Student modeling of physical phenomena as they derivate standard integral formulas.

One way to reduce proof phobias.

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Abstract

Students in a freshmen calculus course should become fluency in the modeling of physical phenomena that are represented by integrals, in particular, geometric formulas for volumes and arc length and physical formulas for work. This paper describes how I successfully trained my students to became fluent in such modeling and derivation of standard integral formulas. These lessons also greatly reduced the students' proof phobias. An appendix provides my lecture notes, which include simple (student level) derivations of integrals for geometric and physical phenomena. The formula for the volume of a "general" cone is included.

Key words: modeling, proof, integral

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Note: The diagrams are NOT included herein.

I believe that fluency in the derivation of geometric formulas for volumes and arc length and physical formulas for work, in the form of integrals, is one of the most important things that students should acquire in a freshmen calculus course. This is my personal view,shared by the editor and presumably many calculus instructors. While textbooks and instructors usually present these derivations, students are rarely required to do any on tests. Normally, students just (rote) memorize the formulas and practice plugging into them. In contrast, I trained my students until they became fluent in derivating standard integral formulas. The purpose of this paper is to provide to those calculus instructors, who share my view, a practical way to implement the training of students to derive integral formulas.

Doing many derivations provides the students with repeatedly practice in the reasoning involved. This should "hard-wire" it into their memories. This is an example of employing Guershon Harel's Repeated Practicing of Reasoning Principle [4].

In the notes, we derive the formula for the volume of the "general" cone, instead of deriving individual formulas for various special cones, namely pyramids, tetrahedrons and ice cream cones. Lyn English and Graeme S. Halford label this "chunking". In [3], they explain why it is very useful for students' learning.

Results. On the chapter test, largely student derivations of standard formulas, (Appendix 2), the median score was 82% and 15 of 19 students scored 70% or higher. It was a university honors (not mathematics honors) class, the students were somewhat better than the regular students.

Students, even "honors" students and students who like math, enter college with much fear of proofs. A major goal was to greatly reduce their proof phobias and make them comfortable with proofs. The students comments (Appendix 3) show that this goal was achieved. Additionally, as two students noted "I enjoyed learning [the proofs] because it helps me to learn the material much more" and "[Doing the proofs enabled me] to understand the [text]book". Reducing proof phobia is crucial for recruiting mathematics majors.

We used this informal definition of the definite integral:

$$\int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{or simply} \quad \int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum f(x)\Delta x.$$

(The formal definition using upper and lower sums had been taught the previous semester.)

The basis for my presentations and the student derivations was training the students in combining "elements of stuff" to make a "whole", in these cases, the whole is an integral. The students used this one page outline:

Doing word problems with integrals.

" Δx" means "a small amount of 'x' ", where x may be any type of quantity.

$$Force = (average) pressure \times area \iff pressure = Force \div area \iff p = \Delta F \div \Delta A$$

$$Weight = density \times volume \iff density = weight \div volume \iff \Delta W = \rho \Delta V$$

Step 1. Draw a useful diagram. Indicate the section and Δx or Δy on the diagram.

Step 2. The Whole $= \sum$ parts:

$$W = \sum \Delta W, \quad F = \sum \Delta F, \quad V = \sum \Delta V, \quad \ell = \sum \Delta s$$

Step 3. State a physical equation verbally for the total physical quantity and/or the part associated with a section. State a geometric equation verbally for the total geometric quantity or the part associated with a section. Label the verbal equations with symbols. For example:

$$Work = Force \times distance, \quad Area \approx base \times height, \quad Pythagorean\ Theorem$$
$$W = Fd, \qquad \Delta A \approx \Delta x\ y, \qquad (\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$Volume\ (of\ thickened\ surface) \approx (Area\ of\ surface) \times thickness$$

$$\Delta Vol = \Delta V \approx A \Delta x \quad (or\ \Delta y \quad or\ \Delta r)$$

Pappus's Theorem $\implies \Delta$ Vol(of a cylindrical shell) $= 2\pi r \Delta A = 2\pi r (y \Delta x)$.

Step 4. Write the physical or geometric quantity in terms of sections. For example:

(i) $\Delta W_i = F_i(\Delta dist.) = F_i\ \Delta x$, when the force varies as it pushes or pulls an entire object.

(ii) $\Delta W_i = \Delta F_i\ h_i$, where ΔF_i is the force required (the weight of the water) to lift the section a height of h. Then

$$\Delta W_i \approx \Delta F_i\ h_i, \quad \text{where} \quad \Delta F_i = (\Delta weight) = \rho \Delta Vol_i \quad \text{and}$$

$\Delta Vol_i \approx A_i \Delta x$.

Combine the equations: $W = \sum \Delta W_i \approx \sum \Delta F_i\ h_i \approx \sum (\rho\ A_i\ \Delta x) h_i$.

