MATH 241 Final Examination

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Instructions. Answer each question **on a *separate* answer sheet**. Show all your work. A correct answer without work to justify it may not receive full credit. *Be sure your name, section number, and problem number are on each answer sheet, and that you have copied and signed the honor pledge on the first answer sheet.* The point value of each problem is indicated. The exam is worth a total of 200 points. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. Please leave answers such as $5\sqrt{2}$ or 3π in terms of radicals and π and *do not convert to decimals*.
You are allowed use of one sheet of notes. Calculators are not permitted.

1. (a) (10 points) Find the area of the triangle with vertices $(1, -1, 1)$, $(1, 2, 3)$, $(2, -1, 2)$.

   (b) (20 points) Show that the line ℓ described by

$$\frac{x+\sqrt{2}}{1} = \frac{y+3}{2\sqrt{2}} = \frac{z}{2}$$

   is parallel to the plane $\mathcal{P}$ that has equation $2\sqrt{2}\,x - 3y + 2\sqrt{2}\,z = 0$. What is the distance D from ℓ to $\mathcal{P}$?

2. (25 points) A particle moves in the x-y plane along the curve $y = (2/3)x^{3/2}$ in such a way that the x coordinate of the particle is $x(t) = \left(\frac{3t}{2}\right)^{2/3} - 1$ for any time $t \geq 2/3$. Find the distance L traveled by this particle for $1 \leq t \leq 2$.

3. (20 points) Find all critical points of the function

$$f(x, y) = 6x^2 - 3xy^2 + 3y^2 + 17$$

   and characterize each one as a local maximum, a local minimum, or a saddle point.

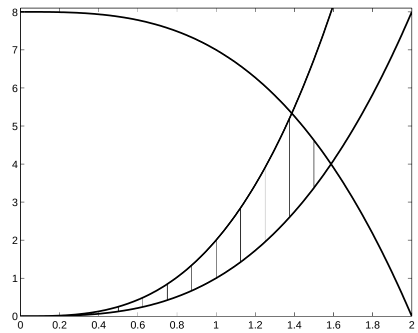
4. (20 points) Suppose

$$f(u, v) = \frac{e^v}{u+1}.$$

   If $u = x^2 + y^2$ and $v = x^2 - y^2$, *use the chain rule* to compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ (as functions of x and y). If you do the problem without using the chain rule, you will only receive partial credit.

5. (25 points) Find the volume of the region D lying above the plane $z = 1$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

6. (30 points) Find the area in the first quadrant enclosed by the curves $y = x^3$, $y = 2x^3$, and $y = 8 - x^3$ (see figure that follows). One way to do the problem is to use the change of variables $u = x^3$, $v = y + x^3$.



7. Consider the vector field
$$\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}\ .$$
   (a) (15 points) By use of Gauss's Theorem (also known as the Divergence Theorem), evaluate the flux integral
   $$I = \iint_\Sigma (\mathbf{F} \cdot \mathbf{n})\, dS\ ,$$
   where Σ is the boundary of the cube D with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$, and $\mathbf{n}$ is the unit normal directed outward from D.

   (b) (10 points) Compute *directly* the flux integral I of part (a). You should show that the result agrees with that found in part (a).

8. Of the vector fields
$$\mathbf{F}_1 = (2xy + z)\,\mathbf{i} + (x^2 + \cos y)\,\mathbf{j} + x\,\mathbf{k} \text{ and } \mathbf{F}_2 = (2xy)\,\mathbf{i} + (x^2 + \cos y)\,\mathbf{j} + x\,\mathbf{k},$$
one is conservative and the other is not.

   (a) (15 points) Determine which vector field is conservative, and write it in the form ∇f for some function f.

   (b) (10 points) If $\mathbf{F}$ $(= \mathbf{F}_1$ or $\mathbf{F}_2)$ is the conservative vector field, compute the line integral
   $$\int_C \mathbf{F} \cdot d\mathbf{r},$$
   where C is the path consisting of a straight line segment from $(1,0,0)$ to $(1,\pi,0)$ followed by a straight line segment from $(1,\pi,0)$ to $(0,\pi,1)$.