Math 241 Chapter 12 **Dr. Justin O. Wyss-Gallifent**

§12.1 Definitions and Examples of Vector Valued Functions

1. Definition: For each t, $\bar{F}(t)$ (or usually, and later, $\bar{r}(t)$) points from the origin to a point on the curve.

2. Classic examples: Circles, helices, lines and line segments, functions.

3. Properties: Without going too far into detail note that VVFs are vectors and so we can do vectorish things with them like $\bar{F} \times \bar{G}$ and $f\bar{F}$ where f is a regular function.

§12.2 Limits and Continuity of VVFs

1. We can define limit the limit of a VVF by taking the limits of the components. We can then define a VVF to be continuous iff the components are continuous. We won't go into detail but it's good to be aware that limits exist so that derivatives do.

§12.3 Derivatives and Integrals of VVFs

1. We can do this formally with limits but we won't. In essence the derivative of a VVF is found by taking the derivatives of the components.

2. Meaning: If $\bar{r}$ is position then $\bar{v}(t) = \bar{r}'(t)$ is the velocity and is a vector which is tangent to the curve, meaning it points in the direction of the curve. The function $s(t) = ||\bar{v}(t)||$ is speed. The vector $\bar{a}(t) = \bar{v}'(t) = \bar{r}''(t)$ is acceleration and points to the inside of the curve.

3. Properties:
   (a) $(\bar{F} \pm \bar{G})' = \bar{F}' \pm \bar{G}'$
   (b) $(\bar{F} \cdot \bar{G})' = \bar{F}' \cdot \bar{G} + \bar{F} \cdot \bar{G}'$
   (c) $(\bar{F} \times \bar{G})' = \bar{F}' \times \bar{G} + \bar{F} \times \bar{G}'$
   (d) $(f\bar{F})' = f'\bar{F} + f\bar{F}'$
   (e) $(\bar{F} \circ f)' = \bar{F}'(f)f'$

4. The integral of the VVF is just the integral of the components. For an indefinite integral we just put a $+\bar{C}$ on the end rather than giving each component its own constant.

5. One classic problem is to give $\bar{a}(t)$, $\bar{v}(t_0)$ and $\bar{r}(t_1)$ and then find $\bar{r}(t)$ by successive integration and plugging-in.

§12.4 Space Curves and their Lengths

1. Definition: *A (space) curve* is just the curve itself, without worrying initially about the parametrization. A *parametrization* of C is an $\bar{r}(t)$ which draws that curve.

2. Associated definitions:
   (a) A curve is *closed* if it has a parametrization $\bar{r}(t)$ where $\bar{r}(a) = \bar{r}(b)$ and the curve only touches itself finitely many times at most. In other words it forms a loop.
   (b) A curve is *smooth* if it has a parametrization $\bar{r}(t)$ such that: If $\bar{r}(t)$ exists for some t then $\bar{r}'(t)$ exists and $\bar{r}'(t) \neq \bar{0}$ except it's allowed to at the end points.
   (c) A curve is *piecewise smooth* if it can be divided up into a number of smooth pieces.
   (d) The length of a curve is $\int_a^b ||\bar{r}'(t)||\ dt$.

§12.5 Tangents and Normals to Curves

1. Defn: The tangent vector is $\bar{T}(t) = \frac{\bar{r}'(t)}{||\bar{r}'(t)||}$. Note that the tangent vector is different from a tangent vector. Also note that $\bar{T}(t)$ is a unit vector.

2. Defn: The normal vector is $\bar{N}(t) = \frac{\bar{T}'(t)}{||\bar{T}'(t)||}$. This points inwards from the curve. If the curve does not bend then the normal vector does not exist.

§12.6 Curvature

1. Defn: The curvature is $\kappa(t) = \frac{||\bar{T}'(t)||}{||\bar{r}'(t)||}$. Another way to get it is $\kappa(t) = \frac{||\bar{v}\times\bar{a}||}{||\bar{v}||^3}$. The first is the formal definition while the second is usually easier to calculate.

2. Meaning: The more bendy the curve is at a point the higher the curvature at that point. A straight line has curvature 0.