A primer on the $(2+1)$ Einstein universe

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Abstract. The Einstein universe is the conformal compactification of Minkowski space. It also arises as the ideal boundary of anti-de Sitter space. The purpose of this article is to develop the synthetic geometry of the Einstein universe in terms of its homogeneous submanifolds and causal structure, with particular emphasis on dimension $2+1$, in which there is a rich interplay with symplectic geometry.

Mathematics Subject Classification (2000). Primary 53-XX; Secondary 83-XX.

Keywords. Minkowski space, spacetime, Lorentzian manifold, conformal structure, Lie algebra, symplectic vector space

Contents

*Goldman is grateful to the Erwin Schrodinger Institute (Vienna) for hospitality during the writing of this paper, and to National Science Foundation grant DMS-0405605. Charette is grateful for support from NSERC.

1. Introduction

We will explore the geometry of the conformal compactification of Minkowski $(n+1)$–space inside of $\mathbb{R}^{n,2}$. We shall call this conformal compactification $\mathrm{Ein}^{n,1}$, or the *Einstein universe*, and its universal cover will be denoted $\widetilde{\mathrm{Ein}}^{n,1}$. The Einstein universe is a homogeneous space G/P, where $G = \mathrm{PO}(n,2)$, and P is a parabolic subgroup. When $n = 3$, then G is locally isomorphic to $\mathrm{Sp}(4,\mathbb{R})$.

The origin of the terminology “Einstein universe” is that A. Einstein himself considered as a paradigmatic universe the product $S^3 \times \mathbb{R}$ endowed with the Lorentz metric $ds_0^2 - dt^2$, where ds_0^2 is the usual constant curvature Riemannian metric on S^3. The conformal transformations preserve the class of lightlike geodesics and provide a more flexible geometry than that given by the metric tensor.

Our motivation is to understand conformally flat Lorentz manifolds and the Lorentzian analog of Kleinian groups. Such manifolds are locally homogeneous geometric structures modeled on $\mathrm{Ein}^{2,1}$.

The Einstein universe $\mathrm{Ein}^{n,1}$ is the *conformal compactification* of Minkowski space $\mathbb{E}^{n,1}$ in the same sense that the n-sphere

$$S^n = \mathbb{E}^n \cup \{\infty\}$$

conformally compactifies *Euclidean space* $\mathbb{E}^n$; in particular, a Lorentzian analog of the following theorem holds (see [11]):

Theorem 1.0.1 (Liouville's theorem). *Suppose $n \geq 3$. Then every conformal map $U \xrightarrow{f} \mathbb{E}^n$ defined on a nonempty connected subdomain $U \subset \mathbb{E}^n$ extends to a conformal automorphism $\bar{f}$ of S^n. Furthermore $\bar{f}$ lies in the group* $\mathrm{PO}(n+1,1)$ *generated by inversions in hyperspheres and Euclidean isometries.*

Our viewpoint involves various geometric objects in Einstein space: points are organized into 1-dimensional submanifolds which we call *photons,* as they are lightlike geodesics. Photons in turn form various subvarieties, such as lightcones and hyperspheres. For example, a lightcone is the union of all photons through a given point. Hyperspheres fall into two types, depending on the signature of the induced conformal metric. *Einstein hyperspheres* are Lorentzian, and are models of $\mathrm{Ein}^{n-1,1}$, while *spacelike hyperspheres* are models of S^n with conformal Euclidean geometry.

The Einstein universe $\mathrm{Ein}^{n,1}$ can be constructed by projectivizing the nullcone in the inner product space $\mathbb{R}^{n+1,2}$ defined by a symmetric bilinear form of type $(n+1,2)$. Thus the points of $\mathrm{Ein}^{n,1}$ are *null lines* in $\mathbb{R}^{n+1,2}$, and photons correspond to isotropic 2-planes. Linear hyperplanes H in $\mathbb{R}^{n+1,2}$ determine lightcones, Einstein hyperspheres, and spacelike hyperspheres, respectively, depending on whether the restriction of the bilinear form to H is degenerate, type $(n,2)$, or Lorentzian, respectively.

Section 4 discusses *causality* in Einstein space. Section 5 is specific to dimension 3, where the conformal Lorentz group is locally isomorphic to the group of linear symplectomorphisms of $\mathbb{R}^4$. This establishes a close relationship between the symplectic geometry of $\mathbb{R}^4$ (and hence the contact geometry of $\mathbb{RP}^3$) and the conformal Lorentzian geometry of $\mathrm{Ein}^{2,1}$. Section 6 reinterprets these synthetic geometries in terms of the structure theory of Lie algebras. Section 7 discusses the dynamical theory of discrete subgroups of $\mathrm{Ein}^{2,1}$ due to Frances [13], and begun by Kulkarni [19]. Section 8 discusses the *crooked planes*, discovered by Drumm [8], in the context of $\mathrm{Ein}^{2,1}$; their closures, called *crooked surfaces* are studied and shown to be Klein bottles invariant under the Cartan subgroup of $\mathrm{SO}(3,2)$. The paper concludes with a brief description of discrete groups of conformal transformations and some open questions.

Much of this work was motivated by the thesis of Charles Frances [11], which contains many constructions and examples, his paper [13] on Lorentzian Kleinian groups, and his note [11] on compactifying crooked planes. We are grateful to Charles Frances and Anna Wienhard for many useful discussions.

We are also grateful to the many institutions where we have been able to meet to discuss the mathematics in this paper. In particular, we are grateful for the hospitality provided by the Banff International Research Station [5] where all of us were able to meet for a workshop in November 2004, the workshop in Oostende, Belgium in May 2005 on “Discrete groups and geometric structures,” the miniconference in Lorentzian geometry at the E.N.S. Lyon in July 2005, the special semester at the Newton Institute in Cambridge in Fall 2005, the special semester at the Erwin Schrödinger Institute in Fall 2005, and a seminar at the University of Maryland in summer 2006, when the writing began.

2. Synthetic geometry of $\mathrm{Ein}^{n,1}$

In this section we develop the basic synthetic geometry of *Einstein space,* or the *Einstein universe*, starting with the geometry of *Minkowski space* $\mathbb{E}^{n,1}$.

2.1. Lorentzian vector spaces.

We consider *real inner product spaces,* that is, vector spaces V over $\mathbb{R}$ with a nondegenerate symmetric bilinear form $\langle , \rangle$. A nonsingular symmetric $n \times n$-matrix B defines a symmetric bilinear form on $\mathbb{R}^n$ by the rule:

$$\langle u, v\rangle_B := u^\dagger B v.$$

where $u^\dagger$ denotes the transpose of the vector u. We shall denote by $\mathbb{R}^{p,q}$ a real inner product space whose inner product is of type (p,q). For example, if

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \\ u_{p+1} \\ \vdots \\ u_{p+q} \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_p \\ v_{p+1} \\ \vdots \\ v_{p+q} \end{bmatrix},$$

then

$$\langle u, v \rangle := u_1 v_1 + \cdots + u_p v_p - u_{p+1} v_{p+1} - \cdots - u_{p+q} v_{p+q}$$

defines a type (p,q) inner product, induced by the matrix $\mathrm{I}_p \oplus -\mathrm{I}_q$ on $\mathbb{R}^{p+q}$. The group of linear automorphisms of $\mathbb{R}^{p,q}$ is $\mathrm{O}(p,q)$.

If B is positive definite—that is, $q = 0$—then we say that the inner product space $(V, \langle , \rangle)$ is *Euclidean.* If $q = 1$, then $(V, \langle , \rangle)$ is *Lorentzian.* We may omit reference to the bilinear form if it is clear from context.

If V is Lorentzian, and $v \in V$, then v is:

- *timelike* if $\langle v, v \rangle < 0$;
- *lightlike* (or *null* or *isotropic*) if $\langle v, v \rangle = 0$;
- *causal* if $\langle v, v \rangle \leq 0$;
- *spacelike* if $\langle v, v \rangle > 0$.

The *nullcone* $\mathfrak{N}(V)$ in V consists of all null vectors.

If $W \subset V$, then define its *orthogonal complement:*

$$W^\perp := \{ v \in V \mid \langle v, w \rangle = 0 \ \forall \ w \in W \}.$$

The hyperplane $v^\perp$ is *null* (respectively, *timelike*, *spacelike*) if v is null (respectively spacelike, timelike).

In the sequel, according to the object of study, we will consider several symmetric $n \times n$-matrices and the associated type (p,q) symmetric bilinear forms. For different bilinear forms, different subgroups of $\mathrm{O}(p,q)$ are more apparent. For example:

- Using the diagonal matrix
  $$\mathrm{I}_p \oplus -\mathrm{I}_q$$
  invariance under the maximal compact subgroup
  $$\mathrm{O}(p) \times \mathrm{O}(q) \subset \mathrm{O}(p,q)$$
  is more apparent.

- Under the bilinear form defined by the matrix

$$\mathrm{I}_{p-q} \oplus \bigoplus^{q} -1/2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(if $p \geq q$), invariance under the *Cartan subgroup*

$$\{\mathrm{I}_{p-q}\} \times \prod^{q} \mathrm{O}(1,1)$$

is more apparent.

- Another bilinear form which we use in the last two sections is:

$$\mathrm{I}_{p-1} \oplus -\mathrm{I}_{q-1} \oplus -1/2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is useful in extending subgroups of $\mathrm{O}(p-1, q-1)$ to $\mathrm{O}(p,q)$.

2.2. Minkowski space. *Euclidean space* $\mathbb{E}^n$ is the model space for *Euclidean geometry,* and can be characterized up to isometry as a simply connected, geodesically complete, flat Riemannian manifold. For us, it will be simpler to describe it as an *affine space* whose underlying vector space of translations is a Euclidean inner product space $\mathbb{R}^n$. That means $\mathbb{E}^n$ comes equipped with a simply transitive vector space of translations

$$p \mapsto p + v$$

where $p \in \mathbb{E}^n$ is a point and $v \in \mathbb{R}^n$ is a vector representing a parallel displacement. Under this simply transitive $\mathbb{R}^n$-action, each tangent space $T_p(\mathbb{E}^n)$ naturally identifies with the vector space $\mathbb{R}^n$. The Euclidean inner product on $\mathbb{R}^n$ defines a positive definite symmetric bilinear form on each tangent space—that is, a *Riemannian metric.*

Minkowski space $\mathbb{E}^{n,1}$ is the Lorentzian analog. It is characterized up to isometry as a simply connected, geodesically complete, flat Lorentzian manifold. Equivalently, it is an affine space whose underlying vector space of translations is $\mathbb{R}^{n,1}$.

The geodesics in $\mathbb{E}^{n,1}$ are paths of the form

$$\begin{aligned} \mathbb{R} &\xrightarrow{\gamma} \mathbb{E}^{n,1} \\ t &\longmapsto p_0 + tv \end{aligned}$$

where $p_0 \in \mathbb{E}^{n,1}$ is a point and $v \in \mathbb{R}^{n,1}$ is a vector. A path γ as above is *timelike, lightlike, or spacelike,* if the *velocity* v is timelike, lightlike, or spacelike, respectively.

Let $p \in \mathbb{E}^{n,1}$. The *affine lightcone* $L^{\mathrm{aff}}(p)$ at p is defined as the union of all lightlike geodesics through p:

$$L^{\mathrm{aff}}(p) := \{p + v \in \mathbb{E}^{n,1} \mid \langle v, v \rangle = 0\}.$$

Equivalently $L^{\mathrm{aff}}(p) = p + \mathfrak{N}$ where $\mathfrak{N} \subset \mathbb{R}^{n,1}$ denotes the nullcone in $\mathbb{R}^{n,1}$. The hypersurface $L^{\mathrm{aff}}(p)$ is ruled by lightlike geodesics; it is singular only at $\{p\}$. The Lorentz form on $\mathbb{E}^{n,1}$ restricts to a degenerate metric on $L^{\mathrm{aff}}(p) \setminus \{p\}$.

A lightlike geodesic $\ell \subset \mathbb{E}^{n,1}$ lies in a unique null affine hyperplane. (We denote this $\ell^{\perp}$, slightly abusing notation.) That is, writing $\ell = p + \mathbb{R}v$, where $v \in \mathbb{R}^{n,1}$ is a lightlike vector, the null hyperplane $p + v^{\perp}$ is independent of the choices of p and v used to define ℓ.

The *de Sitter hypersphere* of radius r centered at p is defined as

$$S_r(p) := \{p + v \in \mathbb{E}^{n,1} \mid \langle v, v\rangle = r^2\}.$$

The Lorentz metric on $\mathbb{E}^{n,1}$ restricts to a Lorentz metric on $S_r(p)$ having constant sectional curvature $1/r^2$. It is geodesically complete and homeomorphic to $S^{n-1} \times \mathbb{R}$. It is a model for *de Sitter space* $\mathrm{dS}^{n-1,1}$.

As in Euclidean space, a *homothety (centered at x_0)* is any map conjugate by a translation to scalar multiplication:

$$\begin{aligned} \mathbb{E}^{n,1} &\longrightarrow \mathbb{E}^{n,1} \\ x &\longmapsto x_0 + r(x - x_0). \end{aligned}$$

A *Minkowski similarity transformation* is a composition of an isometry of $\mathbb{E}^{n,1}$ with a homothety:

$$f : x \longmapsto rA(x) + b.$$

where $A \in \mathrm{O}(n,1), r > 0$ and $b \in \mathbb{R}^{n,1}$ defines a translation. Denote the group of similarity transformations of $\mathbb{E}^{n,1}$ by $\mathrm{Sim}(\mathbb{E}^{n,1})$.

2.3. Einstein space.

Einstein space $\mathrm{Ein}^{n,1}$ is the projectivized nullcone of $\mathbb{R}^{n+1,2}$. The nullcone is

$$\mathfrak{N}^{n+1,2} := \{v \in \mathbb{R}^{n+1,2} \mid \langle v, v\rangle = 0\}$$

and the $(n+1)$-*dimensional Einstein universe* $\mathrm{Ein}^{n,1}$ is the image of $\mathfrak{N}^{n+1,2} - \{0\}$ under projectivization:

$$\mathbb{R}^{n+1,2} - \{0\} \xrightarrow{\mathbb{P}} \mathbb{RP}^{n+2}.$$

In the sequel, for notational convenience, we will denote $\mathbb{P}$ as a map from $\mathbb{R}^{n+1,2}$, implicitly assuming that the origin 0 is removed from any subset of $\mathbb{R}^{n+1,2}$ on which we apply $\mathbb{P}$.

The *double covering* $\widehat{\mathrm{Ein}}^{n,1}$ is defined as the quotient of the nullcone $\mathfrak{N}^{n+1,2}$ by the action by *positive scalar multiplications.* For many purposes the double covering may be more useful than $\mathrm{Ein}^{n,1}$, itself. We will also consider the universal covering $\widetilde{\mathrm{Ein}}^{n,1}$ in §4.

Writing the bilinear form on $\mathbb{R}^{n+1,2}$ as $\mathrm{I}_{n+1} \oplus -\mathrm{I}_2$, that is,

$$\langle v, v\rangle = v_1^2 + \cdots + v_{n+1}^2 - v_{n+2}^2 - v_{n+3}^2,$$

the nullcone is defined by

$$v_1^2 + \cdots + v_{n+1}^2 = v_{n+2}^2 + v_{n+3}^2.$$

This common value is always nonnegative, and if it is zero, then $v = 0$ and v does not correspond to a point in $\mathrm{Ein}^{n,1}$. Dividing by the positive number $\sqrt{v_{n+2}^2 + v_{n+3}^2}$ we may assume that

$$v_1^2 + \cdots + v_{n+1}^2 = v_{n+2}^2 + v_{n+3}^2 = 1$$

which describes the product $S^n \times S^1$. Thus

$$\widehat{\mathrm{Ein}}^{n,1} \approx S^n \times S^1.$$

Scalar multiplication by -1 acts by the antipodal map on both the S^n and the S^1-factor. On the S^1-factor the antipodal map is a translation of order two, so the quotient

$$\mathrm{Ein}^{n,1} = \widehat{\mathrm{Ein}}^{n,1}/\{\pm 1\}$$

is homeomorphic to the mapping torus of the antipodal map on S^n. When n is even, $\mathrm{Ein}^{n,1}$ is nonorientable and $\widehat{\mathrm{Ein}}^{n,1}$ is an orientable double covering. If n is odd, then $\mathrm{Ein}^{n,1}$ is orientable.

The objects in the synthetic geometry of $\mathrm{Ein}^{n,1}$ are the following collections of points in $\mathrm{Ein}^{n,1}$:

- *Photons* are projectivizations of totally isotropic 2-planes. We denote the space of photons by $\mathrm{Pho}^{n,1}$. A photon enjoys the natural structure of a *real projective line*: each photon $\phi \in \mathrm{Pho}^{n,1}$ admits *projective parametrizations*, which are diffeomorphisms of ϕ with $\mathbb{RP}^1$ such that if g is an automorphism of $\mathrm{Ein}^{n,1}$ preserving ϕ, then $g|_\phi$ corresponds to a projective transformation of $\mathbb{RP}^1$. The projective parametrizations are unique up to post-composition with transformations in $\mathrm{PGL}(2, \mathbb{R})$.

