Explain why very little vegetation grows in the ocean at depths greater than 10 meters.

Remark. The formula

$$I(x) = I(0)e^{-\mu x}$$

is the Bouguer-Lambert Law of photometry. Pierre Bouguer (1698-1758) studied the absorption of light in the atmosphere. Johann Lambert (1728-1777) stated this law for homogeneous transparent substances.

One method that supermarkets[1] use for preparing ground beef is to mix ground fatty meat[2] from bin #1 together with ground lean cow meat[3] from bin #2. Gamma rays are shot through the fatty meat in bin #1 and through the lean meat in bin #2 in order to determine the fat content in each bin. (As in Exercise A. 12). The fat content in each bin changes from batch to batch. Then the fat and lean meat is mixed together, in just the right proportions, in order to obtain the exact fat content desired by the supermarket management.

Let us examine the penetration of ground beef by gamma rays. Some of the gamma rays are absorbed by the ground beef in the same mathematical manner as some of the x-rays are absorbed by

---

1 The Safeway supermarkets in the Washington, D.C. area prepare the bulk of their ground beef in the manner described here. (Ref.: 1979 Food issue of the magazine: Washington Consumer's Checkbook).

2 The fatty meat is the left-over meat which was trimmed off the regular meat cuts. It has much more than the 30% fat allowable for "ground beef".

3 The lean cow meat is not of sufficiently high quality to be sold as regular steaks or roasts.

tissue. Therefore Propositions A.2, A.3 and A.4 are still valid and therefore they may be used. Suppose that the gamma ray density for fat is $\rho_f = -1.0$ and for muscle is $\rho_m = -1.03$ where each

$$\rho = \log_{10}\left\{\begin{array}{l}\text{fraction of the gamma} \\ \text{rays which penetrates} \\ \text{1 mm. of tissue.}\end{array}\right\}$$

Exercise A.11. Suppose we have a line consisting of 10 1 mm-squares of fat and 40 1 mm-squares of muscle.

(a) Find the total gamma-ray optical-mass of this line.

(b) Suppose that a 10 unit gamma-ray is fired along this line. What is the strength of the beam which comes out of the other end?

Exercise A.12. Suppose that the bins of chopped meat are 100 mm long and that 10 unit gamma rays are shot through the 100 mm of meat in each bin. Let m be the percentage of muscle in the bin and let f be the percentage of fat. (Therefore $m + f = 1$.)

(a) Find the total gamma-ray optical-mass of the meat in the bin (in terms of m and f).

(b) What is the strength of the beam which comes out of the other end (in terms of m and f)?

(c) Suppose that the gamma ray readings (on the other side) are

(i) 0.9660 for bin #1

(ii) 0.9397 for bin #2.

Calculate the percentage of fat in each bin.

(d) Suppose that the supermarket manager has decided that the "regular" ground beef will be 25% fat and that the "premium" or "lean" ground beef will be 15% fat. In what proportions should the meat from bins #1 and 2 (of part c) be mixed in order to produce "regular" ground beef? "Lean" ground beef?

(f) Find the function F which will calculate the amount of fat, f, in terms of r, the gamma-ray reading on the other side:

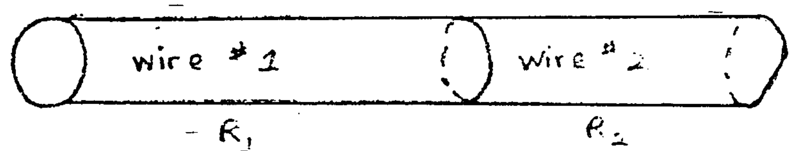
$$f = F(r).$$

We will state the two basic arithmetic rules for electrical networks consisting of resistors. (These rules are consequences of Kirchoff's voltage and current laws.)

Let R_1 and R_2 be the "resistances" of two wires and let R be the total resistance due to the two wires together.

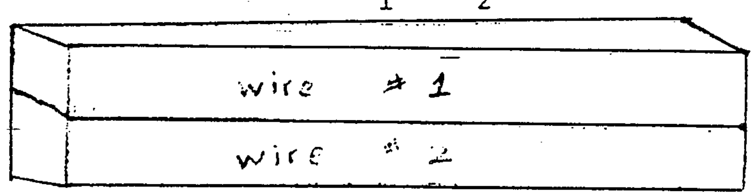
<u>Resistors connected in series rule</u>

$$R = R_1 + R_2$$



<u>Resistors connected in parallel rule</u>

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$



<u>Exercise A.13</u>. Using Proposition A.3 as a "model", show that

$$\left\{\begin{matrix}\text{resistance of}\\ \text{a copper wire}\end{matrix}\right\} = \left\{\begin{matrix}\text{length}\\ \text{of wire}\end{matrix}\right\} \times \left\{\begin{matrix}\text{resistance of}\\ \text{a piece (of the}\\ \text{same wire) with}\\ \text{length equal to}\\ \text{one}\end{matrix}\right\}$$

<u>Exercise A.14</u>. Let R_1 be a piece of copper wire which is 1 meter long and whose cross section is a 1 mm square. Let R_6 be another 1 meter long piece of copper wire and let its cross section be a 2 mm × 3 mm rectangle. Show that

$$R_6 = \frac{1}{6} R_1 .$$

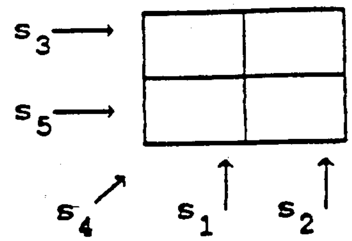
Exercise A.15. Let R_1 and R_4 be two pieces of copper wire with the same lengths. Let 1 and A be the cross-sectional area of the two pieces. Show that

$$R_4 = \frac{1}{A} R_1.$$

Hint: Modify our discussion on optical density in order to show that $\frac{1}{R_1} = \frac{1}{A} \times \frac{1}{R_4}$.

Remark. Electrical engineers usually describe the results of the last three exercises by saying that the resistance of a wire is proportional to its length and is inversely proportional to its cross-sectional area.

Exercise A.16 Suppose we are doing tomography on a square section consisting of 4 squares:



(a) Find the matrix equation when x-rays are shot along the 4 directions s_1, s_2, s_3 and s_4.

Check that this 4×4 matrix is invertible.

(b) Find the matrix equation when x-rays are shot along the 4 directions s_1, s_2, s_3 and s_5.

Check that this 4×4 matrix is not invertible.

Exercise A.17 In the preceding exercise, suppose x-rays are shot along all 5 directions. That the values come from real data, hence they have some measurement errors. Set this up as a least squares fit problem. Check that ker $M = \underline{o}$.

Exercise A.18 In the tomography example worked out in this section, suppose that s_7 is replaced by s_{13} which shoots x-rays straight up the column on the right side.

Check that the resulting 12 x 12 matrix is not invertible.