Step 5. Take the Limit as Δ (whatever)$\to 0$ in order to obtain a definite integral (include the limits of integration); this converts all approximately equals "$\approx$" to "$=$". For example:

$$W = \lim_{\Delta x \to 0} \sum \rho\ A_i\ h_i\ \Delta x = \int \rho\ A(x) h(x) dx$$

I gave a quiz every day; the students were required to reproduce the derivation of the integral formula which I had derived during the previous class; they were permitted to consult the one page outline "Doing word problems with integrals" (Figure 1). These quizzes were crucial for getting the students to take the derivations seriously and my corrections on the quiz papers provided them with useful feedback on their derivations. For me, the breakthrough to success came only after I added the many quizzes. In previous honors classes, without the quizzes, most of the students flunked the chapter test; a retest was necessary – a rarity in an honors class.

One morning, I wrote down the formula for the energy of a stretched spring which obeys Hooke's Law: $E = \frac{1}{2}kx^2$. I enquired who was familiar with the formula; all hands went up. – not surprising since they all had AP calculus in high school and many had taken a physics course. Then I asked who knew where the formula came from; *all hands shot down.* This underscores the fact that many students, even "honors" students learn (rote memorize) many formulas without learning where they came from. Then I derived the formula, and gave a quiz on it two days later. The students did very well on the chapter test question, which demanded a derivation of the integral formula for work, together with the formula $E = \frac{1}{2}kx^2$.

I typed up the notes (Appendix 1) after the material was presented in class. This seemed to be greatly appreciated by the students even though the notes were just a subset of what I had written on the blackboard. At the editors wise request, I added a proof of the formula for area of a surface of revolution. (Now that they are typed, they can be distributed before the chapter is begun.) These notes were meant to enhance and supplement not replace what is in the textbook. These notes are filling in, what I perceive as gaps in, the textbook. Now is a good time for the reader to browse these notes in Appendix 1.

I covered this material in an "honors" section of our second semester freshmen calculus course for engineering and science students (Math 141H) at the University of Maryland at College Park during the spring of 1996. The "honors" section uses the same textbook, [3] and takes the same uniform final as the regular sections. Our "low-level" "honors" calculus students come from the campus's general honors program (only a quarter of these honors students take calculus); they do not include the best students, the ones who receive credit for AP calculus.

Guided group work As an "honors" class, I met my 20 students 4 hours each week, one of the hours was devoted to "guided" small group work ([1]) in which the class divided itself up into teams of three to four students and worked the exercises as a team. This guided group work included much one-on-four tutoring as I visited each group making mathematical comments, answering questions and checking their work. In contrast, the regular (non-honors) classes met 5-7 hours a week (3 hours in a monster lecture hall and 2-4 hours in small sections with a graduate student).

Time constraints. For (the vast majority of) instructors who derive just the integral formulas that are in the textbook, the only extra class time needed is the time devoted to the quizzes on these derivations. (Of course to quiz or not to quiz is instructor's choice; as noted above, my instruction failed without the quizzes.) There is the chore of correcting the quizzes. Proofs presented by the instructor might follow my outline, "Doing word problems with integrals", or the instructor's own improvement/modification of it.

The (optional) additional time that I spent, on the extra theorems in Appendix 1, came at the expense of presentations on how to set up and solve specific problems using the various (memorized) integral formulas.

While I only tried this in an "honors" section, I know of no pedagogical reason for not trying this with regular students.

References

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Appendix 1. The handout VOLUMES [and other quantities] AS DEFINITE INTEGRALS

The handout will follow these notes on it.

The first day, I derived Theorem 1: that the volume of a right cylinder is {area of base} $\times$ {height}. This was somewhat uncomfortable for the students; I did not test on it.

I used Theorem 1 to derive the formula for a volume of a sphere using the "cross-sectional" method. The next class started with the first quiz which required the students to derive the same formula.

As "guided" small group work ([1]), the students derived the formulas for the area of an ellipse and the volume of an ellipsoid (Exercises 2 and 3). A moderate amount of guidance was required during their calculations for the volume of an ellipsoid. Much of that guidance is now included in the discussion following Exercise 3. They seemed quite pleased with themselves when their calculations simplified the complicated formulas to the desired one.

Instead of calculating the volumes for various pyramids and a circular cone, individually, I derived the formula for the volume of a general cone (Theorem **??**). In "guided" small group work, the students established Lemma **??** about the proportionality of lengths of corresponding lines at different levels of the [inverted] cone. Two days later there was a quiz which demanded the derivation of the volume for a general [inverted] cone assuming that the areas of the slices are proportional to the square of the heights of the slices (Lemma **??**).

Next, I derived the natural formula for arc length; again followed by a quiz two days later. Then the derivation was modified to quickly derive the formula for the surface area of a solid of revolution.

The notes end with a page on moments which emphasizes and exploits the fact that moments are a "Whole $= \sum$ parts" phenonmen.

I believe that it is important that students know where standard formulas come from. This is why the exercises ask them to calculate various formulas, from the circumference of a circle to the surface area of a cone, using the already derived integral formulas. The students were also expected to do the standard exercises in the textbook.

VOLUMES [and other quantities] AS DEFINITE INTEGRALS

In these notes, we will evaluate volumes as definite integrals in the manner of the handout "Doing word problems with integrals". We use this method to derive many of the popular formulas of solid geometry including formulas for the volume and surface area of a sphere.