- *Lightcones* are singular hypersurfaces. Given any point $p \in \mathrm{Ein}^{n,1}$, the *lightcone $L(p)$ with vertex p* is the union of all photons containing p:

$$L(p) := \bigcup \{\phi \in \mathrm{Pho}^{n,1} \mid p \in \phi\}.$$

  The lightcone $L(p)$ can be equivalently defined as the projectivization of the orthogonal complement $p^\perp \cap \mathfrak{N}^{n+1,2}$. The only singular point on $L(p)$ is p, and $L(p) \setminus \{p\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

- The *Minkowski patch* $\mathrm{Min}(p)$ determined by an element p of $\mathrm{Ein}^{n,1}$ is the complement of $L(p)$ and has the natural structure of Minkowski space $\mathbb{E}^{n,1}$, as will be explained in §3 below. In the double cover, a point $\hat{p}$ determines *two* Minkowski patches:

$$\mathrm{Min}^+(\hat{p}) := \{\hat{q} \in \widehat{\mathrm{Ein}}^{n,1} \mid \langle p, q\rangle > 0 \ \forall p, q \in \mathbb{R}^{n+1,2} \text{ representing } \hat{p}, \hat{q}\}$$
$$\mathrm{Min}^-(\hat{p}) := \{\hat{q} \in \widehat{\mathrm{Ein}}^{n,1} \mid \langle p, q\rangle < 0 \ \forall p, q \in \mathbb{R}^{n+1,2} \text{ representing } \hat{p}, \hat{q}\}.$$

- There are two different types of *hyperspheres*.
  - *Einstein hyperspheres* are closures in $\mathrm{Ein}^{n,1}$ of de Sitter hyperspheres $S_r(p)$ in Minkowski patches as defined in §2.2. Equivalently, they are projectivizations of $v^\perp \cap \mathfrak{N}^{n+1,2}$ for spacelike vectors v.
  - *Spacelike hyperspheres* are one-point compactifications of spacelike hyperplanes like $\mathbb{R}^n$ in a Minkowski patch $\mathbb{R}^{n,1} \subset \mathrm{Ein}^{n,1}$. Equivalently, they are projectivizations of $v^\perp \cap \mathfrak{N}^{n+1,2}$ for timelike vectors v.
- An *anti-de Sitter space* $\mathrm{AdS}^{n,1}$ is one component of the complement of an Einstein hypersphere $\mathrm{Ein}^{n-1,1} \subset \mathrm{Ein}^{n,1}$. It is homeomorphic to $S^1 \times \mathbb{R}^n$. Its *ideal boundary* is $\mathrm{Ein}^{n-1,1}$.

2.4. 2-dimensional case. Because of its special significance, we discuss in detail the geometry of the 2-dimensional Einstein universe $\mathrm{Ein}^{1,1}$.

- $\mathrm{Ein}^{1,1}$ is diffeomorphic to a 2-torus.
- Each lightcone $L(p)$ consists of two photons which intersect at p.
- $\mathrm{Ein}^{1,1}$ has two foliations F_- and F_+ by photons, and the lightcone $L(p)$ is the union of the leaves through p of the respective foliations.
- The leaf space of each foliation naturally identifies with $\mathbb{RP}^1$, and the mapping
$$\mathrm{Ein}^{1,1} \longrightarrow \mathbb{RP}^1 \times \mathbb{RP}^1$$
is equivariant with respect to the isomorphism
$$\mathrm{O}(2,2) \xrightarrow{\cong} \mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{R}).$$

Here is a useful model (compare Pratoussevitch [26]): The space $\mathrm{Mat}_2(\mathbb{R})$ of 2×2 real matrices with the bilinear form associated to the determinant gives an isomorphism of inner product spaces:

$$\mathrm{Mat}_2(\mathbb{R}) \longrightarrow \mathbb{R}^{2,2}$$
$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \longmapsto \begin{bmatrix} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{bmatrix}$$

where $\mathbb{R}^{2,2}$ is given the bilinear form defined by

$$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The group $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$ acts on $\mathrm{Mat}_2(\mathbb{R})$ by:

$$X \xrightarrow{(A,B)} AXB^{-1}$$

and induces a local isomorphism

$$\mathrm{SL}_{\pm}(2,\mathbb{R}) \times \mathrm{SL}_{\pm}(2,\mathbb{R}) \longrightarrow \mathrm{O}(2,2)$$

where

$$\mathrm{SL}_{\pm}(2,\mathbb{R}) := \{A \in \mathrm{GL}(2,\mathbb{R}) \mid \det(A) = \pm 1\}.$$

Here we will briefly introduce *stems*, which are pieces of crooked planes, as will be discussed in §8 below. Let $p_0, p_\infty \in \mathrm{Ein}^{1,1}$ be two points not contained in a common photon. Their lightcones intersect in two points p_1 and p_2, and the union

$$L(p_0) \cup L(p_\infty) \subset \mathrm{Ein}^{1,1}$$

comprises four photons intersecting in the four points p_0, p_∞, p_1, p_2, such that each point lies on two photons and each photon contains two of these points. This *stem configuration* of four points and four photons can be represented schematically as in Figure 1 below.

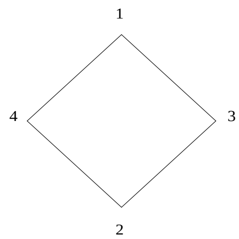

Figure 1. Stem Configuration

The complement

$$\mathrm{Ein}^{1,1} \setminus \big(L(p_0) \cup L(p_\infty)\big)$$

consists of four quadrilateral regions (see Figure 2). In §8 the union S of two non-adjacent quadrilateral regions will be studied; this is the *stem* of a crooked surface. Such a set is bounded by the four photons of $L(p_0) \cup L(p_\infty)$.

2.5. 3-dimensional case.

Here we present several observations particular to the case of $\mathrm{Ein}^{2,1}$.

- We will see that $\mathrm{Pho}^{2,1}$ identifies naturally with a 3-dimensional real projective space (§5.5).
- A lightcone in $\mathrm{Ein}^{2,1}$ is homeomorphic to a pinched torus.

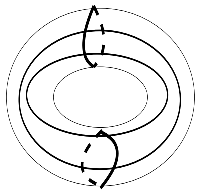

Figure 2. Two lightcones in $\mathrm{Ein}^{1,1}$

- Suppose $p \neq q$. Define

$$C(p,q) := L(p) \cap L(q).$$

  If p and q are *incident,*—that is, they lie on a common photon—then $C(p,q)$ is the unique photon containing them. Otherwise $C(p,q)$ is a submanifold that we will call a *spacelike circle.* Spacelike circles are projectivized nullcones of linear subspaces of $\mathbb{R}^{3,2}$ of type $(2,1)$. The closure of a spacelike geodesic in $\mathbb{E}^{2,1}$ is a spacelike circle.

- A *timelike circle* is the projectivized nullcone of a linear subspace of $\mathbb{R}^{3,2}$ of metric type $(1,2)$.

- Einstein hyperspheres in $\mathrm{Ein}^{2,1}$ are copies of $\mathrm{Ein}^{1,1}$. In addition to their two rulings by photons, they have a foliation by spacelike circles.

- Lightcones may intersect Einstein hyperspheres in two different ways. These correspond to intersections of degenerate linear hyperplanes in $\mathbb{R}^{3,2}$ with linear hyperplanes of type $(2,2)$. Let $u, v \in \mathbb{R}^{3,2}$ be vectors such that $u^\perp$ is degenerate, so u determines a lightcone L, and $v^\perp$ has type $(2,2)$, so v defines the Einstein hypersphere H. In terms of inner products,

$$\langle u,u\rangle = 0,\ \langle v,v\rangle > 0.$$

  If $\langle u,v\rangle \neq 0$, then u, v span a nondegenerate subspace of signature $(1,1)$. In that case $L \cap H$ is a spacelike circle. If $\langle u,v\rangle = 0$, then u, v span a degenerate subspace and the intersection is a lightcone in H, which is a union of two distinct but incident photons.

- Similarly, lightcones intersect spacelike hyperspheres in two different ways. The generic intersection is a spacelike circle, and the non-generic intersection is a single point, such as the intersection of $L(0)$ with the spacelike plane $z = 0$ in $\mathbb{R}^{2,1}$.

- A *pointed photon* is a pair $(p,\phi) \in \mathrm{Ein}^{2,1} \times \mathrm{Pho}^{2,1}$ such that $p \in \phi$. Such a pair naturally extends to a triple

$$p \in \phi \subset L(p)$$

which corresponds to an *isotropic flag,* that is, a linear filtration of $\mathbb{R}^{3,2}$

$$0 \subset \ell_p \subset P_\phi \subset (\ell_p)^\perp \subset \mathbb{R}^{3,2},$$

where ℓ_p is the 1-dimensional linear subspace corresponding to p; P_ϕ is the 2-dimensional isotropic subspace corresponding to ϕ; and $(\ell_p)^\perp$ is the orthogonal subspace of ℓ_p. These objects form a homogeneous space, an incidence variety, denoted $\mathrm{Flag}^{2,1}$, of $\mathrm{O}(3,2)$, which fibers both over $\mathrm{Ein}^{2,1}$ and $\mathrm{Pho}^{2,1}$. The fiber of the fibration $\mathrm{Flag}^{2,1} \longrightarrow \mathrm{Ein}^{2,1}$ over a point p is the collection of all photons through p. The fiber of the fibration $\mathrm{Flag}^{2,1} \longrightarrow \mathrm{Pho}^{2,1}$ over a photon ϕ identifies with all the points of ϕ. Both fibrations are circle bundles.

3. $\mathrm{Ein}^{n,1}$ as the conformal compactification of $\mathbb{E}^{n,1}$

Now we shall describe the geometry of $\mathrm{Ein}^{n,1}$ as the compactification of Minkowski space $\mathbb{E}^{n,1}$. We begin with the Euclidean analog.

3.1. The conformal Riemannian sphere. The standard conformal compactification of Euclidean space $\mathbb{E}^n$ is topologically the one-point compactification, the n-dimensional sphere. The *conformal Riemannian sphere* S^n is the projectivization $\mathbb{P}(\mathfrak{N}^{n+1,1})$ of the nullcone of $\mathbb{R}^{n+1,1}$.

For $U \subset S^n$ an arbitrary open set, any local section

$$U \xrightarrow{\sigma} \mathbb{R}^{n+1,1} \setminus \{0\}$$

of the restriction of the projectivization map to U determines a pullback of the Lorentz metric on $\mathbb{E}^{n+1,1}$ to a Riemannian metric g_σ on U. This metric depends on σ, but its conformal class is independent of σ. Every section is $\sigma' = f\sigma$ for some non-vanishing function $f : U \to \mathbb{R}$. Then

$$g_{\sigma'} = f^2 g_\sigma$$

so the pullbacks are conformally equivalent. Hence the metrics g_σ altogether define a canonical conformal structure on S^n.

The orthogonal group $\mathrm{O}(n+1,1)$ leaves invariant the nullcone $\mathfrak{N}^{n+1,1} \subset \mathbb{R}^{n+1,1}$. The projectivization

$$S^n = \mathbb{P}(\mathfrak{N}^{n+1,1})$$

is invariant under the *projective orthogonal group* $\mathrm{PO}(n+1,1)$, which is its conformal automorphism group.

Let

$$S^n \xrightarrow{\sigma} \mathfrak{N}^{n+1,1} \subseteq \mathbb{R}^{n+1,1} \setminus \{0\}$$

be the section taking values in the unit Euclidean sphere. Then the metric g_σ is the usual $\mathrm{O}(n+1)$-invariant spherical metric.

Euclidean space $\mathbb{E}^n$ embeds in S^n via a *spherical paraboloid* in the nullcone $\mathfrak{N}^{n+1,1}$. Namely consider the quadratic form on $\mathbb{R}^{n+1,1}$ defined by

$$\mathrm{I}_n \oplus -1/2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{I}_n & & \\ & 0 & -1/2 \\ & -1/2 & 0 \end{bmatrix}.$$

The map

$$\begin{aligned} \mathbb{E}^n &\to \mathfrak{N}^{n+1,1} \subset \mathbb{R}^{n+1,1} \\ x &\longmapsto \begin{bmatrix} x \\ \langle x, x \rangle \\ 1 \end{bmatrix} \end{aligned} \tag{1}$$

composed with projection $\mathfrak{N}^{n+1,1} \xrightarrow{\mathbb{P}} S^n$ is an embedding $\mathcal{E}$ of $\mathbb{E}^n$ into S^n, which is *conformal.*

The Euclidean similarity transformation

$$f_{r,A,b} : x \longmapsto rAx + b$$

where $r \in \mathbb{R}_+$, $A \in \mathrm{O}(n)$, and $b \in \mathbb{R}^n$, is represented by

$$F_{r,A,b} := \begin{bmatrix} \mathrm{I}_n & 0 & b \\ 2b^\dagger & 1 & \langle b, b \rangle \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-1} \end{bmatrix} \in \mathrm{O}(n+1,1). \tag{2}$$

That is, for every $x \in \mathbb{E}^n$,

$$F_{r,A,b}\mathcal{E}(x) = \mathcal{E}\big(f_{r,A,b}(x)\big).$$

Inversion in the unit sphere $\langle v, v \rangle = 1$ of $\mathbb{E}^n$ is represented by the element

$$\mathrm{I}_n \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which acts on $\mathbb{E}^n \setminus \{0\}$ by:

$$\iota : x \mapsto \frac{1}{\langle x, x \rangle} x.$$

The origin is mapped to the point (called ∞) having homogeneous coordinates

$$\begin{bmatrix} 0_n \\ 1 \\ 0 \end{bmatrix}$$

where $0_n \in \mathbb{R}^n$ is the zero vector.

The map $\mathcal{E}^{-1}$ is a coordinate chart on the open set

$$\mathbb{E}^n = S^n \setminus \{\infty\}$$

and $\mathcal{E}^{-1} \circ \iota$ is a coordinate chart on the open set $(\mathbb{E}^n \cup \{\infty\}) \setminus \{0\} = S^n \setminus \{0\}$.

3.2. The conformal Lorentzian quadric. Consider now the inner product space $\mathbb{R}^{n+1,2}$. Here it will be convenient to use the inner product

$$\begin{aligned}\langle u, v\rangle &:= u_1v_1 + \ldots + u_nv_n - u_{n+1}v_{n+1} - \frac{1}{2}u_{n+2}v_{n+3} - \frac{1}{2}u_{n+3}v_{n+2} \\ &= u^\dagger \left(\mathrm{I}_n \oplus - \mathrm{I}_1 \oplus - 1/2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) v.\end{aligned}$$

In analogy with the Riemannian case, consider the embedding $\mathcal{E} : \mathbb{E}^{n,1} \to \mathrm{Ein}^{n,1}$ via a *hyperbolic paraboloid* defined by (1) as above, where the Lorentzian inner product on $\mathbb{E}^{n,1}$ is defined by $Q = \mathrm{I}_n \oplus - \mathrm{I}_1$. The procedure used previously in the Riemannian case naturally defines an $\mathrm{O}(n+1,2)$-invariant conformal Lorentzian structure on $\mathrm{Ein}^{n,1}$, and the embedding we have just defined is conformal.

Minkowski similarities $f_{r,A,b}$ map into $\mathrm{O}(n+1,2)$ as in the formula (2), where $r \in \mathbb{R}_+; A \in \mathrm{O}(n+1,1); b \in \mathbb{R}^{n,1}$; $\langle , \rangle$ is the Lorentzian inner product on $\mathbb{R}^{n,1}$; and $2b^\dagger$ is replaced by $2b^\dagger Q$.

The conformal compactification of Euclidean space is the one-point compactification; the compactification of Minkowski space, however, is more complicated, requiring the addition of more than a single point. Let $p_0 \in \mathrm{Ein}^{n,1}$ denote the *origin,* corresponding to

$$\begin{bmatrix} 0_{n+1} \\ 0 \\ 1 \end{bmatrix}.$$

To see what lies *at infinity,* consider the *Lorentzian inversion in the unit sphere* defined by the matrix $\mathrm{I}_{n+1} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is given on $\mathbb{E}^{n,1}$ by the formula

$$\iota : x \longmapsto \frac{1}{\langle x, x\rangle} x. \tag{3}$$

Here the whole affine lightcone $L^{\mathrm{aff}}(p_0)$ is thrown to infinity. We distinguish the points on $\iota(L^{\mathrm{aff}}(p_0))$:

- The *improper point* p_∞ is the image $\iota(p_0)$. It is represented in homogeneous coordinates by

$$\begin{bmatrix} 0_{n+1} \\ 1 \\ 0 \end{bmatrix}.$$

- The *generic point* on $\iota\big(L^{\mathrm{aff}}(p_0)\big)$ has homogeneous coordinates

$$\begin{bmatrix} v \\ 1 \\ 0 \end{bmatrix}$$

where $0 \neq v \in \mathbb{R}^{n,1}$; it equals $\iota\big(\mathcal{E}(v)\big)$.

We have described all the points in

$$\mathbb{E}^{n,1} \cup \iota(\mathbb{E}^{n,1})$$

which are the points defined by vectors $v \in \mathbb{R}^{n+1,2}$ with coordinates $v_{n+2} \neq 0$ or $v_{n+3} \neq 0$. It remains to consider points having homogeneous coordinates

$$\begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$$

where necessarily $\langle v, v\rangle = 0$. This equation describes the nullcone in $\mathbb{R}^{n,1}$; its projectivization is a spacelike sphere S_∞, which we call the *ideal sphere.* When $n = 2$, we call this the *ideal circle* and its elements *ideal points.* Each ideal point is the endpoint of a unique null geodesic from the origin; the union of that null geodesic with the ideal point is a photon through the origin. Every photon through the origin arises in this way. The ideal sphere is fixed by the inversion ι.

