A Frobenius theorem for Cartan geometries, with applications

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August 8, 2011

Abstract

We prove analogues for Cartan geometries of Gromov's major theorems on automorphisms of rigid geometric structures. The starting point is a Frobenius theorem, which says that infinitesimal automorphisms of sufficiently high order integrate to local automorphisms. Consequences include a stratification theorem describing the configuration of orbits for local Killing fields in a compact real-analytic Cartan geometry, and an open-dense theorem in the smooth case, which says that if there is a dense orbit, then there is an open, dense, locally homogeneus subset. Combining the Frobenius theorem with the embedding theorem of Bader, Frances, and the author gives a representation theorem that relates the fundamental group of the manifold with the automorphism group.

1 Introduction

The classical result on local orbits in geometric manifolds is Singer's homogeneity theorem for Riemannian manifolds [1]: given a Riemannian manifold M, there exists k, depending on $\dim M$, such that if every $x, y \in M$ are related by an infinitesimal isometry of order k, then M is locally homogeneous. An *infinitesimal isometry of order k* is a linear isometry from T_xM to T_yM that pulls back the curvature tensor at y and its covariant derivatives up to order k to those at x. An open subset $U \subseteq M$ of a geometric manifold

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is *locally homogeneous* if for every $x, x' \in U$, there is a *local automorphism* f in U with $f(x) = x'$. Such a local automorphism is a diffeomorphism from a neighborhood V of x in U to a neighborhood of x' in U, with f an isomorphism between the geometric structures restricted to V and $f(V)$.

Gromov extended Singer's theorem to manifolds with *rigid geometric structures of algebraic type* in [2, 1.6.G]. He also proved the celebrated open-dense theorem ([2, 3.3.A]) and a stratification for orbits of local automorphisms of such structures on compact real-analytic manifolds (see [2, 3.4] and [3, 3.2.A]). The open-dense theorem says that if M is a smooth manifold with smooth rigid geometric structure of algebraic type, and if there is an orbit for local automorphisms that is dense in M, then M contains an open, dense, locally homogeneous subset. A crucial ingredient for Gromov's theorems is his difficult Frobenius theorem, which says that infinitesimal isometries of sufficiently high order can be integrated to local isometries near any point on a real-analytic manifold, and near *regular* points in the smooth case. Infinitesimal isometries in this context are a suitable generalization of the definition in the previous paragraph.

This article treats *Cartan geometries*, a notion of geometric structure less flexible than Gromov's rigid geometric structures, but still including essentially all classical geometric structures with finite-dimensional automorphism groups, such as pseudo-Riemannian metrics, conformal pseudo-Riemannian structures in dimension at least 3, and a broad class of CR structures. The author does not know whether every Cartan geometry determines a rigid geometric structure à la Gromov, but strongly suspects not. The central result is a Frobenius theorem for Cartan geometries (3.11, 6.3), which is considerably easier in this setting, and is in fact broadly modeled on the paper [4] of Nomizu from 1960 treating Riemannian isometries (see also [5]).

From the Frobenius theorem we obtain the stratification and open-dense theorems as in [2] for local Killing fields of Cartan geometries (4.1, 6.4). The embedding theorem for automorphism groups of Cartan geometries proved in [6], combined with the Frobenius theorem, gives rise to centralizer and π_1-representation theorems for real-analytic Cartan geometries (5.4, 5.9), which can be formulated for actions that do not necessarily preserve a finite volume.

A Cartan geometry infinitesimally models a manifold on a homogeneous space.

Definition 1.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $P < G$ a closed subgroup with Lie algebra $\mathfrak{p}$. A* Cartan geometry *on a manifold M modeled on the pair $(\mathfrak{g}, P)$ is a triple (M, B, ω) where $\pi : B \to M$ is a principal P-bundle over M, and ω is a $\mathfrak{g}$-valued 1-form on B satisfying*

1. *$\omega_b : T_b B \to \mathfrak{g}$ is a linear isomorphism for all $b \in B$*
2. *for all $X \in \mathfrak{p}$, if $X^{\ddagger}$ is the fundamental vector field on B corresponding to X, then $\omega_b(X^{\ddagger}) = X$ at all $b \in B$.*
3. *$R_g^* \omega = Ad\ g^{-1} \circ \omega$ for all $g \in P$*

Definition 1.2. *Let (M, B, ω) be a Cartan geometry. An* automorphism *of (M, B, ω) is a diffeomorphism f of M that lifts to a bundle automorphism $\tilde{f}$ of B satisfying $\tilde{f}^* \omega = \omega$. The group of all such automorphisms will be denoted, somewhat abusively, by Aut M below.*

An important fact that will be used throughout the sequel is that Aut M, when lifted to B, acts freely. In fact, a local automorphism f defined on $U \subset M$ is determined by $\tilde{f}(b)$ for any $b \in \pi^{-1}(U)$; similarly, a local Killing field X defined on $U \subset M$ lifts to $\tilde{X}$ on $\pi^{-1}(U)$ and is determined by $\tilde{X}(b)$ for any $b \in \pi^{-1}(U)$. This freeness follows from the fact that (local) automorphisms preserve a framing on B, so act freely and properly (see [7, I.3.2]).