The definite integral is defined as a limit of Riemann sums, informaly

$$\int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{or simply} \quad \int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum f(x)\Delta x.$$

It is a nuisance to write out the index, i many times, it also slows down the reading; therefore we often omit them and simply understand that the indices are implied. When the function $f(x)$ is differentiable (or continuous) on the closed interval $a \le x \le b$, then the limit always converges as $\Delta x \to 0$. Also $\int_a^b f(x)\ dx \approx \sum f(x)\Delta x$. [1]

Area under a curve is the most popular example of the definite integral, but it is only one of many examples; it is not the definition. Ther are many other examples.

The lower Riemann sums approximate the area under a curve by the sum of the areas of a set of "inscribed" rectangles; this is equivalent to the region under the curve being approximated by a set of "inscribed" rectangles.

A *circular right cylinder* is a solid object which has the shape of a can of food. The top and bottom are disks, the vertical lines are perpendicular to the top and bottom. A circular right cylinder is a special case of the general right cylinder. A *right cylinder* (with height h) is any region in the xy-plane base, together with vertical lines of height h sticking up from each point in the base, or any rotation and/or translation of such an object. A rectanglular box fits this definition.

The volume of a rectanglular box is length x width x height.

Both area and volume are "The whole equals the sum of its parts" or "Whole $= \sum$ parts" phenomenon.

Δ **Notation.** We will use ΔVol and ΔA to represent a small amount of volume and area; ΔVol_i and ΔA_i will represent the volume and area of the "i^{th}" region. Δx will represent both a small change in x and/or a small length in the x-direction. In general, ΔX will represent a small amount of X for any quantity, X.

Theorem 1 (Volume of a Cylinder) *The volume of a right cylinder is* $\{$*area of base*$\}$ $\times$ $\{$*height*$\}$.

[1] "$\approx$" means "approximately equal to" or "very close to".

Figure 1. Cylinder with amoeba-like base, with internal vertical box

Proof. Let us consider a right cylinder with height h and area of base b.

$$\textit{The base} \approx \textit{set of ``inscribed'' rectangles}\{R_i,\ i = 1, 2, ..., n\}$$

We set $\Delta A_i =$ Area of $R_i,\ i = 1, 2, ..., n.$

$$\textit{Area of base} \approx \sum \textit{Areas of these ``inscribed'' rectangles} = \sum_{i=1}^{n} A_i$$

We may obtain a set of "inscribed" rectanglular boxes for the right cylinder by taking rectanglular boxes of height h over the set of "inscribed" rectangles of the base. We set $\Delta Vol_i =$ Volume of the i^{th} rectanglular box. Then

$$\Delta Vol_i = \Delta A_i \times h$$

Thus

$$Volume \approx \sum_{i=1}^{n} \Delta Vol_i = \sum_{i=1}^{n} \Delta A_i \times h = h \sum_{i=1}^{n} \Delta A_i \approx hb.$$

Taking the limit as the rectangles are expanded to fill the base results in the $\approx$ becoming equal signs; which results in

$$Volume = hb.$$

This is a slightly informal argument; a more formal argument is

$$Volume = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \Delta Vol_i = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \Delta A_i \times h = \lim_{\Delta x \to 0} h \sum_{i=1}^{n} \Delta A_i = hb.$$

$\surd$ YEAH

Theorem 2 (Volume of a Sphere) *The volume of a spherical ball with radius* R *is* $\frac{4}{3}\pi R^3$.

Figure 2. Sphere with section

Proof. Consider the spherical ball of radius R about the origin; its equation is $x^2+y^2+z^2 \leq R^2$. If $R > 5$, then the horizontal slice at height $z = 5$ is the set of points which satisfy both conditions ($x^2+y^2+z^2 \leq R^2$ and $z = 5$). This is the disk $x^2+y^2+25 \leq R^2$ or equivalently $x^2+y^2 \leq R^2-25$. Now we partition the z-axis from -R to +R with $\{-R = z_0, z_1 = z_0+\Delta z, ..., z_n = z_0+n\Delta z = +R\}$. We slice the spherical ball hoizontally at each z_i. In general, the horizontal slice at height $z = z_i$ is the set of points which satisfy both conditions ($x^2 + y^2 + z^2 \leq R^2$ and $z = z_i$). This is the disk $x^2 + y^2 \leq R^2 - z_i^2$. Its radius is $r_i = \sqrt{R^2 - z_i^2}$. The area of the slice is $A_i = \pi r_i^2 = \pi(R^2 - z_i^2)$. The section between slices at z_i and z_{i+1} is appoximately a circular right cylinder with height Δz. Thus the volume of the i^{th} section is

$$\Delta Vol_i \approx A_i \Delta z = \pi r_i^2 \Delta z = \pi(R^2 - z_i^2)\Delta z.$$

Since volume is a "Whole $= \sum$ parts" phenonmen.

$$Volume = \sum_{i=1}^{n} \Delta Vol_i \approx \sum_{i=1}^{n} \pi(R^2 - z_i^2)\Delta z$$

Taking the limit as Δz goes to zero yields:

$$Volume = \int_{-R}^{R} \pi(R^2 - z^2)\ dz = \frac{4}{3}\pi R^3.$$

(You need to do the integration in order to check the last equation.)