The union of the ideal sphere S_∞ with $\iota(L^{\mathrm{aff}}(p_0))$ is the lightcone $L(p_\infty)$ of the improper point. Photons in $L(p_\infty)$ are called *ideal photons.* Minkowski space $\mathbb{E}^{n,1}$ is thus the complement of a lightcone $L(p_\infty)$ in $\mathrm{Ein}^{n,1}$. This fact motivated the earlier definition of a Minkowski patch $\mathrm{Min}(p)$ as the complement in $\mathrm{Ein}^{n,1}$ of a lightcone $L(p)$.

Changing a Lorentzian metric by a non-constant scalar factor modifies timelike and spacelike geodesics, but not images of null geodesics (see for example [3], p. 307). Hence the notion of (non-parametrized) null geodesic is well-defined in a conformal Lorentzian manifold. For $\mathrm{Ein}^{n,1}$, the null geodesics are photons.

3.3. Involutions. When n is even, involutions in $\mathrm{SO}(n+1,2) \cong \mathrm{PO}(n+1,2)$ correspond to nondegenerate splittings of $\mathbb{R}^{n+1,2}$. For any involution in $\mathrm{PO}(3,2)$, the fixed point set in $\mathrm{Ein}^{2,1}$ must be one of the following:

- the empty set $\emptyset$;
- a spacelike hypersphere;
- a timelike circle;
- the union of a spacelike circle with two points;
- an Einstein hypersphere.

In the case that $\mathrm{Fix}(f)$ is disconnected and equals

$$\{p_1, p_2\} \cup S$$

where $p_1, p_2 \in \mathrm{Ein}^{2,1}$, and $S \subset \mathrm{Ein}^{2,1}$ is a spacelike circle, then

$$S = L(p_1) \cap L(p_2).$$

Conversely, given any two non-incident points p_1, p_2, there is a unique involution fixing p_1, p_2 and the spacelike circle $L(p_1) \cap L(p_2)$.

3.3.1. Inverting photons. Let p_∞ be the improper point, as above. A photon in $\mathrm{Ein}^{2,1}$ either lies on the ideal lightcone $L(p_\infty)$, or it intersects the spacelike plane S_0 consisting of all

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for which $z = 0$. Suppose ϕ is a photon intersecting S_0 in the point p_0 with polar coordinates

$$p_0 = \begin{bmatrix} r_0 \cos(\psi) \\ r_0 \sin(\psi) \\ 0 \end{bmatrix} \in S_0 \subset \mathbb{E}^{2,1}.$$

Let v_0 be the null vector

$$v_0 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 1 \end{bmatrix}$$

and consider the parametrized lightlike geodesic

$$\phi(t) := p_0 + t v_0$$

for $t \in \mathbb{R}$. Then inversion ι maps $\phi(t)$ to

$$(\iota \circ \phi)(t) = \iota(p_0) + \tilde{t} \begin{bmatrix} -\cos(\theta - 2\psi) \\ \sin(\theta - 2\psi) \\ 1 \end{bmatrix}$$

where

$$\tilde{t} := \frac{t}{r_0^2 + 2r_0 \cos(\theta - \psi) t} \ .$$

Observe that ι leaves invariant the spacelike plane S_0 and acts by Euclidean inversion on that plane.

3.3.2. Extending planes in $\mathbb{E}^{2,1}$ to $\mathrm{Ein}^{2,1}$.

- The closure of a null plane P in $\mathbb{E}^{2,1}$ is a lightcone and its frontier $\bar{P} \setminus P$ is an ideal photon. Conversely a lightcone with vertex on the ideal circle S_∞ is the closure of a null plane containing p_0, while a lightcone with vertex on

$$L(p_\infty) \ \setminus \ (S_\infty \cup \{p_\infty\})$$

  is the closure of a null plane not containing p_0.
- The closure of a spacelike plane in $\mathbb{E}^{2,1}$ is a spacelike sphere and its frontier is the improper point p_∞.
- The closure of a timelike plane in $\mathbb{E}^{2,1}$ is an Einstein hypersphere and its frontier is a union of two ideal photons (which intersect in p_∞).

- The closure of a timelike (respectively spacelike) geodesic in $\mathbb{E}^{2,1}$ is a timelike (respectively spacelike) circle containing p_∞, and p_∞ is its frontier.

Consider the inversion on the lightcone of p_0:

$$\iota\left(\begin{bmatrix} t\sin\theta \\ t\cos\theta \\ t \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} t\sin\theta \\ t\cos\theta \\ t \\ 1 \\ 0 \end{bmatrix}.$$

The entire image of the light cone $L(p_0)$ lies outside the Minkowski patch $\mathbb{E}^{2,1}$.

Let us now look at the image of a timelike line in $\mathbb{E}^{2,1}$ under the inversion. For example,

$$\iota\left(\begin{bmatrix} 0 \\ 0 \\ t \\ -t^2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ t \\ 1 \\ -t^2 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 0 \\ -1/t \\ -1/t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s \\ -s^2 \\ 1 \end{bmatrix}$$

where $s = -1/t$. That is, the inversion maps the timelike line minus the origin to itself, albeit with a change in the parametrization.

4. Causal geometry

In §3.2 we observed that $\mathrm{Ein}^{n,1}$ is naturally equipped with a conformal structure. This structure lifts to the double cover $\widehat{\mathrm{Ein}}^{n,1}$. As in the Riemannian case in §3.1, a global representative of the conformal structure on $\widehat{\mathrm{Ein}}^{n,1}$ is the pullback by a global section $\sigma : \widehat{\mathrm{Ein}}^{n,1} \to \mathbb{R}^{n+1,2}$ of the ambient quadratic form of $\mathbb{R}^{n+1,2}$. The section $\sigma : \widehat{\mathrm{Ein}}^{n,1} \to \mathbb{R}^{n+1,2}$ taking values in the set where

$$v_1^2 + \cdots + v_{n+1}^2 = v_{n+2}^2 + v_{n+3}^2 = 1$$

exhibits a homeomorphism $\widehat{\mathrm{Ein}}^{n,1} \cong S^n \times S^1$ as in §2.3; it is now apparent that $\widehat{\mathrm{Ein}}^{n,1}$ is conformally equivalent to $S^n \times S^1$ endowed with the Lorentz metric $ds_0^2 - d\theta^2$, where ds_0^2 and $d\theta^2$ are the usual round metrics on the spheres S^n and S^1 of radius one.

In the following, elements of $S^n \times S^1$ are denoted by (φ, θ). In these coordinates, we distinguish the timelike vector field $\eta = \partial_\theta$ tangent to the fibers $\{*\} \times S^1$.

4.1. Time orientation. First consider Minkowski space $\mathbb{E}^{n,1}$ with underlying vector space $\mathbb{R}^{n,1}$ equipped with the inner product:

$$\langle u, v\rangle := u_1v_1 + \cdots + u_nv_n - u_{n+1}v_{n+1}.$$

A vector u in $\mathbb{R}^{n,1}$ is *causal* if $u_{n+1}^2 \geq u_1^2 + \ldots + u_n^2$. It is *future-oriented* (respectively *past-oriented*) if the coordinate u_{n+1} is positive (respectively negative); equivalently, u is future-oriented if its inner product with

$$\eta_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

is negative.

The key point is that the choice of the coordinate u_{n+1}—equivalently, of an everywhere timelike vector field like η_0—defines a decomposition of every affine lightcone $L^{\mathrm{aff}}(p)$ in three parts:

- $\{p\}$;
- The future lightcone $L_+^{\mathrm{aff}}(p)$ of elements $p + v$ where v is a future-oriented null vector;
- The past lightcone $L_-^{\mathrm{aff}}(p)$ of elements $p + v$ where v is a past-oriented null vector.

The above choice is equivalent to a continuous choice of one of the connected components of the set of timelike vectors based at each $x \in \mathbb{E}^{n,1}$; timelike vectors in these components are designated future-oriented. In other words, η_0 defines a *time orientation* on $\mathbb{E}^{n,1}$.

To import this notion to $\widehat{\mathrm{Ein}}^{n,1}$, replace η_0 by the vector field η on $\widehat{\mathrm{Ein}}^{n,1}$. Then a causal tangent vector v to $\widehat{\mathrm{Ein}}^{n,1}$ is *future-oriented* (respectively *past-oriented*) if the inner product $\langle v, \eta \rangle$ is negative (respectively positive).

We already observed in §2.3 that the antipodal map is $(\varphi, \theta) \mapsto (-\varphi, -\theta)$ on $S^n \times S^1$; in particular, it preserves the timelike vector field η, which then descends to a well-defined vector field on $\mathrm{Ein}^{n,1}$, so that $\mathrm{Ein}^{n,1}$ is time oriented, for all integers n.

Remark 4.1.1. The Einstein universe does not have a preferred Lorentz metric in its conformal class. The definition above is nonetheless valid since it involves only signs of inner products and hence is independent of the choice of metric in the conformal class.

The group $\mathrm{O}(n+1, 2)$ has four connected components. More precisely, let $\mathrm{SO}(n+1, 2)$ be the subgroup of $\mathrm{O}(n+1, 2)$ formed by elements with determinant 1; these are the orientation-preserving conformal transformations of $\widehat{\mathrm{Ein}}^{n,1}$. Let $\mathrm{O}^+(n+1, 2)$ be the subgroup comprising the elements preserving the time orientation of $\widehat{\mathrm{Ein}}^{n,1}$. The identity component of $\mathrm{O}(n+1, 2)$ is the intersection

$$\mathrm{SO}^+(n+1, 2) = \mathrm{SO}(n+1, 2) \cap \mathrm{O}^+(n+1, 2).$$

Moreover, $\mathrm{SO}(n + 1, 2)$ and $\mathrm{O}^+(n + 1, 2)$ each have two connected components.

The center of $\mathrm{O}(n + 1, 2)$ has order two and is generated by the antipodal map, which belongs to $\mathrm{SO}(n + 1, 2)$ if and only if n is odd. Hence the center of $\mathrm{SO}(n + 1, 2)$ is trivial if n is even—in particular, when $n = 2$. On the other hand, the antipodal map always preserves the time orientation.

The antipodal map is the only element of $\mathrm{O}(n+1, 2)$ acting trivially on $\mathrm{Ein}^{n,1}$. Hence the group of conformal transformations of $\mathrm{Ein}^{n,1}$ is $\mathrm{PO}(n+1, 2)$, the quotient of $\mathrm{O}(n+1, 2)$ by its center. When n is even, $\mathrm{PO}(n+1, 2)$ is isomorphic to $\mathrm{SO}(n+1, 2)$.

4.2. Future and past.

A C^1-immersion

$$[0, 1] \xrightarrow{c} \mathbb{E}^{1,n}$$

is *a causal curve* (respectively *a timelike curve*) if the tangent vectors $c'(t)$ are all causal (respectively timelike). This notion extends to any conformally Lorentzian space—in particular, to $\mathrm{Ein}^{n,1}$, $\widehat{\mathrm{Ein}}^{n,1}$, or $\widetilde{\mathrm{Ein}}^{n,1}$. Furthermore, a causal curve c is future-oriented (respectively past-oriented) if all the tangent vectors $c'(t)$ are future-oriented (respectively past-oriented).

Let A be a subset of $\mathbb{E}^{n,1}$, $\mathrm{Ein}^{n,1}$, $\widehat{\mathrm{Ein}}^{n,1}$, or $\widetilde{\mathrm{Ein}}^{n,1}$. The *future* $\mathrm{I}^+(A)$ (respectively the *past* $\mathrm{I}^-(A)$) of A is the set comprising endpoints $c(1)$ of future-oriented (respectively past-oriented) timelike curves with starting point $c(0)$ in A. The *causal future* $\mathrm{J}^+(A)$ (respectively the *causal past* $\mathrm{J}^-(A)$) of A is the set comprising endpoints $c(1)$ of future-oriented (respectively past-oriented) causal curves with starting point $c(0)$ in A. Two points p, p' are *causally related* if one belongs to the causal future of the other: $p' \in J^{\pm}(p)$. The notion of future and past in $\mathbb{E}^{n,1}$ is quite easy to understand: p' belongs to the future $\mathrm{I}^+(p)$ of p if and only if $p' - p$ is a future-oriented timelike element of $\mathbb{R}^{n,1}$.

Thanks to the conformal model, these notions are also quite easy to understand in $\mathrm{Ein}^{n,1}$, $\widehat{\mathrm{Ein}}^{n,1}$, or $\widetilde{\mathrm{Ein}}^{n,1}$: let d_n be the spherical distance on the homogeneous Riemannian sphere S^n of radius 1. The universal covering $\widetilde{\mathrm{Ein}}^{n,1}$ is conformally isometric to the Riemannian product $S^n \times \mathbb{R}$ where the real line $\mathbb{R}$ is endowed with the negative quadratic form $-d\theta^2$. Hence, the image of any causal, C^1, immersed curve in $\widetilde{\mathrm{Ein}}^{n,1} \approx S^n \times \mathbb{R}$ is the graph of a map $f : I \to S^n$ where I is an interval in $\mathbb{R}$ and where f is 1-*Lipschitz* —that is, for all θ, θ' in $\mathbb{R}$:

$$d_n(f(\theta), f(\theta')) \leq |\theta - \theta'|.$$

Moreover, the causal curve is timelike if and only if the map f is *contracting*—that is, satisfies

$$d_n(f(\theta), f(\theta')) < |\theta - \theta'|.$$

It follows that the future of an element (φ_0, θ_0) of $\widetilde{\mathrm{Ein}}^{n,1} \approx S^n \times \mathbb{R}$ is:

$$\mathrm{I}^+(\varphi_0, \theta_0) = \{(\varphi, \theta) \ \mid \ \theta - \theta_0 > d_n(\varphi, \varphi_0)\}$$

and the causal future $\mathrm{J}^+(p)$ of an element p of $\widetilde{\mathrm{Ein}}^{n,1}$ is the closure of the future $\mathrm{I}^+(p)$:

$$\mathrm{J}^+(\varphi_0, \theta_0) = \{(\varphi, \theta) \ \mid \ \theta - \theta_0 \geq d_n(\varphi, \varphi_0)\}.$$

As a corollary, the future $\mathrm{I}^+(A)$ of a nonempty subset A of $\mathrm{Ein}^{n,1}$ or $\widehat{\mathrm{Ein}}^{n,1}$ is the entire spacetime. In other words, the notion of past or future is relevant in $\widetilde{\mathrm{Ein}}^{n,1}$, but not in $\mathrm{Ein}^{n,1}$ or $\widehat{\mathrm{Ein}}^{n,1}$.

There is, however, a relative notion of past and future still relevant in $\widehat{\mathrm{Ein}}^{n,1}$ that will be useful later when considering crooked planes and surfaces: let $\hat{p}$, $\hat{p}'$ be two elements of $\widehat{\mathrm{Ein}}^{n,1}$ such that $\hat{p}' \neq \pm\hat{p}$. First observe that the intersection $\mathrm{Min}^+(\hat{p}) \cap \mathrm{Min}^+(\hat{p}')$ is never empty. Let p_∞ be any element of this intersection, so $\mathrm{Min}^+(\hat{p}_\infty)$ contains $\hat{p}$ and $\hat{p}'$. The time orientation on $\widehat{\mathrm{Ein}}^{n,1}$ induces a time orientation on such a Minkowski patch $\mathrm{Min}^+(\hat{p}_\infty)$.

Fact 4.2.1. *The points $\hat{p}'$ and $\hat{p}$ are causally related in $\mathrm{Min}^+(\hat{p}_\infty)$ if and only if, for any lifts p, p' of $\hat{p}, \hat{p}'$, respectively, to $\mathbb{R}^{n+1,2}$, the inner product $\langle p, p'\rangle$ is positive.*

Hence, if $\hat{p}$ and $\hat{p}'$ are causally related in some Minkowski patch, then they are causally related in any Minkowski patch containing both of them. Therefore, (slightly abusing language) we use the following convention: two elements $\hat{p}$, $\hat{p}'$ of $\widehat{\mathrm{Ein}}^{n,1}$ are *causally related* if the inner product $\langle p, p'\rangle$ in $\mathbb{R}^{n+1,2}$ is positive for any lifts p, p'.

4.3. Geometry of the universal covering. The geometrical understanding of the embedding of Minkowski space in the Einstein universe can be a challenge. In particular, the closure in $\mathrm{Ein}^{n,1}$ of a subset of a Minkowski patch may be not obvious, as we will see for crooked planes. This difficulty arises from the nontrivial topology of $\mathrm{Ein}^{n,1}$.

On the other hand, the topology of the universal covering $\widetilde{\mathrm{Ein}}^{n,1}$ is easy to visualize; indeed, the map

$$\widetilde{\mathrm{Ein}}^{n,1} \approx S^n \times \mathbb{R} \xrightarrow{\mathcal{S}} \mathbb{R}^{n+1} \setminus \{0\}$$
$$\mathcal{S} : (\varphi, \theta) \longmapsto \exp(\theta)\varphi$$

is an embedding. Therefore, $\widetilde{\mathrm{Ein}}^{n,1}$ can be considered as a subset of $\mathbb{R}^{n+1}$—one that is particularly easy to visualize when $n = 2$. Observe that the map $\mathcal{S}$ is $\mathrm{O}(n+1)$-equivariant for the natural actions on $\widetilde{\mathrm{Ein}}^{n,1}$ and $\mathbb{R}^{n+1}$.