Examples.

Riemannian and pseudo-Riemannian metrics. The canonical Cartan geometry associated to a Riemannian metric on M^n is modeled on the pair $(\mathfrak{isom}(\mathbf{R}^n), \mathrm{O}(n))$, where $\mathfrak{isom}(\mathbf{R}^n)$ is the Lie algebra of $\mathrm{Isom}(\mathbf{R}^n) \cong \mathrm{O}(n) \ltimes \mathbf{R}^n$. The bundle of orthonormal frames on M is the principal $\mathrm{O}(n)$-bundle B. The Levi-Civita connection gives an $\mathfrak{o}(n)$-valued 1-form ν on B with the appropriate $\mathrm{O}(n)$-eqivariance. The Cartan connection ω is the sum of ν and the n tautological 1-forms on B.

Now if M carries a pseudo-Riemannian metric of type (p, q), where $p + q = n$, then there is again a Levi-Civita connection. The model pair is $(\mathfrak{isom}(\mathbf{R}^{p,q}), \mathrm{O}(p, q))$, and B is the analogous bundle of normalized frames, in which the metric takes the standard form. Again, ω is the sum of the Levi-Civita connection with the tautological 1-forms.

The automorphisms of the resulting (M, B, ω) are exactly the isometries of the metric.

Conformal structures. A conformal Riemannian structure on M^n, $n \geq 3$, yields a canonical Cartan geometry modeled on the round sphere S^n, as a homogeneous space corresponding to the pair $(\mathfrak{o}(1, n+1), P)$, where $P < \mathrm{O}(1, n+1) \cong \mathrm{Conf}\, S^n$ is the stabilizer of a point of S^n. It is a parabolic subgroup isomorphic to $(\mathbf{R}^* \times \mathrm{SO}(n)) \ltimes \mathbf{R}^n$. The bundle B is a subset of the second-order frame bundle of M, comprising 2-jets of local diffeomorphisms from $\mathbf{R}^n$ to M that are conformal to order 2 at the origin. The existence of a canonical form ω on B was proved by Cartan.

More generally, if M has a type-(p, q) conformal structure, $p + q \geq 3$, then there is a canonical Cartan geometry modeled on the pseudo-Riemannian generalization of the round sphere, namely the Einstein space $\mathrm{Ein}^{p,q}$. The Lie algebra of $\mathrm{Conf}\, \mathrm{Ein}^{p,q}$ is $\mathfrak{o}(p+1, q+1)$, and the stabilizer of a point is a maximal parabolic $P \cong \mathrm{CO}(p, q) \ltimes \mathbf{R}^{p,q}$.

The automorphisms of the resulting (M, B, ω) are the conformal transformations of M.

Nondegenerate CR-structures. These structures model real hypersurfaces in complex manifolds. A nondegenerate strictly pseudoconvex CR structure on a $(2m - 1)$-dimensional manifold M is the data of a contact subbundle $E \subset TM$ equipped with an almost-complex structure J and a conformal class of positive definite Hermitian metrics. Such a structure is equivalent to a canonical Cartan geometry modeled on $\mathrm{SU}(1, m)/P$, where P is the parabolic subgroup stabilizing an isotropic complex line in $\mathbf{C}^{1,m}$. This homogeneous space is the boundary of complex hyperbolic space $\mathbf{CH}^m$.

More generally, if $p + q = m - 1$, then a nondegenerate CR structure of type (p, q) is as above, but with the conformal class of Hermitian metrics of signature (p, q). One of these structures is equivalent to a canonical Cartan geometry modeled on $\mathrm{SU}(p+1, q+1)/P$, where P is again a maximal parabolic subgroup stabilizing a null line in the standard representation on $\mathbf{C}^{m+1}$.

The equivalence problem for strictly pseudo-convex CR-structures was first solved by E. Cartan in dimension 3, and in the general case in [8], [9], [10].

Let (M, B, ω) be a Cartan geometry modeled on G/P. We will make the following standard assumptions on G/P:

1. G is connected.

2. P contains no nontrivial normal subgroup of G. (Suppose that $N \triangleleft G$ were such a subgroup. Then let $G' = G/N$ and $P' = P/N$. If (M, B, ω) is a Cartan geometry modeled on $(\mathfrak{g}, P)$, then ω descends to a $\mathfrak{g}'$-valued 1-form ω' on $B' = B/N$, giving a Cartan geometry $(M, B/N, \omega')$ modeled on G'/P'.)

3. P is an analytic subgroup of G.

In section 5 we will further assume that $\mathrm{Ad}_{\mathfrak{g}} P$ is an algebraic subgroup of $\mathrm{Aut}\, \mathfrak{g}$. In this case, the Cartan geometry (M, B, ω) is said to be *algebraic type*.

Acknowledgements: I thank David Fisher for helpful conversations about this project, and Ben McKay for bringing to my attention an error in a previous version of this article. I also thank Charles Frances, Gregory Margulis, and Amir Mohammadi for illuminating conversations on related topics. I am grateful to the referees for their helpful comments.