Thus the volume of a sphere with radius R is $\frac{4}{3}\pi R^3$. $\surd$ YEAH

Example 3 (Area of a circle) *Starting with the formula for the circumference of a circle ($2\pi R$), find a formula for the area of a disk.*

Figure 3. Circular disk with section

Calculations. Let D be a disk with radius R centered at the origin. Partition the interval $0 \leq x \leq R$ as $\{0 = x_0, x_1 = x_0 + \Delta x, ..., x_n = x_0 + n\Delta x = R\}$. Cut the disk into circular strips of width Δx with the x_i's as the radii. Each circular strip is a thickened circle, hence its area is appoximately

$$\{area\ of\ strip\} \;\approx\; \{circumference\ of\ circle\} \;\times\; \{thickness\},$$

that is

$$\Delta A_i \approx 2\pi x_i \Delta x.$$

Since area is a "Whole $= \sum$ parts" phenonmen,

$$Area = \sum \Delta A_i \approx \sum 2\pi x_i \times \Delta x$$

Taking the limit as Δx goes to zero yields:

$$Area = \int_0^R 2\pi x\ dx = \pi R^2.$$

$\surd$ YEAH

Theorem 4 (Area of a Sphere) *The surface area of a sphere with radius R is $4\pi R^2$.*

Proof. Let $f(r)$ be (the as yet unknown formula for) the area of a sphere of radius r. Let S be a ball of radius R, centered at the origin. Partition the interval $0 \leq r \leq R$ as $\{0 = r_0, r_1 = r_0 + \Delta r, ..., r_n = r_0 + n\Delta r = R\}$. Imagine the ball as consisting of n hollow concentric balls with radii $\{r_i,\ \ i = 0, 1, 2, ..., n-1\}$ and thickness Δr.

Each hollow ball is a thickened sphere, hence its volume is appoximately

$$\{volume\ of\ hollow\ ball\} \;\approx\; \{area\ of\ sphere\} \;\times\; \{thickness\},$$

that is

$$\Delta Vol_i \approx f(r_i)\Delta r.$$

Since volume is a "Whole $= \sum$ parts" phenonmen,

$$Volume = \sum \Delta Vol_i \approx \sum f(r_i) \times \Delta r$$

Taking the limit as Δr goes to zero yields:

$$Volume(R) = \int_0^R f(r)\ dr.$$

In Theorem **??**, we calculated that

$$Volume(R) = \frac{4}{3}\pi R^3.$$

Let $F(R) = \frac{4}{3}\pi R^3$. Thus

$$F(R) = \int_0^R f(r)\ dr$$

and the Fundamental Theorem of Calculus says that $f(R) = F'(R)$. Hence

$$\{Area\ of\ sphere\ with\ radius\ \ R\} = f(R) = F'(R) = 4\pi R^2.$$

$\surd$ YEAH

GENERAL CONES

A *circular right cone* is a solid object which has the shape of an ice cream cone which is filled just to the top with ice cream, and the top is flat. A circular right cone is a special case of the general cone. A *[general] cone* consists of a point called the vertex, a planar region, [2] called the base, together with all the straight lines from the vertex to the base.

Let C be a cone with the origin as vertex and whose base lies in a plane parallel to the xy-plane and h -units above it. Such a cone is said to have height h. A *slice* of this cone at height z_0, $0 < z_0 < h$ is the intersection of the cone and the plane parallel to the xy-plane and z_0-units above it.

A (mathematical) pyramid fits this definition, its base is a rectangle. A tetrahedron is a pyramid with a triangle as its base. All pyramids are examples of (general) cones.

Lemma 5 *Let C be a cone with the origin (O) as vertex and whose base lies in a plane parallel to the xy-plane. Let cone C have height h. Let points p, q, D, E be the vertices of a rectangle in the base of cone C. Let p_0 be the point where line Op meets the slice at height z_0. Let points q_0, D_0, E_0 be defined similarly. Then*

$$\frac{\{length\ of\ Op_0\}}{\{length\ of\ Op\}} = \frac{z_0}{h}.$$

$$\frac{\{length\ of\ p_0q_0\}}{\{length\ of\ pq\}} = \frac{z_0}{h}.$$

Also quadrilateral p_0, q_0, D_0, E_0 is a rectangle and

$$\frac{\{area\ of\ rectangle\ p_0, q_0, D_0, E_0\}}{\{area\ of\ rectangle\ pqDE\}} = (\frac{z_0}{h})^2.$$

[2] a "planar" region is a region in any plane

Figure 4.