The antipodal map

$$(\varphi, \theta) \longmapsto (-\varphi, -\theta)$$

lifts to the automorphism α of

$$\widetilde{\mathrm{Ein}}^{n,1} \approx S^n \times \mathbb{R},$$

defined by

$$(\varphi, \theta) \overset{\alpha}{\longmapsto} (-\varphi, \theta + \pi).$$

In the coordinates $\widetilde{\mathrm{Ein}}^{n,1} \approx \mathbb{R}^{n+1} \setminus \{0\}$ this lifting α is expressed by $x \to -\lambda x$, where $\lambda = \exp(\pi)$.

Since null geodesics in $\mathrm{Ein}^{n,1}$ are photons, the images by $\mathcal{S}$ of null geodesics of $\widetilde{\mathrm{Ein}}^{n,1}$ are curves in $\mathbb{R}^{n+1} \setminus \{0\}$ characterized by the following properties:

- They are contained in 2-dimensional linear subspaces;
- Each is a logarithmic spiral in the 2-plane containing it.

Hence, for $n = 2$, the lightcone of an element p of $\widetilde{\mathrm{Ein}}^{2,1}$ (that is, the union of the null geodesics containing p) is a singular surface of revolution in $\mathbb{R}^3$ obtained by rotating a spiral contained in a vertical 2-plane around an axis of the plane. In particular, for every x in $\widetilde{\mathrm{Ein}}^{n,1} \approx \mathbb{R}^{n+1} \setminus \{0\}$, every null geodesic containing x contains $\alpha(x) = -\lambda x$. The image $\alpha(x) = -\lambda x$ is uniquely characterized by the following properties, so that it can be called the *first future-conjugate point* to x:

- It belongs to the causal future $J^+(x)$;
- For any $y \in J^+(x)$ such that y belongs to all null geodesics containing x, we have $\alpha(x) \in J^-(y)$.

All these considerations allow us to visualize how Minkowski patches embed in $\mathbb{R}^{n+1} \setminus \{0\}$ (see Figure 3): let $\tilde{p} \in \widetilde{\mathrm{Ein}}^{n,1}$ and $\hat{p}$ be its projection to $\widehat{\mathrm{Ein}}^{n,1}$. The Minkowski patch $\mathrm{Min}^+(\hat{p})$ is the projection in $\widehat{\mathrm{Ein}}^{n,1}$ of $I^+(\tilde{p}) \setminus J^+(\alpha(\tilde{p}))$, which can also be defined as $I^+(\tilde{p}) \cap I^-(\alpha^2(\tilde{p}))$. The projection in $\widehat{\mathrm{Ein}}^{n,1}$ of

$$\widetilde{\mathrm{Ein}}^{n,1} \setminus (J^+(\tilde{p}) \cup J^-(\tilde{p}))$$

is the Minkowski patch $\mathrm{Min}^-(\hat{p})$, which is the set of points non-causally related to $\hat{p}$.

4.4. Improper points of Minkowski patches.

We previously defined the improper point p_∞ associated to a Minkowski patch in $\mathrm{Ein}^{n,1}$: it is the unique point such that the Minkowski patch is $\mathrm{Min}(p_\infty)$.

In the double-covering $\widehat{\mathrm{Ein}}^{n,1}$, to every Minkowski patch are attached two *improper points*:

- the *spatial improper point*, the unique element p_∞^{sp} such that the given Minkowski patch is $\mathrm{Min}^-(p_\infty^{\mathrm{sp}})$;
- the *timelike improper point*, the unique element p_∞^{ti} such that the given Minkowski patch is $\mathrm{Min}^+(p_\infty^{\mathrm{ti}})$.

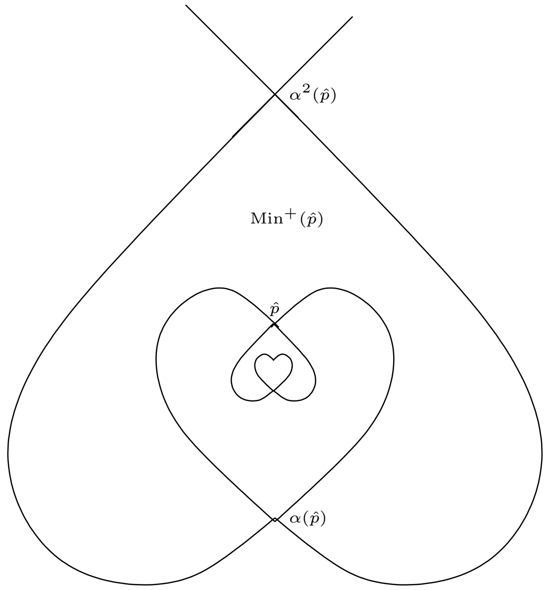

Figure 3. *A Minkowski patch in* $\widetilde{\mathrm{Ein}}^{1,1}$

Let $\mathrm{Min}^+(p_\infty^{\mathrm{ti}}) = \mathrm{Min}^-(p_\infty^{\mathrm{sp}})$ be a Minkowski patch in $\widehat{\mathrm{Ein}}^{n,1}$. Let

$$\mathbb{R} \xrightarrow{\gamma} \mathrm{Min}^+(p_\infty^{\mathrm{ti}}) \approx \mathbb{E}^{n,1}$$

be a geodesic. Denote by Γ the image of γ, and by $\bar{\Gamma}$ the closure in $\widehat{\mathrm{Ein}}^{n,1}$ of Γ.

- If γ is spacelike, then
$$\bar{\Gamma} = \Gamma \cup \{p_\infty^{\mathrm{sp}}\}.$$
- If γ is timelike, then
$$\bar{\Gamma} = \Gamma \cup \{p_\infty^{\mathrm{ti}}\}.$$
- If γ is lightlike, then $\bar{\Gamma}$ is a photon avoiding p_∞^{sp} and p_∞^{ti}.

5. Four-dimensional real symplectic vector spaces

In spatial dimension $n = 2$, Einstein space $\mathrm{Ein}^{2,1}$ admits an alternate description as the *Lagrangian Grassmannian,* the manifold $\mathrm{Lag}(V)$ of Lagrangian 2-planes in a real symplectic vector space V of dimension 4. There results a kind of duality between the conformal Lorentzian geometry of $\mathrm{Ein}^{2,1}$ and the symplectic geometry of $\mathbb{R}^4$. Photons correspond to linear pencils of Lagrangian 2-planes (that is, families of Lagrangian subspaces passing through a given line). The corresponding local isomorphism

$$\mathrm{Sp}(4,\mathbb{R}) \longrightarrow \mathrm{O}(3,2)$$

manifests the isomorphism of root systems of type B_2 (the odd-dimensional orthogonal Lie algebras) and C_2 (the symplectic Lie algebras) of rank 2. We present this correspondence below.

5.1. The inner product on the second exterior power. Begin with a four-dimensional vector space V over $\mathbb{R}$ and choose a fixed generator

$$\mathrm{vol} \in \Lambda^4(V).$$

The group of automorphisms of (V, vol) is the special linear group $\mathrm{SL}(V)$.

The second exterior power $\Lambda^2(V)$ has dimension 6. The action of $\mathrm{SL}(V)$ on V induces an action on $\Lambda^2(V)$ which preserves the bilinear form

$$\Lambda^2(V) \times \Lambda^2(V) \xrightarrow{\mathbb{B}} \mathbb{R}$$

defined by:

$$\alpha_1 \wedge \alpha_2 = -\mathbb{B}(\alpha_1, \alpha_2)\,\mathrm{vol}\,.$$

This bilinear form satisfies the following properties:

- $\mathbb{B}$ is symmetric;
- $\mathbb{B}$ is nondegenerate;
- $\mathbb{B}$ is split—that is, of type $(3,3)$.

(That $\mathbb{B}$ is split follows from the fact that any orientation-reversing linear automorphism of V maps $\mathbb{B}$ to its negative.)

The resulting homomorphism

$$\mathrm{SL}(4,\mathbb{R}) \longrightarrow \mathrm{SO}(3,3) \tag{4}$$

is a local isomorphism of Lie groups, with kernel $\{\pm \mathrm{I}\}$ and image the identity component of $\mathrm{SO}(3,3)$.

Consider a symplectic form ω on V—that is, a skew-symmetric nondegenerate bilinear form on V. Since $\mathbb{B}$ is nondegenerate, ω defines a dual exterior bivector $\omega^* \in \Lambda^2(V)$ by

$$\omega(v_1, v_2) = \mathbb{B}(v_1 \wedge v_2, \omega^*).$$

We will assume that

$$\omega^* \wedge \omega^* = 2\,\mathrm{vol}\,. \tag{5}$$

Thus $\mathbb{B}(\omega^*, \omega^*) = -2 < 0$, so that its symplectic complement

$$W_0 := (\omega^*)^\perp \subset \Lambda^2(V)$$

is an inner product space of type $(3,2)$. Now the local isomorphism (4) restricts to a local isomorphism

$$\mathrm{Sp}(4,\mathbb{R}) \longrightarrow \mathrm{SO}(3,2) \tag{6}$$

with kernel $\{\pm \mathrm{I}_4\}$ and image the identity component of $\mathrm{SO}(3,2)$.

5.2. Lagrangian subspaces and the Einstein universe. Let V, ω, $\mathbb{B}$, ω^*, and W_0 be as above. The projectivization of the null cone in W_0 is equivalent to $\mathrm{Ein}^{2,1}$. Points in $\mathrm{Ein}^{2,1}$ correspond to Lagrangian planes in V—that is, 2-dimensional linear subspaces $P \subset V$ such that the restriction $\omega|_P \equiv 0$. Explicitly, if v_1, v_2 constitute a basis for P, then the line generated by the bivector

$$w = v_1 \wedge v_2 \in \Lambda^2(V)$$

is independent of the choice of basis for P. Furthermore, w is null with respect to $\mathbb{B}$ and orthogonal to ω^*, so w generates a null line in $W_0 \cong \mathbb{R}^{3,2}$, and hence defines a point in $\mathrm{Ein}^{2,1}$.

For the reverse correspondence, first note that a point of $\mathrm{Ein}^{2,1} \cong \mathbb{P}(\mathfrak{N}(W_0))$ is represented by a vector $a \in W_0$ such that $a \wedge a = 0$. Elements $a \in \Lambda^2 V$ with $a \wedge a = 0$ are exactly the decomposable ones—that is, those that can be written $a = v_1 \wedge v_2$ for $v_1, v_2 \in V$. Then the condition $a \perp \omega^*$ is equivalent by construction to $\omega(v_1, v_2) = 0$, so a represents a Lagrangian plane, $\mathrm{span}\{v_1, v_2\}$, in V. Thus Lagrangian 2-planes in V correspond to isotropic lines in $W_0 \cong \mathbb{R}^{3,2}$.

For a point $q \in \mathrm{Ein}^{2,1}$, denote by L_q the corresponding Lagrangian plane in V.

5.2.1. Complete flags. A photon ϕ in $\mathrm{Ein}^{2,1}$ corresponds to a line ℓ_ϕ in V, where

$$\ell_\phi = \bigcap_{p \in \phi} L_p.$$

A pointed photon (p, ϕ), as defined in §2.5, corresponds to a pair of linear subspaces

$$\ell_\phi \subset L_p \tag{7}$$

where $\ell_\phi \subset V$ is the line corresponding to ϕ and where $L_p \subset V$ is the Lagrangian plane of corresponding to p. Recall that the incidence relation $p \in \phi$ extends to

$$p \in \phi \subset L(p),$$

corresponding to the complete linear flag

$$0 \subset \ell_p \subset P_\phi \subset (\ell_p)^\perp \subset W_0$$

where P_ϕ is the null plane projectivizing to ϕ. The linear inclusion (7) extends to a linear flag

$$0 \subset \ell_\phi \subset L_p \subset (\ell_\phi)^\perp \subset V$$

where now $(\ell_\phi)^\perp$ denotes the symplectic orthogonal of ℓ_ϕ. Clearly the lightcone $L(p)$ corresponds to the linear hyperplane $(\ell_\phi)^\perp \subset V$.

5.2.2. Pairs of Lagrangian planes. Distinct Lagrangian subspaces L_1, L_2 may intersect in either a line or in 0. If $L_1 \cap L_2 \neq 0$, the corresponding points $p_1, p_2 \in \mathrm{Ein}^{2,1}$ are incident. Otherwise

$$V = L_1 \oplus L_2$$

and the linear involution of V

$$\theta = \mathrm{I}_{L_1} \oplus -\mathrm{I}_{L_2}$$

is *anti-symplectic* :

$$\omega(\theta(v_1), \theta(v_2)) = -\omega(v_1, v_2).$$

The corresponding involution of $\mathrm{Ein}^{2,1}$ fixes the two points p_1, p_2 and the spacelike circle $L(p_1) \cap L(p_2)$. It induces a time-reversing involution of $\mathrm{Ein}^{2,1}$.

5.3. Symplectic planes. Let $P \subset V$ be a *symplectic plane,* that is, one for which the restriction $\omega|_P$ is nonzero (and hence nondegenerate). Its symplectic complement $P^\perp$ is also a symplectic plane, and

$$V = P \oplus P^\perp$$

is a symplectic direct sum decomposition.

Choose a basis $\{u_1, u_2\}$ for P. We may assume that $\omega(u_1, u_2) = 1$. Then

$$\mathbb{B}(u_1 \wedge u_2, \omega^*) = 1$$

and

$$\upsilon_P := 2u_1 \wedge u_2 + \omega^*$$

lies in $(\omega^*)^\perp$ since $\mathbb{B}(\omega^*, \omega^*) = -2$. Furthermore

$$\begin{aligned} \mathbb{B}(\upsilon_P, \upsilon_P) &= \ \mathbb{B}(2u_1 \wedge u_2, 2u_1 \wedge u_2) \ + \ 2\,\mathbb{B}(2u_1 \wedge u_2, \omega^*) \ + \ \mathbb{B}(\omega^*, \omega^*) \\ &= \ 0 \ + \ 4 \ - \ 2 \\ &= \ 2. \end{aligned}$$

whence υ_P is a positive vector in $W_0 \cong \mathbb{R}^{3,2}$. In particular $\mathbb{P}(\upsilon_P^\perp \cap \mathfrak{N}(W_0))$ is an Einstein hypersphere.

The two symplectic involutions leaving P (and necessarily also $P^\perp$) invariant

$$\pm\big(\mathrm{I}\,|_P \oplus -\mathrm{I}\,|_{P^\perp}\big)$$

induce maps fixing υ_P, and acting by -1 on $(\upsilon_P)^\perp$. The corresponding eigenspace decomposition is $\mathbb{R}^{1,0} \oplus \mathbb{R}^{2,2}$ and the corresponding conformal involution in $\mathrm{Ein}^{2,1}$ fixes an Einstein hypersphere.

5.4. Positive complex structures and the Siegel space. Not every involution of $\mathrm{Ein}^{2,1}$ arises from a linear involution of V. Particularly important are those which arise from *compatible complex structures*, defined as follows. A *complex structure* on V is an automorphism $V \xrightarrow{\mathrm{J}} V$ such that $\mathrm{J} \circ \mathrm{J} = -\mathrm{I}$. The pair (V, J) then inherits the structure of a *complex vector space* for which V is the underlying real vector space. The complex structure J is *compatible* with the symplectic vector space (V, ω) when

$$\omega(\mathrm{J}\,x, \mathrm{J}\,y) = \omega(x, y).$$

(In the language of complex differential geometry, the exterior 2-form ω *has Hodge type* $(1,1)$ on the complex vector space (V,J).) Moreover

$$\begin{aligned} V \times V &\longrightarrow \mathbb{C} \\ (v,w) &\longmapsto \omega(v, \mathrm{J}\, w) + i\omega(v,w) \end{aligned}$$

defines a *Hermitian form* on (V,J).

A compatible complex structure J on (V,ω) is *positive* if $\omega(v,\mathrm{J}\,v) > 0$ whenever $v \neq 0$. Equivalently, the symmetric bilinear form defined by

$$v \cdot w := \omega(v, \mathrm{J}\, w)$$

is positive definite. This is in turn equivalent to the above Hermitian form being positive definite.

The positive compatible complex structures on V are parametrized by the *symmetric space* of $\mathrm{Sp}(4,\mathbb{R})$. A convenient model is the *Siegel upper-half space* $\mathfrak{S}_2$, which can be realized as the domain of 2×2 complex symmetric matrices with positive definite imaginary part (Siegel [29]).

A matrix $M \in \mathrm{Sp}(4,\mathbb{R})$ acts on a complex structure J by

$$\mathrm{J} \mapsto M\,\mathrm{J}\,M^{-1}$$

and the stabilizer of any J is conjugate to $U(2)$, the group of unitary transformations of $\mathbb{C}^2$. Let the symplectic structure ω be defined by the 2×2-block matrix

$$\mathrm{J} := \begin{bmatrix} 0_2 & -\mathrm{I}_2 \\ \mathrm{I}_2 & 0_2 \end{bmatrix}.$$

This matrix also defines a complex structure. Write M as a block matrix with

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the blocks A, B, C, D are 2×2 real matrices. Because $M \in \mathrm{Sp}(4,\mathbb{R})$,

$$M^{\dagger}\,\mathrm{J}\,M = \mathrm{J}\,. \tag{8}$$

The condition that M preserves the complex structure J means that M commutes with J, which together with (8), means that

$$M^{\dagger} M = \mathrm{I}_4,$$

that is, $M \in \mathrm{O}(4)$. Thus the stabilizer of the pair (ω,J) is $\mathrm{Sp}(4,\mathbb{R}) \cap \mathrm{O}(4)$, which identifies with the unitary group $\mathrm{U}(2)$ as follows.