2 Baker-Campbell-Hausdorff formula

The main proposition of this section, proposition 2.1, asserts that the usual BCH formula holds to any finite order with ω-constant vector fields on B in place of left-invariant vector fields. When (M, B, ω) is real-analytic, this formula gives the Taylor series at each point of B for the flow along two successive ω-constant vector fields, in terms of the exponential coordinates.

For $X, Y \in \mathfrak{g}$, define

$$\begin{aligned} \alpha &: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\ \alpha &: (X, Y) \mapsto \log_e(\exp_e X \cdot \exp_e Y) \end{aligned}$$

where $\exp_e$ is the group exponential map $\mathfrak{g} \cong T_e G \to G$, and $\log_e$ the inverse of $\exp_e$. The exponential map of G can be considered a function $G \times \mathfrak{g} \to G$, with

$$\exp(g, X) = \exp_g X = g \cdot \exp_e X$$

It is the flow for time 1 with initial value g along the left-invariant vector field corresponding to X. Note that

$$\exp_e X \cdot \exp_e Y = \exp(\exp(e, X), Y)$$

For any $k \in \mathbf{N}$, there exist functions $a_1, \ldots, a_k$, and R of (X, Y) such that

$$\alpha(tX, tY) = ta_1(X, Y) + \cdots + \frac{t^k}{k!} a_k(X, Y) + t^k R(tX, tY)$$

where

$$\lim_{t \to 0} R(tX, tY) = 0$$

These functions are given by the BCH formula, and they are rational multiples of iterated brackets of X and Y. For example,

$$\begin{aligned} a_1(X, Y) &= X + Y \\ a_2(X, Y) &= [X, Y] \end{aligned}$$

and

$$a_3(X, Y) = \frac{1}{2}([X, [X, Y]] + [Y, [Y, X]])$$

For any Lie algebra $\mathfrak{u}$, not necessarily finite-dimensional, with a linear injection $\rho : \mathfrak{g} \to \mathfrak{u}$, the functions a_k define obvious functions $a_k : \rho(\mathfrak{g}) \to \mathfrak{u}$, evaluated by taking iterated brackets in $\mathfrak{u}$.

In the bundle B of the Cartan geometry, denote by exp the exponential map $B \times \mathfrak{g} \to B$, defined on a neighborhood of $B \times \{0\}$ and by $\log_b$ the inverse of $\exp_b$, defined on a neighborhood of b. For any $b \in B$, define, for sufficiently small $X, Y \in \mathfrak{g}$

$$\zeta_b(X, Y) = \log_b(\exp(\exp(b, X), Y))$$

As above, there exist functions $z_1, \ldots, z_k$, corresponding to the time derivatives of $\zeta_b(tX, tY)$ up to order k, and a remainder function.

Proposition 2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and (M, B, ω) a Cartan geometry modeled on $(\mathfrak{g}, P)$. Let a_k and z_k be the coefficients of $t^k/k!$ in the respective order-k Taylor approximations of the above functions α and ζ_b. Then*

$$z_k(X, Y) = \omega_b(a_k(\widetilde{X}, \widetilde{Y}))$$

where $\widetilde{X}$ and $\widetilde{Y}$ are the ω-constant vector fields on B corresponding to X and Y, respectively.

Proof: Fix $X, Y \in \mathfrak{g}$, and let $Z(t) = \zeta_b(tX, tY)$. The following lemmas give two different ways to compute, for an arbitrary C^k function φ on B and $b \in B$, the derivative

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right|_0 \varphi(\exp(b, Z(t)))$$

Lemma 2.2. *(compare [11, I.1.88]) For $X \in \mathfrak{g}$, $b \in B$, and $\varphi \in C^k(B)$,*

$$\left.\frac{d^k}{dt^k}\right|_0 \varphi(\exp(b, tX)) = \left.\widetilde{X}^k.\varphi\right|_b$$

Proof: For $k = 1$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \varphi(\exp(b, tX)) = \varphi_{*b}((\exp_b)_*(X)) = \left.\widetilde{X}.\varphi\right|_b$$

Now let $n \geq 1$ and suppose that the formula holds for all $k \leq n$. Then

$$\begin{aligned}
\left.\widetilde{X}^{n+1}.\varphi\right|_b &= \left.\widetilde{X}.\widetilde{X}^n.\varphi\right|_b \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\widetilde{X}^n.\varphi)(\exp(b, tX)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}^n}{\mathrm{d}s^n}\right|_0 \varphi(\exp(\exp(b, tX), sX)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}^n}{\mathrm{d}s^n}\right|_0 \varphi(\exp(b, (t+s)X)) \\
&= \left.\frac{\mathrm{d}^{n+1}}{\mathrm{d}u^{n+1}}\right|_0 \varphi(\exp(b, uX))
\end{aligned}$$

where $u = t + s$. $\diamondsuit$

Corollary 2.3. *For $X, Y \in \mathfrak{g}$,*

$$\left.\frac{d^k}{dt^k}\right|_0 \varphi(\exp(\exp(b, tX), tY)) = \sum_{m+n=k} \frac{k!}{m!n!} \left.\widetilde{X}^n.\widetilde{Y}^m.\varphi\right|_b$$