Lemma 6 *Let* C *be a cone with the origin as vertex and whose base lies in a plane parallel to the xy-plane. Let cone* C *have height* h*. Then the area of the slices are proportional to the square of the heights of the slices, that is:*

$$\frac{\{Area\ of\ slice\ at\ height\ z_0\}}{\{Area\ of\ base\}} = (\frac{z_0}{h})^2.$$

Figure 5. Cone with slice

Proof. Approximate the base with a set of inscribed rectangles $\{R_i, i = 1, 2, ..., n\}$, which have areas $\Delta A_i = area\ of\ R_i, i = 1, 2, ..., n\}$. Then:

$$\{Area\ of\ base\} \approx \sum \Delta A_i.$$

In the slice at height z_0, the cone lines will determine a corresponding set of inscribed rectangles $\{R_i^{slice}, i = 1, 2, ..., n\}$, with areas $\{\Delta A_i^{slice} = area\ of\ R_i^{slice}, i = 1, 2, ..., n\}$. Together these rectangles approximate the slice at height z_0. Hence:

$$\{Area\ of\ slice\} \approx \sum \Delta A_i^{slice}.$$

Lemma **??** implies that

$$\frac{\Delta A_i^{slice}}{\Delta A_i} = (\frac{z_0}{h})^2, \forall_{i=1,2,...,n}.$$

Thus

$$\{Area\ of\ slice\} \approx \sum\ \Delta A_i^{slice} = \sum\ (\frac{z_0}{h})^2 \Delta A_i = (\frac{z_0}{h})^2 \sum\ \Delta A_i \approx (\frac{z_0}{h})^2 \{Area\ of\ base\}.$$

Taking the limit, as the rectangles are increased to fill up the base, results in the $\approx$ signs becoming $=$ signs:

$$\{Area\ of\ slice\} = (\frac{z_0}{h})^2 \{Area\ of\ base\}.$$

Thus:

$$\frac{\{Area\ of\ slice\ at\ height\ z_0\}}{\{Area\ of\ base\}} = (\frac{z_0}{h})^2.$$

$\surd$ YEAH

Theorem 7 (Volume of a Cone) *The volume of a cone is:* $\frac{1}{3}\{area\ of\ base\} \times \{height\}$.

Arc Length

I derived the integral formula for arc length, almost in the same manner as in most textbook; except that I avoided using the Mean Value Theorem.

Here an arc is any curve, not necessarily (and in fact rarely) part of a circle.

Theorem 8 (Arc Length) *Let* $f(x)$ *be a twice differentiable function on an interval* $[a, b]$. *Then the length of its graph is*

$$\{arc\ length\ of\ graph\} = \int_a^b \sqrt{1 + (f'(x))^2}\ dx.$$

Proof. Let us consider an arc which is the graph of a twice differentiable function $f(x)$ on an interval $[a, b]$. Partition the interval $[a, b]$, which induces a partition on the arc with points $\{(x_i, y_i) = (x_i, f(x_i)),\ i = 1, 2, ..., n\}$

Figure 6. Section of arc.

Let ℓ be the length of the arc and hence $\Delta\ell_i$ is the small arc length of the i^{th} part of the arc. The geometry of the diagram shows that:

$$\Delta\{arc\ length\} = \Delta\ell \approx \Delta s \quad and \quad (\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.$$

Thus:

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

But, in order to obtain a Riemann sum, we need Δx as a factor. We obtain this by multiplying by one in the form $\frac{\Delta x}{\Delta x}$:

$$\Delta s = \sqrt{\Delta x)^2 + (\Delta y)^2} = \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}}\ \Delta x = \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}\ \Delta x. \tag{1}$$

Arc length is a Whole $= \sum$ parts phenonmen. Hence:

$$\{arc\ length\} = \ell = \sum \Delta\ell \approx \sum \Delta s = \sum \sqrt{1 + (\frac{\Delta y}{\Delta x})^2}\ \Delta x.$$

Taking the limit as $\Delta x \to 0$ yields $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = f'(x)$ and the $\approx$ becoming an $=$ sign:

$$\{arc\ length\} = \ell = \int_a^b \sqrt{1 + (f'(x))^2}\ dx.$$

$\surd$ YEAH

Surface Area of Revolution

Let ℓ be the length of a straight line segment in the xy-plane, which does not cross the x-axis. Let r be the distance from the midpoint of this line segment to the x-axis. Revolve this line segment 360 degrees about the x-axis. This produces a "frustum" of a circular cone, namely an ice cream cone (without the ice cream) with a section, including its tip, sliced off. Then ℓ will also be the (slant) length of this frustum and $2\pi r$ is the circumference, of the center-circle of this frustum. Let A be the area of this frustum of a cone.

Lemma 9 *(Surface area of a frustum of a cone).* The area of the frustum of a cone, just described, is its {slant length} x {length of its center-circle}, that is: $A = \ell(2\pi r)$.

Proof. Break up (partition) the frustum of a cone into n isomorphic mildly-curved trapezoids, each with "average" width, (which is the width along its part of the centercircle) of $\Delta w = (2\pi r) \div n$. Let ΔA be the area of each of these mildly-curved trapezoids. Then

$$\Delta A \approx \ell\ \Delta w.$$

Area is a "Whole $= \sum$ parts" quantity. Hence:

$$A = \sum \Delta A \approx n[\ell\ \Delta w] = \ell(2\pi r).$$

Taking the limit as $\Delta w \to 0$ yields $A = \ell(2\pi r)$. $\surd$ YEAH

(This lemma is a special case of Pappus's Theorem for surfaces of revolution.)