If M commutes with J, then its block entries satisfy

$$B = -C, \qquad D = A.$$

Relabelling $X = A$ and $Y = C$, then

$$M = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$$

corresponds to a complex matrix $Z = X + iY$. This matrix is symplectic if and only if Z is unitary,

$$\bar{Z}^{\dagger} Z = \mathrm{I}_2\,.$$

5.5. The contact projective structure on photons. The points of a photon correspond to Lagrangian planes in V intersecting in a common line. Therefore, photons correspond to linear 1-dimensional subspaces in V, and the photon space $\mathrm{Pho}^{2,1}$ identifies with the projective space $\mathbb{P}(V)$. This space has a natural contact geometry defined below.

Recall that a *contact structure* on a manifold M^{2n+1} is a vector subbundle $E \subset TM$ of codimension one that is *maximally non-integrable*: E is locally the kernel of a nonsingular 1-form α such that $\alpha \wedge \big(d\alpha\big)^n$ is nondegenerate at every point. This condition is independent of the 1-form α defining E, and is equivalent to the condition that any two points in the same path-component can be joined by a smooth curve with velocity field in E. The 1-form α is called a *contact 1-form* defining E. For more details on contact geometry, see [23, 16, 30].

The restriction of $d\alpha$ to E is a nondegenerate exterior 2-form, making E into a *symplectic vector bundle.* Such a vector bundle always admits a compatible complex structure $J_E : E \longrightarrow E$ (an automorphism such that $J_E \circ J_E = -\mathrm{I}$), which gives E the structure of a *Hermitian vector bundle.* The contact structure we define on photon space $\mathbb{P}(\mathbb{R}^4) \cong \mathrm{Pho}^{2,1}$ will have such Hermitian structures and contact 1-forms arising from compatible complex structures on the symplectic vector space $\mathbb{R}^4$.

5.5.1. Construction of the contact structure. Let $v \in V$ be nonzero, and denote the corresponding line by $[v] \in \mathbb{P}(V)$. The tangent space $T_{[v]}\mathbb{P}(V)$ naturally identifies with $\mathrm{Hom}([v], V/[v])$ ($[v] \subset V$ denotes the 1-dimensional subspace of V, as well). If $V_1 \subset V$ is a hyperplane complementary to $[v]$, then an *affine patch* for $\mathbb{P}(V)$ containing $[v]$ is given by

$$\begin{aligned} \mathrm{Hom}([v], V_1) &\xrightarrow{A_{V_1}} \mathbb{P}(V) \\ \phi &\longmapsto [v + \phi(v)]. \end{aligned}$$

That is, $A_{V_1}(\phi)$ is the graph of the linear map ϕ in $V = [v] \oplus V_1$. This affine patch defines an isomorphism

$$T_{[v]}\mathbb{P}(V) \longrightarrow \mathrm{Hom}([v], V_1) \cong \mathrm{Hom}([v], V/[v])$$

that is independent of the choice of V_1. Now, since ω is skew-symmetric, symplectic product with v defines a linear functional

$$\begin{aligned} V/[v] &\xrightarrow{\alpha_v} \mathbb{R} \\ u &\longmapsto \omega(u, v). \end{aligned}$$

The hyperplane field

$$[v] \longmapsto \{\varphi \ : \ \alpha_v \circ \varphi = 0\}$$

is a well-defined *contact plane field* on $\mathbb{P}(V)$. It posseses a unique transverse orientation; we denote a contact 1-form for this hyperplane field by α.

5.5.2. The contact structure and polarity. The contact structure and the projective geometry of $\mathbb{P}(V)$ interact with each other in an interesting way. If $p \in \mathbb{P}(V)$, then the contact structure at p is a hyperplane $E_p \subset T_p\mathbb{P}(V)$. There is a unique projective hyperplane $H = H(p)$ tangent to E_p at p. Conversely, suppose $H \subset \mathbb{P}(V)$ is a projective hyperplane. The contact plane field is transverse to H everywhere but one point, and that point p is the unique point for which $H = H(p)$. This correspondence results from the correspondence between a line $\ell \subset V$ and its symplectic orthogonal $\ell^\perp \subset V$.

The above correspondence is an instance of a *polarity* in projective geometry. A *polarity* of a projective space $\mathbb{P}(V)$ is a projective isomorphism between $\mathbb{P}(V)$ and its dual $\mathbb{P}(V)^* := \mathbb{P}(V^*)$, arising from a nondegenerate bilinear form on V, which can be either symmetric or skew-symmetric.

Another correspondence is between the set of photons through a given point $p \in \mathrm{Ein}^{2,1}$ and the set of 1-dimensional linear suspaces of the Lagrangian plane $L_p \subset V$. The latter set projects to a projective line in $\mathbb{P}(V)$ tangent to the contact plane field, a *contact projective line.* All contact projective lines arise from points in $\mathrm{Ein}^{2,1}$ in this way.

5.5.3. Relation with positive complex structures on $\mathbb{R}^4$. A compatible positive complex structure J defines a *contact vector field* for the contact structure as follows. Let $\Omega \subset \mathbb{P}(V)$ be a subdomain. For any nonzero $v \in V$, the map $v \longmapsto \mathrm{J}(v)$ defines an element of $\mathrm{Hom}([v], V/[v])$, that is, a tangent vector in $T_{[v]}\mathbb{P}(V)$. The resulting vector field ξ_{J} satisfies $\alpha(\xi_J) > 0$ for any 1-form α defining the contact structure, since $\omega(v, \mathrm{J}\, v) > 0$ for nonzero $v \in V$. More generally, for any smooth map $\mathrm{J} : \Omega \longrightarrow \mathfrak{S}_2$, this construction defines a contact vector field.

5.6. The Maslov cycle.

Given a $2n$-dimensional symplectic vector space V over $\mathbb{R}$, the set $\mathrm{Lag}(V)$ of Lagrangian subspaces of V is a compact homogeneous space. It identifies with $\mathrm{U}(n)/\mathrm{O}(n)$, given a choice of a positive compatible complex structure on $V \cong \mathbb{R}^{2n}$. The fundamental group

$$\pi_1\big(\mathrm{Lag}(V)\big) \cong \mathbb{Z}.$$

An explicit isomorphism is given by the *Maslov index,* which associates to a loop γ in $\mathrm{Lag}(V)$ an integer. (See McDuff-Salamon [23], §2.4 for a general discussion.)

Let $W \in \mathrm{Lag}(V)$ be a Lagrangian subspace. The *Maslov cycle* $\mathrm{Maslov}_W(V)$ associated to W is the subset of $\mathrm{Lag}(V)$ consisting of W' such that

$$W \cap W' \neq 0.$$

Although it is not a submanifold, $\mathrm{Maslov}_W(V)$ carries a natural co-orientation (orientation of its conormal bundle) and defines a cycle whose homology class generates $H_{N-1}(\mathrm{Lag}(V), \mathbb{Z})$ where

$$N = \frac{n(n+1)}{2} = \dim\big(\mathrm{Lag}(V)\big).$$

The Maslov index of a loop γ is the oriented intersection number of γ with the Maslov cycle (after γ is homotoped to be transverse to $\mathrm{Maslov}_W(V)$). If $p \in \mathrm{Ein}^{2,1}$ corresponds to a Lagrangian subspace $W \subset V$, then the Maslov cycle $\mathrm{Maslov}_W(V)$ corresponds to the lightcone $L(p)$. (We thank A. Wienhard for this observation.)

5.7. Summary. We now have a dictionary between the symplectic geometry of $\mathbb{R}^4_\omega$ and the orthogonal geometry of $\mathbb{R}^{3,2}$:

Symplectic $\mathbb{R}^4_\omega$ and contact $\mathbb{P}(V)$	Pseudo-Riemannian $\mathbb{R}^{3,2}$ and $\mathrm{Ein}^{2,1}$
Lagrangian planes $L \subset \mathbb{R}^4_\omega$	Points $p \in \mathrm{Ein}^{2,1}$
Contact projective lines in $\mathbb{P}(V)$	Points $p \in \mathrm{Ein}^{2,1}$
Lines $\ell \subset \mathbb{R}^4_\omega$	Photons ϕ
Hyperplanes $\ell^\perp \subset \mathbb{R}^4_\omega$	Lightcones
Symplectic planes (splittings) in $\mathbb{R}^4_\omega$	Einstein hyperspheres
Linear symplectic automorphisms	time-preserving conformal automorphisms
Linear anti-symplectic automorphisms	time-reversing conformal automorphisms
Flags $\ell \subset L \subset \ell^\perp$ in $\mathbb{R}^4_\omega$	Incident pairs $p \in \phi \subset L(p)$
Positive compatible complex structures	Free involutions of $\mathrm{Ein}^{2,1}$
Lagrangian splittings $V = L_1 \oplus L_2$	Nonincident pairs of points
Lagrangian splittings $V = L_1 \oplus L_2$	Spacelike circles

6. Lie theory of $\mathrm{Pho}^{2,1}$ and $\mathrm{Ein}^{2,1}$

This section treats the structure of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ and the isomorphism with $\mathfrak{o}(3,2)$. We relate differential-geometric properties of the homogeneous spaces $\mathrm{Ein}^{2,1}$ and $\mathrm{Pho}^{2,1}$ with the Lie algebra representations corresponding to the isotropy. This section develops the structure theory (Cartan subalgebras, roots, parabolic subalgebras) and relates these algebraic notions to the synthetic geometry of the three parabolic homogenous spaces $\mathrm{Ein}^{2,1}$, $\mathrm{Pho}^{2,1}$ and $\mathrm{Flag}^{2,1}$. Finally, we discuss the geometric significance of the Weyl group of $\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{SO}(2,3)$.

6.1. Structure theory. Let $V \cong \mathbb{R}^4$, equipped with the symplectic form ω, as above. We consider a *symplectic basis* e_1, e_2, e_3, e_4 in which ω is

$$\mathrm{J} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Lie algebra $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{R})$ consists of all 4×4 real matrices M satisfying

$$M^\dagger \, \mathrm{J} + \mathrm{J}\, M = 0,$$

that is,

$$M = \begin{bmatrix} a & a_{12} & r_{11} & r_{12} \\ a_{21} & -a & r_{21} & r_{22} \\ -r_{22} & r_{12} & b & b_{12} \\ r_{21} & -r_{11} & b_{21} & -b \end{bmatrix} \tag{9}$$

where $a, b, a_{ij}, b_{ij}, r_{ij} \in \mathbb{R}$.

6.1.1. Cartan subalgebras. A *Cartan subalgebra* $\mathfrak{a}$ of $\mathfrak{sp}(4,\mathbb{R})$ is the subalgebra stabilizing the four coordinate lines $\mathbb{R}e_i$ for $i = 1, 2, 3, 4$, and comprises the diagonal matrices

$$H(a,b) := \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{bmatrix}$$

for $a, b \in \mathbb{R}$. The calculation

$$[H, M] = \begin{bmatrix} 0 & (2a)a_{12} & (a-b)r_{11} & (a+b)r_{12} \\ (-2a)a_{21} & 0 & (-a-b)r_{21} & (-a+b)r_{22} \\ (a-b)r_{22} & (-a-b)r_{12} & 0 & (2b)b_{12} \\ (a+b)r_{21} & (-a+b)r_{11} & (-2b)b_{21} & 0 \end{bmatrix}$$

implies that the eight linear functionals assigning to $H(a,b)$ the values

$$2a, -2a, 2b, -2b, a-b, a+b, -a-b, -a+b$$

define the *root system*

$$\begin{aligned} \Delta := \{&(2,0), (-2,0), (0,2), (0,-2), \\ &(1,-1), (1,1), (-1,-1), (-1,1)\} \subset \mathfrak{a}^* \end{aligned}$$

pictured below.

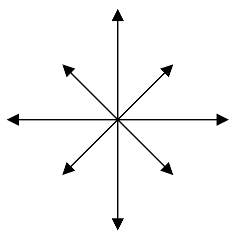

Figure 4. Root diagram of $\mathfrak{sp}(4,\mathbb{R})$

6.1.2. Positive and negative roots. A vector $v_0 \in \mathfrak{a}$ such that $\lambda(v_0) \neq 0$ for all roots $\lambda \in \Delta$ partitions Δ into *positive roots* Δ_+ and *negative roots* Δ_- depending on whether $\lambda(v_0) > 0$ or $\lambda(v_0) < 0$ respectively. For example,

$$v_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

partitions Δ into

$$\begin{aligned} \Delta_+ &= \{ (2,0), (1,1), (0,2), (-1,1) \} \\ \Delta_- &= \{ (-2,0), (-1,-1), (0,-2), (1,-1) \}. \end{aligned}$$

The positive roots

$$\alpha := (2,0), \ \beta := (-1,1)$$

form a pair of *simple positive roots* in the sense that every $\lambda \in \Delta_+$ is a positive integral linear combination of α and β. Explicitly:

$$\Delta_+ = \{\alpha, \alpha+\beta, \alpha+2\beta, \beta\}.$$

6.1.3. Root space decomposition. For any root $\lambda \in \Delta$, define the *root space*

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X\}.$$

In $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{R})$, each root space is one-dimensional, and the elements $X_\lambda \in \mathfrak{g}_\lambda$ are called *root elements.* The Lie algebra decomposes as a direct sum of vector spaces:

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda.$$

For more details, see Samelson [28].

6.2. Symplectic splittings.

The basis vectors e_1, e_2 span a symplectic plane $P \subset V$ and e_3, e_4 span its symplectic complement $P^\perp \subset V$. These planes define a *symplectic direct sum decomposition*

$$V = P \oplus P^\perp.$$

The subalgebra $\mathfrak{h}_P \subset \mathfrak{sp}(4,\mathbb{R})$ preserving P also preserves $P^\perp$ and consists of matrices of the form (9) that are block-diagonal:

$$\begin{bmatrix} a & a_{12} & 0 & 0 \\ a_{21} & -a & 0 & 0 \\ 0 & 0 & b & b_{12} \\ 0 & 0 & b_{21} & -b \end{bmatrix}.$$

Thus

$$\begin{aligned} \mathfrak{h}_P &\cong \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R}) \\ &\cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}). \end{aligned}$$

The Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{sp}(4,\mathbb{R})$ is also a Cartan subalgebra of $\mathfrak{h}_P$, but only the four long roots

$$\Delta' = \{(\pm 2, 0), (0, \pm 2)\} = \{\pm\alpha, \pm(\alpha + 2\beta)\}$$

are roots of $\mathfrak{h}_P$. In particular $\mathfrak{h}_P$ decomposes as

$$\mathfrak{h}_P = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta'} \mathfrak{g}_\lambda.$$

6.3. The Orthogonal Representation of $\mathfrak{sp}(4,\mathbb{R})$. Let $e_1, \dots, e_4$ be a symplectic basis for V as above and

$$\mathrm{vol} := e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

a volume element for V. A convenient basis for $\Lambda^2 V$ is:

$$\begin{aligned} f_1 &:= e_1 \wedge e_3 \\ f_2 &:= e_2 \wedge e_3 \\ f_3 &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4) \\ f_4 &:= e_4 \wedge e_1 \\ f_5 &:= e_2 \wedge e_4 \end{aligned} \tag{10}$$

for which the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

defines the bilinear form $\mathbb{B}$ associated to this volume element.

The matrix M defined in (9) above maps to

$$\widetilde{M} = \begin{bmatrix} a+b & a_{12} & r_{12} & -b_{12} & 0 \\ a_{21} & -a+b & r_{22} & 0 & b_{12} \\ r_{21} & r_{11} & 0 & -r_{22} & -r_{12} \\ -b_{21} & 0 & -r_{11} & a-b & -a_{12} \\ 0 & b_{21} & -r_{21} & -a_{21} & -a-b \end{bmatrix} \in \mathfrak{so}(3,2). \tag{11}$$

For a fixed symplectic plane $P \subset V$, such as the one spanned by e_1 and e_2, denote by $P \wedge P^\perp$ the subspace of $\Lambda^2 V$ of elements that can be written in the form $\sum_i v_i \wedge w_i$, where $v_i \in P$ and $w_i \in P^\perp$ for all i. The restriction of the bilinear form $\mathbb{B}$ to this subspace, which has basis $\{f_1, f_2, f_4, f_5\}$, is type $(2,2)$. Its stabilizer is the image $\widetilde{\mathfrak{h}_P}$ of $\mathfrak{h}_P$ in $\mathfrak{o}(3,2)$. Note that this image is isomorphic to

$$\mathfrak{o}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}).$$

6.4. Parabolic subalgebras.

The homogeneous spaces $\mathrm{Ein}^{2,1}$, $\mathrm{Pho}^{2,1}$ and $\mathrm{Flag}^{2,1}$ identify with quotients G/P of $G = \mathrm{Sp}(4,\mathbb{R})$ where $P \subset G$ is a proper *parabolic subgroup.* When G is algebraic, then any parabolic subgroup P of G is algebraic, and the quotient G/P is a compact projective variety. See Chapter 7 of [18] for more details.

As usual, working with Lie algebras is more convenient. We denote the corresponding *parabolic subalgebras* by $\mathfrak{p}$, and they are indexed by subsets $S \subset \Pi^-$ of the set $\Pi^- := \{-\alpha, -\beta\}$ of simple negative roots, as follows.