Proof: By two applications of lemma 2.2,

$$\begin{aligned}
\left.\widetilde{X}^n.\widetilde{Y}^m.\varphi\right|_b &= \left.\frac{\mathrm{d}^n}{\mathrm{d}s^n}\right|_0 (\widetilde{Y}^m.\varphi)(\exp(b, sX)) \\
&= \left.\frac{\mathrm{d}^n}{\mathrm{d}s^n}\right|_0 \left.\frac{\mathrm{d}^m}{\mathrm{d}t^m}\right|_0 \varphi(\exp(\exp(b, sX), tY))
\end{aligned}$$

The desired formula follows. $\diamondsuit$

Lemma 2.4. *Let $Z(t)$ be a curve in $\mathfrak{g}$ with $Z(0) = 0$, $b \in B$, and $\varphi \in C^k(B)$. Then*

$$\frac{d^k}{dt^k}\bigg|_0 \varphi(\exp_b Z(t)) = \frac{d^k}{dt^k}\bigg|_0 \left[\sum_{n=0}^{k} \frac{1}{n!}\, [t\widetilde{Z}_0 + \cdots + t^k \widetilde{Z}_k]^n.\varphi\Big|_b\right]$$

where

$$Z(t) = tZ_1 + \cdots + t^k Z_k$$

is the k^{th} Taylor polynomial for Z, and $\widetilde{Z}_i$ is the ω-constant vector field on B with value Z_i.

Proof: Taylor's formula for Lie groups, which also applies in the total space of a Cartan geometry, says that for X in a bounded neighborhood of 0 in $\mathfrak{g}$,

$$\varphi(\exp_b(X)) = \sum_{n=0}^{k} \frac{1}{n!} (\widetilde{X}^n.\varphi)(b) + R_k(X)$$

where, in any norm on $\mathfrak{g}$, there is C_k such that $|R_k(X)| \leq C_k |X|^{k+1}$ (see [11, I.1.88]).

To compute the derivative $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\big|_0 \varphi(\exp_b Z(t))$, one can substitute the k^{th} Taylor polynomial for Z in the Taylor formula above and ignore the remainder term. $\diamondsuit$

Now

$$\exp(\exp(b, tX), tY) = \exp(b, \zeta_b(tX, tY)) = \exp_b(Z(t))$$

for $Z(t) = \zeta_b(tX, tY)$. Note that $Z_k = z_k(X, Y)/k!$. Corollary 2.3 gives

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\bigg|_0 \varphi(\exp_b Z(t)) = \sum_{m+n=k} \frac{k!}{m!n!}\, \widetilde{X}^n.\widetilde{Y}^m.\varphi\Big|_b$$

On the other hand, lemma 2.4 gives for the same derivative

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\bigg|_0 \varphi(\exp_b Z(t)) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\bigg|_0 \left[\sum_{n=0}^{k} \frac{1}{n!}\, [t\widetilde{Z}_1 + \cdots + t^k \widetilde{Z}_k]^n.\varphi\Big|_b\right]$$

With these two formulas, the coefficients $z_k(X, Y) = k!Z_k$ can be recursively computed in terms of products of $\widetilde{X}$ and $\widetilde{Y}$. Of course, these formulas hold in the group G with the usual exponential map, so they yield the same expressions, actually involving brackets of $\widetilde{X}$ and $\widetilde{Y}$, for $a_k(X, Y)$ and $z_k(X, Y)$. $\diamondsuit$

3 Frobenius theorem

Throughout this section, (M, B, ω) is a Cartan geometry modeled on $(\mathfrak{g}, P)$. Soon we will impose the assumption that (M, B, ω) is C^ω.

The exponential map of (M, B, ω) satisfies the following P-equivariance relation, which is an easy consequence of part 3 of definition 1.1:

$$\exp(bp^{-1}, X) = \exp(b, (\mathrm{Ad}\, p^{-1})X)p^{-1}$$

The curvature of a Cartan geometry is a $\mathfrak{g}$-valued 2-form on B defined by

$$\Omega(X, Y) = \mathrm{d}\omega(X, Y) + [\omega(X), \omega(Y)]$$

If $X \in \mathfrak{p}$, then $\Omega(X, Y)$ vanishes [12, 5.3.10]. Let

$$V = (\wedge^2(\mathfrak{g}/\mathfrak{p})^*) \otimes \mathfrak{g}$$

The form ω gives an identification $TB \cong B \times \mathfrak{g}$, under which the curvature corresponds to a function $K : B \to V$

$$K : b \mapsto (\omega_b^{-1} \circ \sigma)^* \Omega_b$$

where σ is any linear section $\mathfrak{g}/\mathfrak{p} \to \mathfrak{g}$.