Theorem 10 (Surface area of revolution) Let $f(x) > 0$ be a twice differentiable function on an interval $[a, b]$. Its graph is rotated about the x-axis. Let A be the area generated by this. Then

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

Proof. Let A be the area of the surface generated by revolving, about the x-axis, the graph of a twice differentiable function $f(x)$ on an interval $[a, b]$. Partition the interval $[a, b]$ with points $\{x_i, \ \ i = 1, 2, \ldots, n\}$, which induces a partition of the surface into thin circular strips that are approximately the type presented in Lemma **??**. By Equation (**??**), the width of such a thin strip is

$$\Delta s \approx \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x.$$

Then Lemma **??** says that the area of the i^{th} thin strip is approximately:

$$\Delta S \approx \Delta s(2\pi y) \approx \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x(2\pi y).$$

Area is a "Whole $= \sum$ parts" quantity. Hence:

$$A \approx \sum \Delta S \approx \sum \Delta s(2\pi y) \approx \sum (2\pi f(x))\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x.$$

Taking the limit as $\Delta x \to 0$ results in $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = f'(x)$ and the "$\approx$" becoming an "$=$" sign:

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

$\surd$ YEAH

Notes on Moments and Centers of Gravity

Analogous to Newton's Law of motion for an object, $(F = ma)$, is the corresponding law for rotation: $\tau = I\alpha$, where τ is the "torque" acting on an object, I is the "moment of inertia" of the object, and α is the "rotational" acceleration of the object. When there is a single force acting at a point, torque = {force} $\times$ {distance to axis of rotation}. When there are several forces, each produces a torque and the total torque is the sum of the individual ones, that is torque is a "Whole $= \sum$ parts" quantity.

The force due to gravity on an object is called weight. $\Delta weight = \{(mass) density\} g \times \Delta Vol$. For a thin metal sheet or lamina,

$$\Delta Vol = \Delta Area \times \{thickness\}.$$

For a section of a thin metal sheet or lamina,

$$\Delta\tau = \{distance\ to\ axis\ of\ rotation\} \times \Delta Force,$$

where $\Delta Force = \Delta weight = \{density\} g \times \{thickness\} \times \Delta Area.$
When $\{density\} g \times \{thickness\} = 1$, $\Delta\tau$ reduces to

$$\Delta\tau = \{distance\ to\ axis\ of\ rotation\} \times \Delta Area.$$

This is the same as a small amount of "moment" in calculus class:

$$\Delta M = \{distance\ to\ axis\ \} \times \Delta Area.$$

Occasionally, different terminology is used in mathematics than in physics. The " second moment" in mathematics is similar to the "moment of inertia" of physics and the "first moment" or just "moment" in mathematics is similar to "torque" of physics.

Moments are "Whole $= \sum$ parts" quantities, hence:

$$M_{y-axis} = \sum \Delta M \approx \sum x \Delta A \approx \sum xy \Delta x = \sum x f(x) \Delta x$$

Taking the limit as $\Delta x \to 0$ results in the sum becoming an integral:

$$M_{y-axis} = \lim_{\Delta x \to 0} \sum x f(x) \Delta x = \int x f(x) dx.$$

Average values. For a small region: $\Delta M_{y-axis} \approx x \Delta A$; hence, a reasonable concept for the "average" value $\bar{x}$ of x, for any region A, is one that satisfies a similar equation:

$$M_{y-axis} = \bar{x} A.$$

Therefore, we define the *average value*, $\bar{x}$, of x, for a region with area A, and moment about the y-axis of M_{y-axis} as

$$\bar{x} = M_{y-axis} \div A.$$

Example 11 *Find* $\overline{x}$ *for the letter "L":*

Figure 7: Thick letter "L". *Assume that the y-axis is the left side of the letter "L".*

Calculations. First we will calculate M_{y-axis} which is a "Whole $= \sum$ parts" quantity. Extend the vertical line downward to cut up the letter "L" into an 8x2 "vertical" rectangle and a 2x4 "horizontal" rectangle. Hence

$$M_{vert.\ rect.} = \{avg.\ distance\ to\ y - axis\} \ \times\ area\ = 1 \times 16 = 16$$

$$M_{horiz.\ rect.} = \{avg.\ distance\ to\ y - axis\} \ \times\ area\ = 3 \times 8 = 24$$

$$M_{y-axis} = M_{vert.\ rect.} + M_{horiz.\ rect.} = 16 + 24 = 40$$

Since area is a "Whole $= \sum$ parts" quantity, $Area = 16 + 8 = 24$. Finally, since $M_{y-axis} = \overline{x} \times area$, we see that

$$\overline{x} = \frac{M_{y-axis}}{area} = \frac{40}{24}.$$

$\overline{y}$ may be calcuated in the same manner.

EXERCISES

Exercises 2, and 6-9 are routine; I assigned them as homework. Exercises 3-5 and 10-12 are not routine; I assigned them as guided group board work.

Exercise 1 *State the informal definition of the definite integral.*

Exercise 2 (Area of an Ellipse) *Show that the formula for the area of the (inside of an) ellipse with equation:* $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$, *is* πab.