The *Borel subalgebra* or *minimal parabolic subalgebra* corresponds to $S = \emptyset$ and is defined as

$$\mathfrak{p}_\emptyset := \mathfrak{p}_S = \mathfrak{a} \oplus \bigoplus_{\lambda \in \Delta^+} \mathfrak{g}_\lambda.$$

In general, let $\widetilde{S}$ be the set of finite sums of elements of S. The parabolic subalgebra determined by S is

$$\mathfrak{p}_S := \mathfrak{p}_\emptyset \oplus \bigoplus_{\lambda \in \widetilde{S}} \mathfrak{g}_\lambda.$$

6.4.1. The Borel subalgebra and $\mathrm{Flag}^{2,1}$.

Let $\mathfrak{p}_\emptyset$ be the Borel subalgebra defined above. The corresponding Lie subgroup $P_\emptyset$ is the stabilizer of a unique pointed photon, equivalently, an isotropic flag, in $\mathrm{Flag}^{2,1}$; thus $\mathrm{Flag}^{2,1}$ identifies with the homogeneous space $G/P_\emptyset$. The subalgebra

$$\mathfrak{u}_\emptyset := \sum_{\lambda \in \Delta_+} \mathfrak{g}_\lambda \subset \mathfrak{sp}(4,\mathbb{R})$$

is the Lie algebra of the *unipotent radical* of $P_\emptyset$ and is 3-step nilpotent. A realization of the corresponding group is the group generated by the translations of $\mathbb{E}^{2,1}$ and a unipotent one-parameter subgroup of $\mathrm{SO}(2,1)$.

6.4.2. The parabolic subgroup corresponding to $\mathrm{Pho}^{2,1}$.

Now let $S = \{-\alpha\}$; the corresponding parabolic subalgebra $\mathfrak{p}_\alpha$ is the stabilizer subalgebra of a line in V, or, equivalently, of a point in $\mathbb{P}(V)$. In $\mathfrak{o}(3,2)$ this parabolic is the stabilizer of a null plane in $\mathbb{R}^{3,2}$, or, equivalently, of a photon in $\mathrm{Ein}^{2,1}$.

6.4.3. The parabolic subgroup corresponding to $\mathrm{Ein}^{2,1}$.

Now let $S = \{-\beta\}$; the corresponding parabolic subalgebra $\mathfrak{p}_\beta$ is the stabilizer subalgebra of a Lagrangian plane in V, or, equivalently, a contact projective line in $\mathbb{P}(V)$. In $\mathfrak{o}(3,2)$, this parabolic is the stabilizer of a null line in $\mathbb{R}^{3,2}$, or, equivalently, of a point in $\mathrm{Ein}^{2,1}$.

6.5. Weyl groups.

The Weyl group W of $\mathrm{Sp}(4,\mathbb{R})$ is isomorphic to a dihedral group of order 8 (see Figure 4). It acts by permutations on elements of the quadruples in $\mathbb{P}(V)$ corresponding to a basis of V.

Let A be the connected subgroup of $\mathrm{Sp}(4,\mathbb{R})$, with Lie algebra $\mathfrak{a}$. In the symplectic basis $e_1,\dots,e_4$, it consists of matrices of the form

$$\begin{bmatrix} a_1 & & & \\ & a_1^{-1} & & \\ & & a_2 & \\ & & & a_2^{-1} \end{bmatrix} \qquad a_1, a_2 > 0.$$

The semigroup $A^+ \subset A$ with $a_2 > a_1 > 1$ corresponds to an open Weyl chamber in $\mathfrak{a}$. For $i = 1,2,3,4$, let H_i be the image in $\mathbb{P}(V)$ of the hyperplane spanned by e_j for $j \neq i$. The point $[e_3] \in \mathbb{P}(V)$ is an attracting fixed point for all sequences in A^+, and $[e_4]$ is a repelling fixed point: Any unbounded $a_n \in A^+$ converges uniformly on compact subsets of $\mathbb{P}(V)\backslash H_3$ to the constant map $[e_3]$, while a_n^{-1} converges to $[e_4]$ uniformly on compact subsets of $\mathbb{P}(V)\backslash H_2$. On $H_3\backslash(H_3 \cap H_1)$, an unbounded sequence $\{a_n\}$ converges to $[e_1]$, while on $H_4\backslash(H_4 \cap H_2)$, the inverses a_n^{-1} converge to $[e_2]$.

We will call the point $[e_1]$ a *codimension-one attracting fixed point* for sequences in A^+ and $[e_2]$ a *codimension-one repelling fixed point.* Every Weyl chamber has associated to it a *dynamical quadruple* like $([e_3],[e_4],[e_1],[e_2])$, consisting of an attracting fixed point, a repelling fixed point, a codimension-one attracting fixed point, and a codimension-one repelling fixed point.

Conversely, given a symplectic basis $v_1,\dots,v_4$, the intersection of the stabilizers in $\mathrm{Sp}(4,\mathbb{R})$ of the lines $\mathbb{R}v_i$ is a Cartan subgroup A. The elements $a \in A$ such that $([v_1],\dots,[v_4])$ is a dynamical quadruple for the sequence a^n form a semigroup A^+ that is an open Weyl chamber in A.

The Weyl group acts as a group of permutations of such a quadruple. These permutations must preserve a stem configuration as in Figure 1, where now two points are connected by an edge if the corresponding lines in V are in a common Lagrangian plane, or, equivalently, the two points of $\mathbb{P}(V)$ span a line tangent to the contact structure. The permissible permutations are those preserving the partition $\{v_1,v_2\}|\{v_3,v_4\}$.

In $\mathrm{O}(3,2)$, the Weyl group consists of permutations of four points $p_1,\dots,p_4$ of $\mathrm{Ein}^{2,1}$ in a stem configuration that preserve the configuration. A Weyl chamber again corresponds to a dynamical quadruple $(p_1,\dots,p_4)$ of fixed points, where now sequences $a_n \in A^+$ converge to the constant map p_1 on the complement of $L(p_2)$ and to p_3 on $L(p_2)\backslash(L(p_4) \cap L(p_2))$; the inverse sequence converges to p_2 on the complement of $L(p_1)$ and to p_4 on $L(p_1)\backslash(L(p_1) \cap L(p_3))$.

7. Three kinds of dynamics

In this section, we present the ways sequences in $\mathrm{Sp}(4,\mathbb{R})$ can diverge to infinity in terms of projective singular limits. In [13], Frances defines a trichotomy for sequences diverging to infinity in $\mathrm{O}(3,2)$: they have *bounded*, *mixed*, or *balanced distortion.* He introduces limit sets for such sequences and finds maximal domains of proper discontinuity for certain subgroups of $\mathrm{O}(3,2)$. We translate Frances' trichotomy to $\mathrm{Sp}(4,\mathbb{R})$, along with the associated limit sets and maximal domains of properness.

7.1. Projective singular limits. Let E be a finite-dimensional vector space, and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathrm{GL}(E)$. This sequence induces a sequence $(\bar{g}_n)_{n\in\mathbb{N}}$ of projective transformations of $\mathbb{P}(E)$. Let $\|\cdot\|$ be an auxiliary Euclidean norm on E and let $\|\cdot\|_\infty$ be the associated operator norm on the space of endomorphisms $\mathrm{End}(E)$. The division of g_n by its norm $\|g_n\|_\infty$ does not modify the projective transformation $\bar{g}_n$. Hence we can assume that g_n belongs to the $\|\cdot\|_\infty$-unit sphere of $\mathrm{End}(E)$. This sphere is compact, so $(g_n)_{n\in\mathbb{N}}$ admits accumulation points. Up to a subsequence, we can assume that $(g_n)_{n\in\mathbb{N}}$ converges to an element g_∞ of the $\|\cdot\|_\infty$-unit sphere. Let I be the image of g_∞, and let L be the kernel of g_∞. Let

$$\bar{g}_\infty : \mathbb{P}(E) \setminus \mathbb{P}(L) \to \mathbb{P}(I) \subset \mathbb{P}(E)$$

be the induced map.

Proposition 7.1.1. *For any compact $K \subset \mathbb{P}(E) \setminus \mathbb{P}(L)$, the restriction of the sequence $(\bar{g}_n)_{(n\in\mathbb{N})}$ on K converges uniformly to the restriction on K of $\bar{g}_\infty$.*

Corollary 7.1.2. *Let Γ be a* discrete *subgroup of* $\mathrm{PGL}(E)$. *Let Ω be the open subset of $\mathbb{P}(E)$ formed by points admitting a neighborhood U such that, for any sequence (g_n) in Γ with accumulation point g_∞ having image I and kernel L,*

$$U \cap \mathbb{P}(L) = U \cap \mathbb{P}(I) = \emptyset.$$

Then Γ acts properly discontinuously on Ω.

In fact, the condition $U \cap \mathbb{P}(L) = \emptyset$ is sufficient to define Ω (as is $U \cap \mathbb{P}(I) = \emptyset$). To see this, note that if $g_n \to \infty$ with

$$g_n/\|g_n\|_\infty \longrightarrow g_\infty$$
$$g_n^{-1}/\|g_n^{-1}\|_\infty \longrightarrow g_\infty^{-},$$

then

$$g_\infty \circ g_\infty^{-} = g_\infty^{-} \circ g_\infty = 0.$$

Hence

$$\mathrm{Im}(g_\infty) \subseteq \mathrm{Ker}(g_\infty^{-}) \text{ and } \mathrm{Im}(g_\infty^{-}) \subseteq \mathrm{Ker}(g_\infty).$$

7.2. Cartan's decomposition $G = KAK$. When $(g_n)_{n\in\mathbb{N}}$ is a sequence in a semisimple Lie group G, a very convenient way to identify the accumulation points $\bar{g}_\infty$ is to use the KAK-decomposition in G: first select the Euclidean norm $\|\cdot\|$ so that it is preserved by the maximal compact subgroup K of G. Decompose every g_n in the form $k_n a_n k_n'$, where k_n and k_n' belong to K, and a_n belongs to a fixed Cartan subgroup. We can furthermore require that a_n is the image by the exponential of an element of the closure of a Weyl chamber. Up to a subsequence, k_n and k_n' admit limits k_∞ and k_∞', respectively. Composition on the right or on the left by an element of K does not change the operator norm, so g_n has $\|\cdot\|_\infty$-norm 1 if and only if a_n has $\|\cdot\|_\infty$-norm 1. Let a_∞ be an accumulation point of $(a_n)_{n\in\mathbb{N}}$. Then

$$g_\infty = k_\infty a_\infty k'_\infty.$$

The kernel of g_∞ is the image by $(k'_\infty)^{-1}$ of the kernel of a_∞, and the image of g_∞ is the image by k_∞ of the image of a_∞. Hence, in order to find the singular projective limit $\bar{g}_\infty$, the main task is to find the limit a_∞, and this problem is particularly easy when the rank of G is small.

7.2.1. Sequences in $\mathbf{Sp(4,\mathbb{R})}$. The image by the exponential map of a Weyl chamber in $\mathfrak{sp}(4,\mathbb{R})$ is the semigroup $A^+ \subset A$ of matrices (see §6.5):

$$A(\alpha_1,\alpha_2) = \begin{bmatrix} \exp(\alpha_1) & & & \\ & \exp(-\alpha_1) & & \\ & & \exp(\alpha_2) & \\ & & & \exp(-\alpha_2) \end{bmatrix} \qquad \alpha_2 > \alpha_1 > 0.$$

The operator norm of $A(\alpha_1,\alpha_2)$ is $\exp(\alpha_2)$. We therefore can distinguish three kinds of dynamical behaviour for a sequence $(A(\alpha_1^{(n)},\alpha_2^{(n)}))_{n\in\mathbb{N}}$:

- *no distortion:* when $\alpha_1^{(n)}$ and $\alpha_2^{(n)}$ remain bounded,
- *bounded distortion:* when $\alpha_1^{(n)}$ and $\alpha_2^{(n)}$ are unbounded, but the difference $\alpha_2^{(n)} - \alpha_1^{(n)}$ is bounded,
- *unbounded distortion:* when the sequences $\alpha_1^{(n)}$ and $\alpha_2^{(n)} - \alpha_1^{(n)}$ are unbounded.

This distinction extends to any sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{Sp}(4,\mathbb{R})$. Assume that the sequence $(g_n/\|g_n\|_\infty)_{n\in\mathbb{N}}$ converges to a limit g_∞. Then:

- For no distortion, the limit g_∞ is not singular—the sequence $(g_n)_{n\in\mathbb{N}}$ converges in $\mathrm{Sp}(4,\mathbb{R})$.
- For bounded distortion, the kernel L and the image I are 2-dimensional. More precisely, they are Lagrangian subspaces of V. The singular projective transformation $\bar{g}_\infty$ is defined in the complement of a projective line and takes values in a projective line; these projective lines are both tangent everywhere to the contact structure.
- For unbounded distortion, the singular projective transformation $\bar{g}_\infty$ is defined in the complement of a projective hyperplane and admits only one value.

7.2.2. Sequences in $\mathrm{SO}^+(3,2)$. The Weyl chamber of $\mathrm{SO}^+(3,2)$ is simply the image of the Weyl chamber of $\mathfrak{sp}(4,\mathbb{R})$ by the differential of the homomorphism

$$\mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SO}^+(3,2)$$

defined in §6.2. More precisely, the image of an element $A(\alpha_1,\alpha_2)$ of A^+ is $A'(a_1,a_2)$ where

$$a_1 = \alpha_1 + \alpha_2, a_2 = \alpha_2 - \alpha_1$$

and:

$$A'(a_1,a_2) = \begin{bmatrix} \exp(a_1) & & & & \\ & \exp(a_2) & & & \\ & & 1 & & \\ & & & \exp(-a_2) & \\ & & & & \exp(-a_1) \end{bmatrix} \qquad a_1 > a_2 > 0.$$

The KAK decomposition of $\mathrm{Sp}(4,\mathbb{R})$ above corresponds under the homomorphism to a KAK decomposition of $\mathrm{SO}^+(3,2)$. Reasoning as in the previous section, we distinguish three cases:

- *no distortion:* when $a_1^{(n)}$ and $a_2^{(n)}$ remain bounded,
- *balanced distortion:* when $a_1^{(n)}$ and $a_2^{(n)}$ are unbounded, but the difference $a_1^{(n)} - a_2^{(n)}$ is bounded,
- *unbalanced distortion:* when the sequences $a_1^{(n)}$ and $a_1^{(n)} - a_2^{(n)}$ are unbounded.

The dynamical analysis is similar, but we restrict to the closed subset $\mathrm{Ein}^{2,1}$ of $\mathbb{P}(\mathbb{R}^{3,2})$:

- No distortion corresponds to sequences $(g_n)_{n\in\mathbb{N}}$ converging in $\mathrm{SO}^+(3,2)$.
- For balanced distortion, the intersection between $\mathbb{P}(L)$ and $\mathrm{Ein}^{2,1}$, and the intersection between $\mathbb{P}(I)$ and $\mathrm{Ein}^{2,1}$ are both photons. Hence the restriction of the singular projective transformation $\bar{g}_\infty$ to $\mathrm{Ein}^{2,1}$ is defined in the complement of a photon and takes value in a photon.
- For unbalanced distortion, the singular projective transformation $\bar{g}_\infty$ is defined in the complement of a lightcone and admits only one value.

7.3. Maximal domains of properness.

Most of the time, applying directly Proposition 7.1.1 and Corollary 7.1.2 to a discrete subgroup Γ of $\mathrm{Sp}(4,\mathbb{R})$ or $\mathrm{SO}^+(3,2)$ in order to find domains where the action of Γ is proper is far from optimal.

Through the morphism $\mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SO}^+(3,2)$, a sequence in $\mathrm{Sp}(4,\mathbb{R})$ can also be considered as a sequence in $\mathrm{SO}(3,2)$. Observe that our terminology is coherent:

a sequence has no distortion in $\mathrm{Sp}(4,\mathbb{R})$ if and only if it has no distortion in $\mathrm{SO}^+(3,2)$. Observe also that since

$$a_1 = \alpha_1 + \alpha_2,$$
$$a_2 = \alpha_2 - \alpha_1,$$

a sequence with bounded distortion in $\mathrm{Sp}(4,\mathbb{R})$ is unbalanced in $\mathrm{SO}^+(3,2)$, and a sequence with balanced distortion in $\mathrm{SO}^+(3,2)$ is unbounded in $\mathrm{Sp}(4,\mathbb{R})$. In summary, we distinguish three different kinds of non-converging dynamics, covering all the possibilities:

Definition 7.3.1. A sequence $(g_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{Sp}(4,\mathbb{R})$ escaping from any compact subset in $\mathrm{Sp}(4,\mathbb{R})$ has:

- *bounded distortion* if the coefficient $a_2^{(n)} = \alpha_2^{(n)} - \alpha_1^{(n)}$ is bounded,
- *balanced distortion* if the coefficient $\alpha_2^{(n)} = (a_1^{(n)} + a_2^{(n)})/2$ is bounded,
- *mixed distortion* if all the coefficients $a_1^{(n)}$, $a_2^{(n)}$, $\alpha_1^{(n)}$, $\alpha_2^{(n)}$ are unbounded.