The group P acts on V linearly by

$$(p.\varphi)(u, v) = (\mathrm{Ad}\, p \circ \varphi)((\overline{\mathrm{Ad}}\, p^{-1})u, (\overline{\mathrm{Ad}}\, p^{-1})v)$$

where $\overline{\mathrm{Ad}}$ is the quotient representation of $\mathrm{Ad}\, P$ on $\mathfrak{g}/\mathfrak{p}$. The curvature map is P-equivariant [12, 5.3.23]:

$$K(bp^{-1}) = p.K(b)$$

For $m \in \mathbf{N}$, define the ω-*derivative* of order m of K

$$\begin{aligned} D^m K &: B \to \mathrm{Hom}(\otimes^m \mathfrak{g}, V) \\ D^m K(b) &: X_1 \otimes \cdots \otimes X_m \mapsto (\widetilde{X}_1 \ldots \widetilde{X}_m.K)(b) \end{aligned}$$

where, as above, $\widetilde{X}$ is the ω-constant vector field on B with value X. Note that $D^m K(b)$ is not a symmetric homomorphism, because the ω-constant vector fields $\widetilde{X}$ do not come from coordinates on B. Neither can it be interpreted as a tensor on B, because $D^m K(b)$ is not linear over the ring of functions $C^\infty(B)$. It does suffice, however, to determine the m-jet $j_b^m K$, because any vector field on B is a $C^\infty(B)$-linear combination of ω-constant vector fields.

Proposition 3.1. *The ω-derivative is P-equivariant for each $m \geq 0$:*

$$D^m K(bp^{-1}) = p \circ D^m K(b) \circ Ad\ ^m p^{-1}$$

where $Ad\ ^m$ is the tensor representation on $\otimes^m \mathfrak{g}$ of $Ad\ P$.

Proof: The assertion holds for $m = 0$ by the equivariance of K cited above. Suppose it holds for all $m \leq r$. Then for any $X_1, \ldots, X_{r+1} \in \mathfrak{g}$,

$$\begin{aligned}
(\widetilde{X}_1 \ldots \widetilde{X}_{r+1}.K)(bp^{-1}) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 (\widetilde{X}_2 \ldots \widetilde{X}_{r+1}.K)(\exp(bp^{-1}, tX_1)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 (\widetilde{X}_2 \ldots \widetilde{X}_{r+1}.K)(\exp(b, (\mathrm{Ad}\ p^{-1})tX_1)p^{-1}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 p \circ (D^r K(\exp(b, (\mathrm{Ad}\ p^{-1})tX_1))((\mathrm{Ad}\ p^{-1})X_2, \ldots, (\mathrm{Ad}\ p^{-1})X_{r+1})) \\
&= p \circ (D^{r+1} K(b)((\mathrm{Ad}\ p^{-1})X_1, \ldots, (\mathrm{Ad}\ p^{-1})X_{r+1}))
\end{aligned}$$

so by induction it is true for all $m \geq 0$. $\diamondsuit$

Definition 3.2. *For $m \geq 1$, two points b, b' of B are m-*related *if*

$$D^r K(b) = D^r K(b')$$

*for all $1 \leq r \leq m$. They are ∞-*related *if they are m-related for all m.*

For $\varphi \in \mathrm{Hom}(\otimes^r \mathfrak{g}, V)$ and $X \in \mathfrak{g}$, the contraction $\varphi \llcorner X \in \mathrm{Hom}(\otimes^{r-1}\mathfrak{g}, V)$ is given by

$$(\varphi \llcorner X)(X_1, \ldots, X_{r-1}) = \varphi(X, X_1, \ldots, X_{r-1})$$

Definition 3.3. *For $m \geq 1$, the* Killing generators of order m *at $b \in B$, denoted $Kill^m(b)$, comprise all $A \in \mathfrak{g}$ such that, for all $1 \leq r \leq m$, the contraction*

$$D^r K(b) \llcorner A = 0 \in Hom(\otimes^{r-1}\mathfrak{g}, V)$$

The Killing generators *at $b \in B$ are*

$$Kill^\infty(b) = \bigcap_m Kill^m(b)$$

Note that $\mathrm{Kill}^m(b)$ is a subspace of $\mathfrak{g}$ for all $m \in \mathbf{N} \cup \{\infty\}$. Moreover,

$$\mathrm{Kill}^m(bp^{-1}) = (\mathrm{Ad}\ p)(\mathrm{Kill}^m(b))$$

Then define

$$k_m(x) = \dim \mathrm{Kill}^m(b) \qquad k(x) = \dim \mathrm{Kill}^\infty(b)$$

for any $b \in \pi^{-1}(x)$. Note that for each m, the function $k_m(x)$ is lower semicontinuous—that is, each $x \in B$ has a neighborhood U with $k_m(y) \leq k_m(x)$ for all $y \in U$. The same is true for $k(x)$.

The goal is to show that m-related points, for m sufficiently large, are actually related by local automorphisms, and that Killing generators of sufficiently high order give rise to local Killing fields.