Remark. When the ellipse is a circle, $a = b$ and this area formula becomes πa^2 which is the area of a circle of radius a.

Exercise 3 (Volume of an Ellipsoid) *Show that the formula for the volume (of the inside) of the ellipsoid:* $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$, *is* $\frac{4}{3}\pi abc$.

This is *not* a solid of revolution. Before starting this exercise, you are advised to review the proof of Theorem **??**. The calculations of Exercise **??** will be helpful.
Set up. Consider this ellipsoid.

If $c > 5$, then the horizontal slice at height $z = 5$ is the set of points which satisfy the conditions ($(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 \leq 1$ and $z = 5$). This is the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 \leq 1 - (\frac{5}{c})^2$

Now we partition the z-axis from -R to +R with $-R = z_0, z_1 = z_0 + \Delta z, ..., z_n = z_0 + n\Delta z = +R$. We slice the spherical ball horizontally at each z_i. In general, the horizontal slice at height $z = z_i$ is the set of points which satisfy both conditions ($(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 \leq 1$ and $z = z_i$). Combining these conditions yields: $(\frac{x}{a})^2 + (\frac{y}{b})^2 \leq 1 - (\frac{z_i}{c})^2$. We set $g^2(z) = 1 - (\frac{z}{c})^2$. Then this inequality may be rewritten as $(\frac{x}{ag(z_i)})^2 + (\frac{y}{bg(z_i)})^2 \leq 1$, which we recognize as the equation of an ellipse. Using the formula you calculated in Exercise **??**, the area is $\Delta A_i = \pi abg^2(z_i) = \pi ab(1 - (\frac{z_i}{c})^2)$. The section between slices heights at z_i and z_{i+1} is appoximately a (non-circular) right cylinder with height Δz. Now you may proceed as in the proof of Theorem **??**.

Remark. Note that when the ellipsoid is a sphere, $a = b = c$ and this volume formula becomes $\frac{4}{3}\pi a^3$ which is the volume of a sphere of radius a.

Exercise 4 *Using plane geometry establish Lemma* **??**.

Exercise 5 *Using Lemma* **??** *and the methods of the handout "Doing word problems with integrals" establish Theorem* **??**.

Exercise 6 (Circumference of a Circle) *Using an integral formula for length of a curve, check that the circumference of a circle is* $2\pi R$.

Exercise 7 (Surface Area of a Sphere) *Considering the sphere as a surface of revolution, show that the (surface) area of a sphere with radius* R *is* $4\pi R^2$.

Exercise 8 (Area of a cone) *Considering an ice cream cone (without ice cream) as a surface of revolution, about the x-axis, of the straight line of length* l, *with one end at the origin and the other end at point* (x, y), *with* $y = R$. *Check that the formula, for the area of the cone, produced by using an integral formula for surface area is* πRl.

Exercise 9 (Area of an cyclinder) *Considering a cyclinder as a surface of revolution, about the x-axis, of a straight line of length* l *in the line* $y = R$*. Check that the formula, for the area of the cylinder, produced by using an integral formula for surface area is* $2\pi Rl = \{circumference\}$ *x* $\{length\}$*.*

Exercise 10 (Change of scale) *Let* R *be the region under the curve* $y = \sin^2 x$ *and between the lines* $x = 0$ *and* $x{=}\pi$*. Let* $(\overline{x}, \overline{y})$ *be the c.m. for this Region* R*. Let* R_7 *be the region under the curve* $y = \sin^2 7x$ *and between the lines* $x = 0$ *and* $x{=}\frac{\pi}{7}$*. Let* $(\overline{x}_7, \overline{y}_7)$ *be the c.m. for this Region* R_7*. Show that* $\overline{y}_7 = \overline{y}$*.*

Draw diagrams for this exercise. Can you use the diagrams to explain the results?

Note: Evaluating 4 integrals is the long way to do this exercise. The short way uses the change of variable: $\theta = 7x$.

Exercise 11 *Let* $f(x) = e^{\sin(\cos[\ln\ x])}$*. Let* R_f *be the region under the curve* $y = f(x)$ *and between the lines* $x = 3$ *and* $x{=}9$*.Let* $(\overline{x}_f, \overline{y}_f)$ *be the c.m. for Region* R_f*.*

(a) (Scale) *Let* $g(x) = 7f(x)$*. Let* R_g *be the region under the curve* $y = g(x)$ *and between the lines* $x = 3$ *and* $x{=}9$*.Let* $(\overline{x}_g, \overline{y}_g)$ *be the c.m. for Region* R_g*. Using the integral formulas, show that* $\overline{x}_g = \overline{x}_f$ *and* $\overline{y}_g = \overline{7}y_f$*.*

Trying to find anti-derivatives for these functions is like walking into a trap.