7.3.1. Action on $\mathbf{Ein}^{2,1}$. The dynamical analysis can be refined in the mixed distortion case. In [13], C. Frances proved:

Proposition 7.3.2. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathrm{SO}^+(3,2)$ with mixed distortion, such that the sequence $(g_n/\|g_n\|_\infty)_{n\in\mathbb{N}}$ converges to an endomorphism g_∞. Then there are photons Δ^- and Δ^+ in $\mathrm{Ein}^{2,1}$ such that, for any sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathrm{Ein}^{2,1}$ converging to an element of $\mathrm{Ein}^{2,1}\setminus\Delta^-$, all the accumulation points of $(g_n(p_n))_{n\in\mathbb{N}}$ belong to Δ^+.*

As a corollary (§4.1 in [13]):

Corollary 7.3.3. *Let Γ be a* discrete *subgroup of $\mathrm{SO}^+(3,2)$. Let Ω_0 be the union of all open domains U in $\mathrm{Ein}^{2,1}$ such that, for any accumulation point g_∞, with kernel L and image I, of a sequence $(g_n/\|g_n\|_\infty)_{n\in\mathbb{N}}$ with $g_n \in \mathrm{SO}^+(3,2)$:*

- *When $(g_n)_{n\in\mathbb{N}}$ has balanced distortion, U is disjoint from the photons $\mathbb{P}(L)\cap \mathrm{Ein}^{2,1}$ and $\mathbb{P}(I)\cap\mathrm{Ein}^{2,1}$;*
- *When $(g_n)_{n\in\mathbb{N}}$ has bounded distortion, U is disjoint from the lightcone $\mathbb{P}(L)\cap \mathrm{Ein}^{2,1}$;*
- *When $(g_n)_{n\in\mathbb{N}}$ has mixed distortion, U is disjoint from the photons Δ_- and Δ_+.*

Then the action of Γ on Ω_0 is properly discontinuous.

Observe that the domain Ω_0 is in general bigger than the domain Ω appearing in Corollary 7.1.2. An interesting case is that in which Ω_0 is obtained by removing only photons:

Proposition 7.3.4 (Frances [13]). *A discrete subgroup Γ of $\mathrm{SO}^+(3,2)$ does not contain sequences with bounded distortion if and only if its action on $\mathbb{P}(\mathbb{R}^{3,2})\backslash\mathrm{Ein}^{2,1}$ is properly discontinuous.*

Frances calls such a subgroup a *of the first kind.* The following suggests that the domain Ω_0 is optimal.

Proposition 7.3.5 (Frances [13]). *Let Γ be a discrete, Zariski dense subgroup of $\mathrm{SO}^+(3,2)$ which does not contain sequences with bounded distortion. Then Ω_0 is the unique maximal open subset of $\mathrm{Ein}^{2,1}$ on which Γ acts properly.*

7.3.2. Action on $\mathbb{P}(V)$. A similar analysis should be done when Γ is considered a discrete subgroup of $\mathrm{Sp}(4,\mathbb{R})$ instead of $\mathrm{SO}^+(3,2)$. The following proposition is analogous to Proposition 7.3.2:

Proposition 7.3.6. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathrm{Sp}(V)$ with mixed distortion, such that the sequence $(g_n/\|g_n\|_\infty)_{n\in\mathbb{N}}$ converges to an endomorphism g_∞ of V. Then there are contact projective lines Δ^- and Δ^+ in $\mathbb{P}(V)$ such that, for any sequence*

$$(p_n)_{n\in\mathbb{N}}\in\mathbb{P}(V)$$

converging to an element of $\mathbb{P}(V)\setminus\Delta^-$, all the accumulation points of $(g_n(p_n))_{n\in\mathbb{N}}$ belong to Δ^+.

We can then define a subset Ω_1 of $\mathbb{P}(V)$ as the interior of the subset obtained after removing limit contact projective lines associated to subsequences of Γ with bounded or mixed distortion, and removing projective hyperplanes associated to subsequences with balanced distortion. Then it is easy to prove that the action of Γ on Ω_1 is properly discontinuous.

An interesting case is that in which we remove only projective lines, and no hypersurfaces—the case in which Γ has no subsequence with balanced distortion. Frances calls such Γ *groups of the second kind.* The following questions arise from comparison with Propositions 7.3.5 and 7.3.4:

Question: Can groups of the second kind be defined as groups acting properly on some associated space?

Question: Is Ω_1 the unique maximal open subset of $\mathbb{P}(V)$ on which the action of Γ is proper, at least if Γ is Zariski dense?

7.3.3. Action on the flag manifold. Now consider the action of $\mathrm{Sp}(4,\mathbb{R})$ on the flag manifold $\mathrm{Flag}^{2,1}$. Let $v,w\in V$ be such that $\omega(v,w)=0$, so v and w span a Lagrangian plane. Let

$$\mathrm{Flag}^{2,1}\xrightarrow{\rho_1}\mathrm{Pho}^{2,1}$$
$$\mathrm{Flag}^{2,1}\xrightarrow{\rho_2}\mathrm{Ein}^{2,1}$$

be the natural projections. Let g_n be a sequence in $\mathrm{Sp}(4,\mathbb{R})$ diverging to infinity with mixed distortion. We invite the reader to verify the following statements:

- There are a flag $q^+ \in \mathrm{Flag}^{2,1}$ and points $[v] \in \mathbb{P}(V)$ and $z \in \mathrm{Ein}^{2,1}$ such that, on the complement of
$$\rho_1^{-1}([v^\perp]) \cup \rho_2^{-1}(L(z))$$
the sequence g_n converges uniformly to the constant map q^+.
- There are contact projective lines α^+,α^- in $\mathbb{P}(V)$ and photons β^+,β^- in $\mathrm{Ein}^{2,1}$ such that, on the complement of
$$\rho_1^{-1}(\alpha^-) \cup \rho_2^{-1}(\beta^-)$$
all accumulation points of g_n lie in
$$\rho_1^{-1}(\alpha^+) \cap \rho_2^{-1}(\beta^+).$$
This intersection is homeomorphic to a wedge of two circles.

8. Crooked surfaces

Crooked planes were introduced by Drumm [8, 9, 10] to investigate discrete groups of Lorentzian transformations which act freely and properly on $\mathbb{E}^{2,1}$. He used crooked planes to construct fundamental polyhedra for such actions; they play a role analogous to equidistant surfaces bounding Dirichlet fundamental domains in Hadamard manifolds. This section discusses the conformal compactification of a crooked plane and its automorphisms.

8.1. Crooked planes in Minkowski space. For a detailed description of crooked planes, see Drumm-Goldman [10]. We quickly summarize the basic results here.

Consider $\mathbb{E}^{2,1}$ with the Lorentz metric from the inner product $I_2 \oplus -I_1$ on $\mathbb{R}^{2,1}$. A crooked plane C is a surface in $\mathbb{E}^{2,1}$ that divides $\mathbb{E}^{2,1}$ into two cells, called *crooked half-spaces.* It is a piecewise linear surface composed of four 2-dimensional faces, joined along four rays, which all meet at a point p, called the *vertex.* The four rays have endpoint p, and form two lightlike geodesics, which we denote ℓ_1 and ℓ_2. Two of the faces are null half-planes $\mathcal{W}_1$ and $\mathcal{W}_2$, bounded by ℓ_1 and ℓ_2 respectively, which we call *wings.* The two remaining faces consist of the intersection between $J^\pm(p)$ and the timelike plane P containing ℓ_1 and ℓ_2; their union is the *stem* of C. The timelike plane P is the orthogonal complement of a unique spacelike line $P^\perp(p)$ containing p, called the *spine* of C.

To define a crooked plane, we first define the wings, stem, and spine. A lightlike geodesic $\ell = p + \mathbb{R}v$ lies in a unique null plane $\ell^\perp$ (§2.2). The ambient orientation of $\mathbb{R}^{2,1}$ distinguishes a component of $\ell^\perp \setminus \ell$ as follows. Let $u \in \mathbb{R}^{2,1}$ be a timelike vector such that $\langle u, v\rangle < 0$. Then each component of $\ell^\perp \setminus \ell$ defined by
$$\mathcal{W}^+(\ell) := \{p + w \in \ell^\perp \mid \det(u, v, w) > 0\}$$
$$\mathcal{W}^-(\ell) := \{p + w \in \ell^\perp \mid \det(u, v, w) < 0\}$$

is independent of the choices above. In particular, every orientation-preserving isometry f of $\mathbb{E}^{2,1}$ maps

$$\mathcal{W}^+(\ell) \longrightarrow \mathcal{W}^+(f(\ell))$$
$$\mathcal{W}^-(\ell) \longrightarrow \mathcal{W}^-(f(\ell))$$

and every orientation-reversing isometry f maps

$$\mathcal{W}^+(\ell) \longrightarrow \mathcal{W}^-(f(\ell))$$
$$\mathcal{W}^-(\ell) \longrightarrow \mathcal{W}^+(f(\ell)).$$

Given two lightlike geodesics ℓ_1, ℓ_2 containing p, the stem is defined as

$$\mathcal{S}(\ell_1, \ell_2) := \mathrm{J}^{\pm}(p) \cap (p + \mathrm{span}\{\ell_1 - p, \ell_2 - p\}).$$

The spine is

$$\sigma = p + (\mathcal{S}(\ell_1, \ell_2) - p)^{\perp}.$$

Compare Drumm-Goldman [10].

The *positively-oriented crooked plane* with vertex p and stem $\mathcal{S}(\ell_1, \ell_2)$ is the union

$$\mathcal{W}^+(\ell_1) \cup \mathcal{S}(\ell_1, \ell_2) \cup \mathcal{W}^+(\ell_2).$$

Similarly, the *negatively-oriented crooked plane* with vertex p and stem $\mathcal{S}(\ell_1, \ell_2)$ is

$$\mathcal{W}^-(\ell_1) \cup \mathcal{S}(\ell_1, \ell_2) \cup \mathcal{W}^-(\ell_2).$$

Given an orientation on $\mathbb{E}^{2,1}$, a positively-oriented crooked plane is determined by its vertex and its spine. Conversely, every point p and spacelike line σ containing p determines a unique positively- or negatively-oriented crooked plane.

A crooked plane C is homeomorphic to $\mathbb{R}^2$, and the complement $\mathbb{E}^{2,1} \backslash C$ consists of two components, each homeomorphic to $\mathbb{R}^3$. The components of the complement of a crooked plane are called *open crooked half-spaces* and their closures *closed crooked half-spaces.* The spine of C is the unique spacelike line contained in C.

8.2. An example. Here is an example of a crooked plane with vertex the origin and spine the x-axis:

$$p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \sigma = \mathbb{R} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The lightlike geodesics are

$$\ell_1 = \mathbb{R} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \ell_2 = \mathbb{R} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

the stem is

$$\left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} \ : \ y^2 - z^2 \leq 0 \right\}$$

and the wings are

$$\mathcal{W}_1 = \left\{ \begin{bmatrix} x \\ y \\ -y \end{bmatrix} \ : \ x \geq 0, y \in \mathbb{R} \right\}$$
$$\mathcal{W}_2 = \left\{ \begin{bmatrix} x \\ y \\ y \end{bmatrix} \ : \ x \leq 0, y \in \mathbb{R} \right\}.$$

The identity component of $\mathrm{Isom}(\mathbb{E}^{2,1})$ acts transitively on the space of pairs of vertices and unit spacelike vectors, so it is transitive on positively-oriented and negatively-oriented crooked planes. An orientation-reversing isometry exchanges positively- and negatively-oriented crooked planes, so $\mathrm{Isom}(\mathbb{E}^{2,1})$ acts transitively on the set of all crooked planes.

8.3. Topology of a crooked surface. The closures of crooked planes in Minkowski patches are *crooked surfaces.* These were studied in Frances [12]. In this section we describe the topology of a crooked surface.

Let $C \subset \mathbb{E}^{2,1}$ be a crooked plane.

Theorem 8.3.1. *The closure $\overline{C} \in \mathrm{Ein}^{2,1}$ is a topological submanifold homeomorphic to a Klein bottle. The lift of $\overline{C}$ to the double covering $\widehat{\mathrm{Ein}}^{2,1}$ is the oriented double covering of $\overline{C}$ and is homeomorphic to a torus.*

Proof. Since the isometry group of Minkowski space acts transitively on crooked planes, it suffices to consider the single crooked plane C defined in §8.2.

Recall the stratification of $\mathrm{Ein}^{2,1}$ from §3.2. Write the nullcone $\mathfrak{N}^{3,2}$ of $\mathbb{R}^{3,2}$ as

$$\begin{bmatrix} X \\ Y \\ Z \\ U \\ V \end{bmatrix} \text{ where } X^2 + Y^2 - Z^2 - UV = 0.$$

The homogeneous coordinates of points in the stem $\mathcal{S}(C)$ satisfy

$$X = 0, \qquad Y^2 - Z^2 \leq 0, \qquad V \neq 0$$

and thus the closure of the stem $\overline{\mathcal{S}(C)}$ is defined by (homogeneous) inequalities

$$X = 0, \qquad Y^2 - Z^2 \leq 0.$$

The two lightlike geodesics

$$\ell_1 = \mathbb{R}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \ell_2 = \mathbb{R}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

defining $\mathcal{S}(C)$ extend to photons ϕ_1, ϕ_2 with ideal points represented in homogeneous coordinates

$$p_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad p_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

The closures of the corresponding wings $\mathcal{W}_1, \mathcal{W}_2$ are described in homogeneous coordinates by:

$$\overline{\mathcal{W}_1} = \left\{ \begin{bmatrix} X \\ -Y \\ Y \\ U \\ V \end{bmatrix} : X^2 - UV = 0, \quad XV \geq 0 \right\}$$

$$\overline{\mathcal{W}_2} = \left\{ \begin{bmatrix} X \\ Y \\ Y \\ U \\ V \end{bmatrix} : X^2 - UV = 0, \quad XV \leq 0 \right\}.$$

The closure of each wing intersects the ideal lightcone $L(p_\infty)$ (described by $V = 0$) in the photons:

$$\psi_1 = \left\{ \begin{bmatrix} 0 \\ -Y \\ Y \\ U \\ 0 \end{bmatrix} : Y, U \in \mathbb{R} \right\}$$

$$\psi_2 = \left\{ \begin{bmatrix} 0 \\ Y \\ Y \\ U \\ 0 \end{bmatrix} : Y, U \in \mathbb{R} \right\}.$$

Thus the crooked surface $\overline{C}$ decomposes into the following strata:

- four points in a stem configuration: the vertex p_0, the improper point p_∞, and the two ideal points p_1 and p_2;

- eight line segments, the components of

$$\begin{aligned}&\phi_1 \setminus \{p_0, p_1\}\\&\phi_2 \setminus \{p_0, p_2\}\\&\psi_1 \setminus \{p_\infty, p_1\}\\&\psi_2 \setminus \{p_\infty, p_2\};\end{aligned}$$

- two null-half planes, the interiors of the wings $\mathcal{W}_1, \mathcal{W}_2$;

- the two components of the interior of the stem $\mathcal{S}$.

Recall that the inversion in the unit sphere $\iota = \mathrm{I}_3 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ fixes p_1 and p_2, and interchanges p_0 and p_∞. Moreover ι interchanges ϕ_i with ψ_i, $i = 1, 2$. Finally ι leaves invariant the interior of each $\mathcal{W}_i$ and interchanges the two components of the interior of $\mathcal{S}$.

The original crooked plane equals

$$\{p_0\} \ \cup \ \phi_1 \setminus \{p_1\} \ \cup \ \phi_2 \setminus \{p_2\} \ \cup \ \mathrm{int}(\mathcal{W}_1) \ \cup \ \mathrm{int}(\mathcal{W}_2) \ \cup \ \mathrm{int}(\mathcal{S})$$

and is homeomorphic to $\mathbb{R}^2$. The homeomorphism is depicted schematically in Figure 5. The interiors of $\mathcal{W}_1, \mathcal{W}_2$, and $\mathcal{S}$ correspond to the four quadrants in $\mathbb{R}^2$. The wing $\mathcal{W}_i$ is bounded by the two segments of ϕ_i, whereas each component of $\mathcal{S}$ is bounded by one segment of ϕ_1 and one segment of ϕ_2. These four segments correspond to the four coordinate rays in $\mathbb{R}^2$.

Now we can see that C is a topological manifold: points in $\mathrm{int}(\mathcal{W}_1), \mathrm{int}(\mathcal{W}_2)$, or $\mathrm{int}(\mathcal{S})$ have coordinate neighborhoods in these faces. Interior points of the segments have two half-disc neighborhoods, one from a wing and one from the stem. The vertex p_0 has four quarter-disc neighborhoods, one from each wing, and one from each component of the stem. (See Figure 5.)

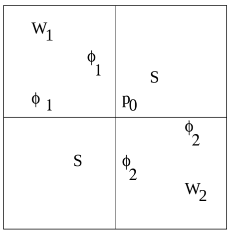

Figure 5. Flattening a crooked plane around its vertex

Coordinate charts for the improper point p_∞ and points in $\psi_i \setminus \{p_\infty, p_i\}$ are obtained by composing the above charts with the inversion ι. It remains to find coordinate charts near the ideal points p_1, p_2. Consider first the case of p_1. The linear functionals on $\mathbb{R}^{3,2}$ defined by

$$T = Y - Z$$
$$W = Y + Z$$

are null since the defining quadratic form factors:

$$X^2 + Y^2 - Z^2 - UV = X^2 + TW - UV.$$

Working in the affine patch defined by $T \neq 0$ with inhomogeneous coordinates

$$\xi := \frac{X}{T}$$
$$\eta := \frac{Y}{T}$$
$$\omega := \frac{W}{T}$$
$$\upsilon := \frac{U}{T}$$
$$\nu := \frac{V}{T}$$

the nullcone is defined by:

$$\xi^2 + \omega - \upsilon\nu = 0$$

whence

$$\omega = -\xi^2 + \upsilon\nu$$

and $(\xi, \upsilon, \nu) \in \mathbb{R}^3$ is a coordinate chart for this patch on $\mathrm{Ein}^{2,1}$.