Definition 3.4. *A* local automorphism *between points b and b' of B is a diffeomorphism f from a neighborhood of b to a neighborhood of b' such that $f^*\omega = \omega$. A* local automorphism *between x and x' in M is a diffeomorphism from a neighborhood U of x to a neighborhood U' of x' inducing an isomorphism of the Cartan geometries $(U, \pi^{-1}(U), \omega)$ and $(U', \pi^{-1}(U'), \omega)$.*

Definition 3.5. *A* local Killing field *near $b \in B$ is a vector field $\widetilde{A}$ defined on a neighborhood of b such that the flow along $\widetilde{A}$, where it is defined, preserves ω. A* local Killing field *near $x \in M$ is a vector field A near x such that the flow φ^t_A along A, if it is defined on a neighborhood $U \times (-\epsilon, \epsilon)$ of $(x, 0)$, gives an isomorphism of the restricted Cartan geometry on U with the restricted Cartan geometry on $\varphi^t_A(U)$ for all $t \in (-\epsilon, \epsilon)$.*

Note that a local automorphism between $x, x' \in M$ lifts to a local automorphism from any $b \in \pi^{-1}(x)$ to some $b' \in \pi^{-1}(x')$. Similarly, a local Killing field near $x \in M$ lifts to a local Killing field near any $b \in \pi^{-1}(x)$. Local automorphisms and Killing fields on B also descend to M; further, the resulting correspondences are bijective, as the next two propositions show.

Proposition 3.6. *Let f be a nontrivial local automorphism between points b and b' of B. Then f descends to a nontrivial local automorphism $\bar{f}$ between $\pi(b)$ and $\pi(b')$ in M.*

Proof: Denote by P^0 the identity component of P. In order to ensure that f commutes with P, we assume it is defined on a connected neighborhood U of b with $U \cap UP \subseteq UP^0$. The P^0-action is generated by flows along the ω-constant vector fields $X^\ddagger$ with $X \in \mathfrak{p}$. Because f_* preserves all ω-constant vector fields, it commutes with the P^0-action. Now there is a well-defined extension of f to UP with $f(qp) = f(q)p$ for any $q \in B, p \in P$. Note that the extended f still preserves ω: if $q \in U$, $p \in P$, then

$$\begin{aligned} \omega_{f(qp)} \circ f_{*qp} &= \omega_{f(q)p} \circ (R_p)_* \circ f_{*q} \circ (R_p)_*^{-1} \\ &= (\mathrm{Ad}\ p^{-1}) \circ \omega_{f(q)} \circ f_{*q} \circ (R_p)_*^{-1} \\ &= (\mathrm{Ad}\ p^{-1}) \circ \omega_q \circ (R_p)_*^{-1} \\ &= \omega_{qp} \end{aligned}$$

Now f descends to a diffeomorphism $\bar{f}$ on $\pi(U) \subset M$, and this diffeomorphism is a local automorphism carrying $\pi(b)$ to $\pi(b')$.

Suppose that $\bar{f}$ were the identity on $\pi(U)$. Then f would have the form

$$f(b) = b \cdot (\rho \circ \pi)(b)$$

for $\rho : \pi(U) \to P$. Let N be the subgroup of P generated by the image of ρ. We will show N is a normal subgroup of G contained in P, contradicting the global assumptions on G and P.

On one hand, $f^*\omega = \omega$, while also

$$(f^*\omega)_b = (\mathrm{Ad}\ \circ \rho \circ \pi)(b)^{-1} \circ \omega_b + (\rho \circ \pi)_*$$

(see [12, 3.4.12]). Then for any $Y \in \mathfrak{g}$ and $x \in \pi(U)$,

$$Y = ((\mathrm{Ad}\ \circ \rho)(x))^{-1} Y + (\rho \circ \pi)_* Y$$

So $(\mathrm{Ad}\ g)(Y) - Y \in \mathfrak{n}$, the Lie algebra of N, for all $g \in N$. Since G is connected, it follows that $hgh^{-1} \in N$ for all $h \in G$, $g \in N$. $\diamondsuit$

Similarly, because local Killing fields in B commute with ω-constant vector fields, they commute with the P^0-action and descend to M. The local Killing fields near $b \in B$ or $x \in M$ are finite-dimensional vector spaces, and will be denoted $\mathrm{Kill}^{loc}(b)$ and $\mathrm{Kill}^{loc}(x)$, respectively. Let

$$l(x) = \dim \mathrm{Kill}^{loc}(x)$$

Proposition 3.7. *For $x = \pi(b)$,*

$$Kill^{loc}(b) \cong Kill^{loc}(x)$$

Moreover, each $x \in M$ has a neighborhood U_x such that $l(y) \geq l(x)$ for all $y \in U_x$.

Proof: It was observed above that a local Killing field near x lifts to a unique local Killing field near any $b \in \pi^{-1}(x)$, and it is clear that this map is linear. It was also noted above that local Killing fields on B descend to M. This map is linear, and it is injective by an argument essentially the same as that in the proof of proposition 3.6 above. The desired isomorphism follows.