(b) (Shift) *Let* $h(x) = f(x{-}7) = e^{\sin(\cos[\ln(x-7)])}$*. Let* R_h *be the region under the curve* $y = h(x)$ *and between the lines* $x = 10$ *and* $x{=}16$*.Let* $(\overline{x}_h, \overline{y}_h)$ *be the c.m. for Region* R_h*. Show that* $\overline{x}_g - 7 = \overline{x}_h$ *and* $\overline{y}_g = \overline{y}_h$*. A change of variable* $u = x - 7$ *may help.*

(c) *Draw diagrams for this exercise. Can you use the diagrams to explain the results?*

Exercise 12 *Generalize the results of the last two exercises.*

Appendix 2. Professor Dancis's honors-calculus exam on deriving integral formulas.

Prof. Dancis **MATH 141H** **Test 3** **April 10, 1996**

Directions. For word problems with integrals: Use the method of the Green Notes. This includes: stating a physical equation verbally and/or a geometric equation verbally. Draw a useful diagram which indicates the section and Δx or Δy. Calculate a formula for ΔVol, $\Delta Work$, or $\Delta whatever$.

1. Derive the (parametric) integral formula for the arc length of the curve defined by
$x = 7\cos 6t$ and y=7sin $6t$ when $0 \leq t \leq \frac{\pi}{3}$.
Evaluate the integral.
Sketch the curve. *20 points*

2. The force exerted by a stretched spring is $F = -kx$, where k is Hooke's spring constant and x is the extra length of the spring over its "rest" length. Derive the integral formula for the work needed to stretch the spring from its rest length to a length that is x_0 units longer than the rest length.
Evaluate the integral. *15 points*

3. State the Theorem of Pappus and Gulden on volumes of revolution. Use it to find the volume of a bagel (doughnut) with inner radius r and outer radius R. Draw a diagram. *10 points*

Lemma. The area of a cross section of a cone is proportional to its height squared.
4. Using this lemma, show that the volume of all cones (even those with kidney-shaped bases) are $\frac{1}{3}\{area\ of\ base\} \times \{height\}$. Use this result to quickly find the volume of a pyramid with height 30 feet and a square base with 20 feet on each side. *20 points*

5. The village's water tank is a hemisphere with radius 70 feet (with a flat top and round bottom). Derive the integral formula for the work needed to fill the tank from a lake whose water surface is 100 feet below the bottom of the water tank. Water (weight) density is 62.5 $lbs./ft^3$. Do not integrate. *18 points*

6. Without using integrals, find the average value ($\overline{y}$) of y for the figure below. *12 points*

7. On average, how many hours per week did you study for this course? How did you feel about proofs last year? How did you feel about proofs now? *5 points*

Total 100 points

Appendix 3. Professor Dancis's MATH 141H students' answers to Question 7 on the exam

See Appendix 2. (Just/only the number of hours studied was removed)

I feel much at ease with proofs this year. In fact, thanks to those proofs, I understand the formulas much better, and can reason them out better. Thanks.

Last year I hated proofs but now they don't seem so bad.

Last year I didn't feel very comfortable about proofs, but now I am at least relaxed about them. I enjoy learning them because it helps me to learn the material much more.

Last year I was a little scared; and now I feel more confident. I like the we prove in class more than in the book. It's easier to understand the problem (and later to understand the book).

Last year proofs were somewhat scary. This year they are much easier.... I really like your "informal proof" approach to understanding the proofs.

I thought this chapter was hard to study because the book wasn't much help since it only taught you how to plug into a formula and that isn't what we are tested on. Last year I hated proofs. After this chapter I am more comfortable with them because I can actually follow them and they make sense.

We did a lot of proofs last year, but I feel more confident with them now.

I didn't understand proofs, [now] I understand them when we go over them in class in groups or in lecture.

I wasn't very comfortable with proofs last year, but now they no longer intimidate me because of the amount I have done them this year.

The proof are EXTREMELY useful. Proof last year were not as cool.

The proofs of this kind is somewhat new. I do feel comfortable about it.

I am much more comfortable with proofs now than I was at the beginning of the semester or last year.

Last year, I didn't care about proofs one way or another. This year, I was reminded that I really didn't like them. [This student scored 32 on test]

I did not like them but they were easy. I did not like them at the beginning of this year because they were harder than before but after doing so many they don't seem as hard.

I think I was quite easy with proofs last year; and same is now.

Last year I wasn't so confident about proofs, but I am becoming more comfortable.

Last year I didn't do many proofs and the ones that I did I felt comfortable with. However, I now feel comfortable doing proofs and ?? better than actually using what we prove in math.

Biographical sketch: Jerome Dancis taught himself calculus from an outline book to avoid the boredom of his high school solid geometry class. Deriving integral formulas like the ones in this article was the best part.

Dancis earned a B.S. degree in Applied Mathematics at the Polytechnic Institute of Brooklyn (1961). He earned his Ph.D. in geometric topology from Academician R. H. Bing at the Univ. of Wisconsin in Madison (1966). He has written research papers on geometric topology, matrix theory and matrix numerical analysis.

In graduate school, Dancis was trained in the R. L. Moore method of guided discovery learning with students deriving the proofs of the course's theorems. He is interested in mathematics pedagogy at both the college and high school levels. He remains a strong proponent of guided discovery learning. He is a math education reformer but not a Reformer as he is an opponent of the current math education Reform movement.

Dancis teaches mathematics at the University of Maryland (College Park).