Math 241 Chapter 11 **Dr. Justin O. Wyss-Gallifent**

§11.1 Cartesian Coordinates in Space

1. Preliminaries: How to plot points in 3-space, the coordinate planes, the first octant. Emphasize how perspective can be confusing at first.

2. Distance between points: $|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$

3. Equation of a circle, a closed disk, a sphere and a closed ball. Pictures of all.

§11.2 Vectors in Space

1. Definition of a vector as a triple of numbers. The notation $\bar{a}$ or $\vec{a}$. We can add and subtract vectors by adding and subtracting components and we can multiply a scalar by a vector by multiplying by all the components. Three special vectors are $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$. Then every vector can be written as $\bar{a} = a_1\,\hat{\imath} + a_2\,\hat{\jmath} + a_3\,\hat{k}$. Vectors are not necessarily anchored anywhere though often we anchor them somewhere (the origin, for example) for some reason.

2. Basic properties and associated definitions:

   (a) The *zero vector* is $\bar{0} = 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}$.
   (b) The *length* of a vector is $||\bar{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
   (c) A *unit vector* has length 1. If a vector $\bar{a}$ is given we can create a unit vector in the same direction by doing $\bar{a}/||\bar{a}||$.
   (d) Two vectors are *parallel* if they are nonzero multiple of one another. In other words $\bar{a} = c\bar{b}$ with $c \neq 0$.
   (e) The vector pointing from $P = (a_1, a_2, a_3)$ to $Q = (b_1, b_2, b_3)$ is $\vec{PQ} = (b_1 - a_1)\,\hat{\imath} + (b_2 - a_2)\,\hat{\jmath} + (b_3 - a_3)\,\hat{k}$.
   (f) $\bar{0} + \bar{a} = \bar{a} = \bar{a} + \bar{0}$
   (g) $\bar{a} + \bar{b} = \bar{b} + \bar{a}$
   (h) $c(\bar{a} + \bar{b}) = c\bar{a} + c\bar{b}$
   (i) $0\bar{a} = \bar{0}$
   (j) $1\bar{a} = \bar{a}$
   (k) $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$

3. Geometric interpretation of $\bar{a} + \bar{b}$, of $\bar{a} - \bar{b}$ and $c\bar{a}$.

§11.3 The Dot Product

1. Definition of $\bar{a} \cdot \bar{b}$.

2. Basic properties:

   (a) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$
   (b) $\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$
   (c) $(\bar{b} + \bar{c}) \cdot \bar{a} = \bar{b} \cdot \bar{a} + \bar{c} \cdot \bar{a}$
   (d) $c(\bar{a} \cdot \bar{b}) = (c\bar{a}) \cdot \bar{b} = \bar{a} \cdot (c\bar{b})$

3. Additional Properties:

   (a) If θ is the angle between $\bar{a}$ and $\bar{b}$ (anchored at the same point) then $\bar{a} \cdot \bar{b} = ||\bar{a}||||\bar{b}|| \cos\theta$.
   (b) $\bar{a}$ and $\bar{b}$ are perpendicular iff $\bar{a} \cdot \bar{b} = 0$.
   (c) $\bar{a} \cdot \bar{a} = ||\bar{a}||^2$.
   (d) Definition of projection of $\bar{b}$ onto $\bar{a}$ and formula $\text{Pr}_{\bar{a}}\bar{b} = \left(\frac{\bar{a}\cdot\bar{b}}{\bar{a}\cdot\bar{a}}\right)\bar{a}$.

§11.4 The Cross Product

1. Definition of $\bar{a} \times \bar{b}$.
2. Basic properties:
   (a) $\bar{a} \times \bar{a} = \bar{0}$
   (b) $\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$
   (c) $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$
   (d) $(\bar{b} + \bar{c}) \times \bar{a} = \bar{b} \times \bar{a} + \bar{c} \times \bar{a}$
   (e) $c(\bar{a} \times \bar{b}) = (c\bar{a}) \times \bar{b} = \bar{a} \times (c\bar{b})$
3. Additional properties:
   (a) $\bar{a} \times \bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$. This is extremely useful.
   (b) $||\bar{a} \times \bar{b}|| = ||\bar{a}|| ||\bar{b}|| \sin\theta$
   (c) $\bar{a}$ and $\bar{b}$ are parallel iff $\bar{a} \times \bar{b} = \bar{0}$ but this is not a particularly good way to check.

§11.5 Lines in Space

1. Intro: What determines a line? What can we use for an equation? The fundamental way to define a line is to have a point on the line and a vector pointing along (parallel to) the line. If $\bar{a}\,\hat{\imath} + \bar{b}\,\hat{\jmath} + \bar{c}\,\hat{k}$ is parallel to the line and if the line contains the point $P = (x_0, y_0, z_0)$ then:
2. Parametric Equations: $x = x_0 + at$, $y = y_0 + bt$ and $z = z_0 + ct$. Each t gives a point on the line.
3. Vector Equation: $\bar{r} = (x_0 + at)\,\hat{\imath} + (y_0 + bt)\,\hat{\jmath} + (z_0 + ct)\,\hat{k}$. Each t gives a vector which points from the origin to a point on the line. This is far from unique since on any given line there are many points and many vectors pointing along the line.
4. Symmetric Equations: Solve for t in each of the parametric equations and set them all equal. If one doesn't have a t in it just leave it be. If two do not then leave those alone and don't even write the third because that variable could be anything.
5. If a line has point P and vector $\bar{L}$ then the distance from another point Q to the line is $\frac{||\bar{L} \times \overrightarrow{PQ}||}{||\bar{L}||}$.

§11.6 Planes in Space

1. A plane is determined by a point $P = (x_0, y_0, z_0)$ and a normal vector $\bar{N} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$ which is perpendicular to the plane. A point $Q = (x, y, z)$ is on the plane iff the vector from P to Q is perpendicular to $\bar{N}$, meaning $(a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}) \cdot \overrightarrow{PQ} = 0$ which is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. This is often rearranged to get $ax + by + cz = d$.
2. If a plane has point P and normal vector $\bar{N}$ then the distance from another point Q to the plane is $\frac{|\bar{N} \cdot \overrightarrow{PQ}|}{||\bar{N}||}$.
3. Sketching planes:
   - Those like $ax + by + cz = d$, draw a little triangle using the intercepts.
   - Those like $z = 0$ or $x = 2$ or $y = -3$, parallel to the coordinate planes.
   - Those like $2x + y = 10$, draw a line and extend in the direction of the missing variable.