In these coordinates, p_1 is the origin $(0,0,0)$, ϕ_1 is the line $\xi = \upsilon = 0$, and ψ_1 is the line $\xi = \nu = 0$. The wing $\mathcal{W}_2$ misses this patch, but both $\mathcal{S}$ and $\mathcal{W}_1$ intersect it. In these coordinates $\mathcal{S}$ is defined by

$$\xi = 0, \quad \omega \leq 0$$

and $\mathcal{W}_1$ is defined by

$$\xi \leq 0, \quad \omega = 0.$$

Since on $\mathcal{W}_1$

$$\upsilon\nu = \xi^2 \geq 0$$

this portion of $\mathcal{W}_1$ in this patch has two components

$$\upsilon, \nu < 0$$
$$\upsilon, \nu > 0$$

and the projection (υ, ν) defines a coordinate chart for a neighborhood of p_1. (Compare Figure 6.)

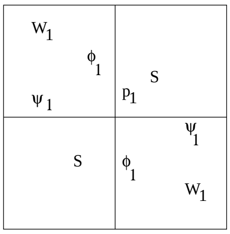

Figure 6. Flattening a crooked surface around an ideal point p_1

The case of p_2 is completely analogous. It follows that $\overline{C}$ is a closed surface with cell decomposition with four 0-cells, eight 1-cells and four 2-cells. Therefore

$$\chi(\overline{C}) = 4 - 8 + 4 = 0$$

and $\overline{C}$ is homeomorphic to either a torus or a Klein bottle.

To see that $\overline{C}$ is nonorientable, consider a photon, for example ϕ_1. Parallel translate the null geodesic $\phi_1 \setminus \{p_1\}$ to a null geodesic ℓ lying on the wing $\mathcal{W}_1$ and disjoint from $\phi_1 \setminus \{p_1\}$. Its closure $\bar{\ell} = \ell \cup \{p_1\}$ is a photon on $\overline{\mathcal{W}_1} \subset \overline{C}$ which intersects ϕ_1 transversely with intersection number 1. Thus the self-intersection number

$$\phi_1 \cdot \phi_1 = 1$$

so $\phi_1 \subset \overline{C}$ is an orientation-reversing loop. Thus $\overline{C}$ is nonorientable, and homeomorphic to a Klein bottle. ☐

Next we describe the stratification of a crooked surface in the double covering $\widehat{\mathrm{Ein}}^{2,1}$. Recall from §4.4 that a Minkowski patch in $\widehat{\mathrm{Ein}}^{2,1}$ has both a spatial and a timelike improper point. Let C be a crooked plane of $\mathbb{E}^{2,1}$, embedded in a Minkowski patch $\mathrm{Min}^+(p_\infty)$, so $p_\infty = p_\infty^{\mathrm{ti}}$, the timelike improper point of this patch. Denote by p_∞^{sp} the spatial improper point.

The closure $\overline{C}$ of C in $\widehat{\mathrm{Ein}}^{2,1}$ decomposes into the following strata:

- seven points: $p_0, p_\infty^{\mathrm{ti}}, p_\infty^{\mathrm{sp}}, p_1^{\pm}, p_2^{\pm}$;
- twelve photon segments:

$$\begin{gathered} \phi_i^{\pm},\ \text{connecting } p_0 \text{ to } p_i^{\pm} \\ \alpha_i^{\pm},\ \text{connecting } p_\infty^{\mathrm{ti}} \text{ to } p_i^{\pm} \\ \beta_i^{\pm},\ \text{connecting } p_\infty^{\mathrm{sp}} \text{ to } p_i^{\pm}; \end{gathered}$$

- two null half-planes, the interiors of $\mathcal{W}_1$ and $\mathcal{W}_2$. The wing $\mathcal{W}_i$ is bounded by the curves $\phi_i^{\pm}$ and $\beta_i^{\pm}$;
- the two components of the interior of the stem $\mathcal{S}$. The stem is bounded by the curves $\phi_i^{\pm}$ and $\alpha_i^{\pm}$, for $i = 1, 2$.

The saturation of $\overline{C}$ by the antipodal map on $\widehat{\mathrm{Ein}}^{2,1}$ is the lift of a crooked surface from $\mathrm{Ein}^{2,1}$. The interested reader can verify that it is homeomorphic to a torus.

8.4. Automorphisms of a crooked surface. Let C be the positively-oriented crooked plane of Section 8.2, and $\overline{C}$ the associated crooked surface in $\mathrm{Ein}^{2,1}$. First, C is invariant by all positive homotheties centered at the origin, because each of the wings and the stem are. Second, it is invariant by the 1-dimensional group of linear hyperbolic isometries of Minkowski space preserving the lightlike lines bounding the stem. The subgroup A, which can be viewed as the subgroup of $\mathrm{SO}(3,2)$ acting by positive homotheties and positive linear hyperbolic isometries of Minkowski space, then preserves C, and hence $\overline{C}$. The element

$$s_0 = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

is a reflection in the spine, and also preserves $\overline{C}$. Note that s_0 is time-reversing. Then we have

$$\mathbb{Z}_2 \ltimes A \cong \mathbb{Z}_2 \ltimes (\mathbb{R}^*_{>0})^2 \subset \mathrm{Aut}(\overline{C}).$$

Next let ℓ_1, ℓ_2 be the two lightlike geodesics bounding the stem (alternatively bounding the wings) of C. As above, the inversion ι leaves invariant $C \setminus (\ell_1 \cup \ell_2)$. In fact, the element

$$s_1 = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}$$

is an automorphism of $\overline{C}$. The involution

$$s_2 = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

also preserves $\overline{C}$ and exchanges the ideal points p_1 and p_2. The involutions s_0, s_1, and s_2 pairwise commute, and each product is also an involution, so we have

$$G := \mathbb{Z}_2^3 \ltimes (\mathbb{R}^*_{>0})^2 \subset \mathrm{Aut}(\overline{C})$$

To any crooked surface can be associated a quadruple of points in a stem configuration. The stabilizer of a stem configuration in $\mathrm{SO}(3,2) \cong \mathrm{PO}(3,2)$ is $N(A)$, the normalizer of a Cartan subgroup A. Suppose that the points $(p_0, p_1, p_2, p_\infty)$

are associated to $\overline{C}$. As above, a neighborhood of p_0 in $\overline{C}$ is not diffeomorphic to a neighborhood of p_1 in $\overline{C}$, so any automorphism must in fact belong to the subgroup $N'(A)$ preserving each pair $\{p_0, p_\infty\}$ and $\{p_1, p_2\}$. Each $g \in N'(A)$ either preserves $\overline{C}$ or carries it to its *opposite*, the closure of the negatively-oriented crooked plane having the same vertex and spine as C. Now it is not hard to verify that the full automorphism group of $\overline{C}$ in $\mathrm{SO}(3, 2)$ is G.

9. Construction of discrete groups

A *complete flat Lorentzian manifold* is a quotient $\mathbb{E}^{n,1}/\Gamma$, where Γ acts freely and properly discontinuously on $\mathbb{E}^{n,1}$. When $n = 2$, Fried and Goldman [15] showed that unless Γ is solvable, projection on $\mathrm{O}(2, 1)$ is necessarily injective and, furthermore, this linear part is a discrete subgroup $\Gamma_0 \subset \mathrm{O}(2, 1)$[1, 6, 24].

In this section we identify $\mathbb{E}^{2,1}$ with its usual embedding in $\mathrm{Ein}^{2,1}$, so that we consider such Γ as discrete subgroups of $\mathrm{SO}(3, 2)$. We will look at the resulting actions on Einstein space, as well as on photon space. At the end of the section, we list some open questions.

9.1. Spine reflections. In §8.4, we described the automorphism group of a crooked surface. We recall some of the basic facts about the reflection in the spine of a crooked surface, which is discussed in §3.3 and §5.2.2, and which is denoted s_0 in the example above. Take the inner product on $\mathbb{R}^{3,2}$ to be given by the matrix

$$\mathrm{I}_2 \oplus -\mathrm{I}_1 \oplus \left(-\frac{1}{2}\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and identify $\mathbb{E}^{2,1}$ with its usual embedding in the Minkowski patch determined by the improper point p_∞. Let C be the crooked plane determined by the stem configuration $(p_0, p_1, p_2, p_\infty)$ as in §8.2, with

$$p_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } p_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then s_0 is an orientation-preserving, time-reversing involution having fixed set

$$\mathrm{Fix}(s_0) \;=\; \{p_1, p_2\} \,\cup\, \big(L(p_1) \cap L(p_2)\big).$$

In the Minkowski patch, $\langle s_0 \rangle$ interchanges the two components of the complement of C.

If a set of crooked planes in $\mathbb{E}^{2,1}$ is pairwise disjoint, then the group generated by reflections in their spines acts properly discontinuously on the entire space [7, 8, 10]. Thus spine reflections associated to disjoint crooked planes give rise to discrete subgroups of $\mathrm{SO}(3,2)$. We will outline a way to construct such groups; see [4], for details.

Let $S_1,\ S_2 \subset \mathrm{Ein}^{2,1}$ be a pair of spacelike circles that intersect in a point; conjugating if necessary, we may assume that this point is p_∞. Each circle S_i, $i = 1,2$, is the projectivized nullcone of a subspace $V_i \subset \mathbb{R}^{3,2}$ of type (2,1); $V_1 + V_2$ can be written as the direct sum

$$\mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus W,$$

where $v_1,\ v_2$ are spacelike vectors and $W = V_1 \cap V_2$ is of type (1,1). We call $\{S_1, S_2\}$ an *ultraparallel pair* if $v_1^\perp \cap v_2^\perp$ is spacelike. Alternatively, we can define the pair to be ultraparallel if they are parallel to vectors $u_1,\ u_2 \in \mathbb{R}^{2,1}$ such that $u_1^\perp \cap u_2^\perp$ is a spacelike line in $\mathbb{E}^{2,1}$.

Let S_1, S_2 be an ultraparallel pair of spacelike circles in $\mathrm{Ein}^{2,1}$. Denote by ι_1 and ι_2 the spine reflections fixing the respective circles. (Note that ι_1 and ι_2 are conjugate to s_0, since $\mathrm{SO}(3,2)$ acts transitively on crooked surfaces.) Identifying the subgroup of $\mathrm{SO}(3,2)$ fixing p_0 and p_∞ with the group of Lorentzian linear similarities

$$\mathrm{Sim}(\mathbb{E}^{2,1}) = \mathbb{R}_+ \cdot \mathrm{O}(2,1),$$

then $\gamma = \iota_2 \circ \iota_1$ has hyperbolic linear part—that is, it has three, distinct real eigenvalues. The proof of this fact and the following proposition may be found, for instance, in [4].

Proposition 9.1.1. *Let S_1 and S_2 be an ultraparallel pair of spacelike circles as above. Then S_1 and S_2 are the spines of a pair of disjoint crooked planes, bounding a fundamental domain for $\langle\gamma\rangle$ in $\mathbb{E}^{2,1}$.*

Note that while $\langle\gamma\rangle$ acts freely and properly discontinuously on $\mathbb{E}^{2,1}$, it fixes p_∞ as well as two points on the ideal circle.

Next, let S_i, $i = 1,2,3$ be a triple of pairwise ultraparallel spacelike circles, all intersecting in p_∞, and let $\Gamma = \langle \iota_1, \iota_2, \iota_3 \rangle$ be the associated group of spine reflections. Then Γ contains an index-two free group generated by hyperbolic isometries of $\mathbb{E}^{2,1}$ (see [4]). Conversely, we have the following generalization of a well-known theorem in hyperbolic geometry.

Theorem 9.1.2. *[4] Let $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1\gamma_2\gamma_3 = Id \rangle$ be a subgroup of isometries of $\mathbb{E}^{2,1}$, where each γ has hyperbolic linear part and such that their invariant lines are pairwise ultraparallel. Then there exist spine reflections ι_i, $i = 1,2,3$, such that $\gamma_1 = \iota_1\iota_2$, $\gamma_2 = \iota_2\iota_3$ and $\gamma_3 = \iota_3\iota_1$.*

Note that Γ as above is discrete. Indeed, viewed as a group of affine isometries of $\mathbb{E}^{2,1}$, its linear part $G \leq \mathrm{O}(2,1)$ acts on the hyperbolic plane and is generated by reflections in three ultraparallel lines. As mentioned before, if the spacelike circles are spines of pairwise disjoint crooked planes, then Γ acts properly discontinuously on the Minkowski patch. Applying this strategy, we obtain that the set of all properly discontinuous groups Γ, with linear part generated by three ultraparallel reflections, is non-empty and open [4].

Here is an example. For $i = 1, 2, 3$, let $V_i \subset \mathbb{R}^{3,2}$ be the $(2,1)$-subspace

$$V_i = \left\{ \begin{bmatrix} au_i + cp_i \\ a\langle u_i, p_i\rangle + b + c\langle p_i, p_i\rangle \\ c \end{bmatrix} \;\mid\; a, b, c \in \mathbb{R} \right\},$$

where

$$\begin{aligned} u_1 &= \begin{bmatrix} \sqrt{2} & 0 & 1 \end{bmatrix}^\dagger & u_2 &= \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & 1 \end{bmatrix}^\dagger & u_3 &= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{2} & 1 \end{bmatrix}^\dagger \\ p_1 &= \begin{bmatrix} 0 & \sqrt{2} & 1 \end{bmatrix}^\dagger & p_2 &= \begin{bmatrix} -\frac{\sqrt{6}}{2} & -\frac{\sqrt{2}}{2} & 1 \end{bmatrix}^\dagger & p_3 &= \begin{bmatrix} \frac{\sqrt{6}}{2} & -\frac{\sqrt{2}}{2} & 1 \end{bmatrix}^\dagger. \end{aligned}$$

Then the projectivized nullcone of V_i is a spacelike circle—in fact, it corresponds to the spacelike geodesic in $\mathbb{E}^{2,1}$ passing through p_i and parallel to u_i. The crooked planes with vertex p_i and spine $p + \mathbb{R}u_i$, respectively, are pairwise disjoint (one shows this using inequalities found in [10]).

9.2. Actions on photon space.

Still in the same Minkowski patch as above, let G be a finitely generated discrete subgroup of $\mathrm{O}(2,1)$ that is free and *purely hyperbolic*—that is, every nontrivial element is hyperbolic. Considered as a group of isometries of the hyperbolic plane, G is a convex cocompact free group. By Barbot [2],

Theorem 9.2.1. *Let Γ be a subgroup of isometries of $\mathbb{E}^{2,1}$ with convex cocompact linear part. Then there is a pair of non-empty, Γ-invariant, open, convex sets $\Omega^\pm \subset \mathbb{E}^{2,1}$ such that*

- *The action of Γ on $\Omega^\pm$ is free and proper;*
- *The quotient spaces $\Omega^\pm/\Gamma$ are globally hyperbolic;*
- *Each $\Omega^\pm$ is maximal among connected open domains satisfying these two properties;*
- *The only open domains satisfying all three properties above are $\Omega^\pm$.*

The notion of global hyperbolicity is central in General Relativity, see for example [3]. The global hyperbolicity of $\Omega^\pm/\Gamma$ implies that it is homeomorphic to the product $(\mathbf{H}^2/G) \times \mathbb{R}$. It also implies that no element of Γ preserves a null ray in $\Omega^\pm$.

Let Γ be as in Theorem 9.2.1 and consider its action, for instance, on Ω^+. Since Ω^+/Γ is globally hyperbolic, it admits a *Cauchy hypersurface*, a spacelike surface S_0 which meets every complete causal curve and with complement consisting of two connected components. The universal covering $\widetilde{S_0}$ is Γ-invariant. The subset $\mathrm{Pho}_0^{2,1} \subset \mathrm{Pho}^{2,1}$ comprising photons which intersect $\widetilde{S_0}$ is open.

We claim that Γ acts freely and properly on $\mathrm{Pho}_0^{2,1}$. Indeed, let $K \leq \mathrm{Pho}_0^{2,1}$ be a compact set. Then K is contained in a product of compact subsets $K_1 \times K_2$, where $K_1 \subset \widetilde{S}_0$ and $K_2 \subset S^1$, the set of photon directions. The action of Γ restricts to a Riemannian action on $\widetilde{S}_0$. Thus the set

$$\{\gamma \in \Gamma \mid \gamma(K_1) \cap K_1 \neq \emptyset\}$$

is finite. As $\widetilde{S}_0$ is spacelike, it follows that $\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\}$ is finite too. Finally, global hyperbolicity of Ω^+/Γ implies that no photon intersecting Ω^+ is invariant under the action of any element of Γ.

Corollary 9.2.2. *There exists a non-empty open subset of* $\mathrm{Pho}^{2,1}$ *on which* Γ *acts freely and properly discontinuously.*

9.3. Some questions. So far we have considered groups of transformations of $\mathrm{Ein}^{2,1}$ and $\mathrm{Pho}^{2,1}$ arising from discrete groups of Minkowski isometries. Specifically, we have focused on groups generated by spine reflections associated to spacelike circles intersecting in a point.

Question: Describe the action on $\mathrm{Ein}^{2,1}$ of a group generated by spine reflections corresponding to non-intersecting spacelike circles. In particular, determine the possible dynamics of such an action.

A related question is:

Question: What does a crooked surface look like when its spine does not pass through p_∞, or the lightcone at infinity altogether? Describe the action of the associated group of spine reflections.

More generally, we may wish to consider other involutions in the automorphism group of a crooked surface.

Question: Describe the action on $\mathrm{Ein}^{2,1}$ of a group generated by involutions, in terms of their associated crooked surfaces.

As for the action on photon space, here is a companion question to those asked in §7:

Question: Given a group generated by involutions, what is the maximal open subset of $\mathrm{Pho}^{2,1}$ on which the group acts properly discontinuously?

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