To prove the second statement of the proposition, take a countable nested sequence of neighborhoods U_i of x with $\cap_i U_i = \{x\}$. Let $\mathrm{Kill}_i^{loc}(x)$ be the subspace of local Killing fields defined on U_i. Because $\mathrm{Kill}_i^{loc}(x) \subseteq \mathrm{Kill}_{i+1}^{loc}(x)$ and $\cup_i \mathrm{Kill}_i^{loc}(x) = \mathrm{Kill}^{loc}(x)$, these subspaces eventually stabilize to the finite-dimensional space $\mathrm{Kill}^{loc}(x)$. Set $U_x = U_i$ once $\mathrm{Kill}_i^{loc}(x) = \mathrm{Kill}^{loc}(x)$.

For any $y \in U_x$, every $A \in \mathrm{Kill}^{loc}(x)$ determines an element of $\mathrm{Kill}^{loc}(y)$. If $A \in \mathrm{Kill}^{loc}(x)$ has trivial germ at y, then the lift $\widetilde{A}$ to B has trivial germ at any $b \in \pi^{-1}(y)$, in which case it is trivial everywhere it is defined. Thus the map $\mathrm{Kill}^{loc}(x) \to \mathrm{Kill}^{loc}(y)$ is injective for all $y \in U_x$, so $l(y) \geq l(x)$. $\diamondsuit$

Proposition 3.8. *Let (M, B, ω) be real-analytic. For any compact subset $L \subset B$, there exists $m = m(L)$ such that whenever $b, b' \in L$ are m-related, then there is a unique local automorphism sending b to b'.*

Proof: For each $m \geq 1$, denote by $\mathcal{R}^m$ the C^ω subset of $B \times B$ consisting of pairs (b, b') with $D^m K(b) = D^m K(b')$. Note that $\mathcal{R}^{m+1} \subseteq \mathcal{R}^m$. By the Noetherian property of analytic sets, there exists $m = m(L)$ such that $\mathcal{R}^k \cap (L \times L) = \mathcal{R}^m \cap (L \times L)$ for all $k \geq m$.

Now let $b, b' \in L$ be m-related, so they are in fact ∞-related. Recall that an automorphism is determined by the image of one point, so an automorphism carrying b to b' is unique. Define a map f from an exponential neighborhood of b to a neighborhood of b' by

$$f(\exp_b Y) = \exp_{b'} Y$$

Note that $f(b) = b'$ and $(f^*\omega)_b = \omega_b$. For $Y \in \mathfrak{g}$, denote by $\widetilde{Y}$ the corresponding ω-constant vector feld on B. Now f is a local automorphism if for all $X, Y \in \mathfrak{g}$ and sufficiently small t,

$$f_*(\widetilde{Y}(\exp(b, tX))) = \widetilde{Y}(\exp(b', tX))$$

This equation is equivalent to

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \log_{b'} \circ f(\varphi^s_{\widetilde{Y}} \varphi^t_{\widetilde{X}} b) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \log_{b'}(\varphi^s_{\widetilde{Y}} \varphi^t_{\widetilde{X}} b')$$

Because M is C^ω, it suffices to show that for all $k \geq 0$,

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \log_{b'} \circ f(\varphi^s_{\widetilde{Y}}\varphi^t_{\widetilde{X}} b) = \left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \log_{b'}(\varphi^s_{\widetilde{Y}}\varphi^t_{\widetilde{X}} b') \tag{1}$$

By the BCH formula (proposition 2.1), the right-hand side is

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \frac{1}{(k+1)!}\omega_{b'}(a_{k+1}(t\widetilde{X}, s\widetilde{Y}))$$

Each $a_{k+1}(tX, sY)$ is a sum of $(k+1)$-fold brackets of X and Y with coefficients $t^i s^{k+1-i}/c_i$, where i is the multiplicity of X, and c_i an integer. Then

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 a_{k+1}(tX, sY) = \frac{k!}{c_k}[X, \ldots, X, Y]$$

and the right-hand side of equation (1) is

$$\frac{1}{(k+1)\cdot c_k}\omega_{b'}[\widetilde{X}, \ldots, \widetilde{X}, \widetilde{Y}]$$

where $\widetilde{X}$ appears k times in the iterated bracket.

The left-hand side can be written

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 (\log_{b'} \circ f \circ \exp_b) \circ \log_b(\varphi^s_{\widetilde{Y}}\varphi^t_{\widetilde{X}} b) = \left.\frac{\mathrm{d}^k}{\mathrm{d}t^k}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \log_b(\varphi^s_{\widetilde{Y}}\varphi^t_{\widetilde{X}} b)$$

which, by the BCH formula again, equals

$$\frac{1}{(k+1)\cdot c_k}\omega_b[\widetilde{X}, \ldots, \widetilde{X}, \widetilde{Y}]$$

So it remains to show that these brackets are the same when b and b' are ∞-related. The following lemma completes the proof. $\diamondsuit$

Lemma 3.9. *Let*

$$\Delta_k(b) = [X, \ldots, X, Y] - \omega_b[\widetilde{X}, \ldots, \widetilde{X}, \widetilde{Y}]$$

where X occurs k times in each iterated bracket. Then Δ_k obeys the recursive formula for all $k \geq 1$

$$\Delta_{k+1}(b) \quad = \quad K_b(X, [X, \ldots, X, Y] - \Delta_k(b)) - (\widetilde{X}.\Delta_k)(b) + [X, \Delta_k(b)]$$

If b and b' are ∞-related, then

$$(\widetilde{X}^r.\Delta_k)(b) = (\widetilde{X}^r.\Delta_k)(b') \qquad \textit{for all } r \geq 0$$

If A is a Killing generator at b, then

$$(\widetilde{A}.\widetilde{X}^r.\Delta_k)(b) = 0 \qquad \textit{for all } r \geq 0$$

Proof: We begin with the recursive formula for Δ_k when $k = 1$:

$$\begin{aligned}
\Delta_1(b) &= [X,Y] - \omega_b[\widetilde{X},\widetilde{Y}] \\
&= K_b(X,Y)
\end{aligned}$$

For any $r \geq 0$ and ∞-related b and b',

$$\begin{aligned}
(\widetilde{X}^r.\Delta_1)(b) &= (\widetilde{X}^r.K)_b(X,Y) \\
&= (D^r K_b(X,\ldots,X))(X,Y) \\
&= (D^r K_{b'}(X,\ldots,X))(X,Y) \\
&= (\widetilde{X}^r.\Delta_1)(b')
\end{aligned}$$

where X occurs r times in $(X,\ldots,X)$.

Similarly, if A is a Killing generator at b, then

$$(\widetilde{A}.\widetilde{X}^r.\Delta_1)(b) = ((D^{r+1}K_b \llcorner A)(X,\ldots,X))(X,Y) = 0$$

Next suppose the recursive formula for Δ_k holds up to step k. At the next step,

$$\begin{aligned}
\Delta_{k+1}(b) &= [X,[X,\ldots,X,Y]] - \omega_b[\widetilde{X},[\widetilde{X},\ldots,\widetilde{X},\widetilde{Y}]] \\
&= [X,[X,\ldots,X,Y]] - \omega_b[\widetilde{X},\widetilde{[X,\ldots,X,Y]}] + \omega_b[\widetilde{X},\omega^{-1}\circ\Delta_k] \\
&= K_b(X,[X,\ldots,X,Y]) + \omega_b[\widetilde{X},\omega^{-1}\circ\Delta_k] \\
&= K_b(X,[X,\ldots,X,Y]) - K_b(X,\Delta_k(b)) + (\widetilde{X}.\Delta_k)(b) + [X,\Delta_k(b)] \\
&= K_b(X,[X,\ldots,X,Y] - \Delta_k(b)) + (\widetilde{X}.\Delta_k)(b) + [X,\Delta_k(b)]
\end{aligned}$$

as desired.

Suppose $(\widetilde{X}^r.\Delta_k)(b) = (\widetilde{X}^r.\Delta_k)(b')$ for all $r \geq 0$. Compute

$$\begin{aligned}
(\widetilde{X}^r.\Delta_{k+1})(b) &= \widetilde{X}^r.(K(X,[X,\ldots,X,Y] - \Delta_k))(b) \\
&+ (\widetilde{X}^{r+1}.\Delta_k)(b) + [X,(\widetilde{X}^r.\Delta_k)(b)]
\end{aligned}$$

Compute inductively

$$\begin{aligned}\widetilde{X}^r.(K(X,[X,\ldots,X,Y]-\Delta_k))(b) &= (\widetilde{X}^r.K)_b(X,[X,\ldots,X,Y]-\Delta_k(b)) \\ &- \sum_{i=1}^{r}(\widetilde{X}^{r-i}.K)_b(X,(\widetilde{X}^i.\Delta_k)(b))\end{aligned}$$

By the induction hypothesis on $\widetilde{X}^i.\Delta_k$, and because b and b' are ∞-related, each term in the above sum is the same at b as at b'. Therefore

$$\widetilde{X}^r.(K(X,[X,\ldots,X,Y]-\Delta_k))(b) = \widetilde{X}^r.(K(X,[X,\ldots,X,Y]-\Delta_k))(b')$$

and

$$\begin{aligned}(\widetilde{X}^r.\Delta_{k+1})(b) &= \widetilde{X}^r.(K(X,[X,\ldots,X,Y]-\Delta_k))(b') \\ &+ (\widetilde{X}^{r+1}.\Delta_k)(b') + [X,(\widetilde{X}^r.\Delta_k)(b')] \\ &= (\widetilde{X}^r.\Delta_{k+1})(b')\end{aligned}$$

We leave to the reader the verification that if $(\widetilde{A}.\widetilde{X}^r.\Delta_k)(b) = 0$ for all $r \geq 0$ and A is a Killing generator at b, then

$$(\widetilde{A}.\widetilde{X}^r.\Delta_{k+1})(b) = 0 \qquad \text{for all } r \geq 0$$

$\diamondsuit$

Here is the analogue of proposition 3.8 relating Killing generators and local Killing fields.

Proposition 3.10. *Suppose that (M,B,ω) is real-analytic. Then for all $b \in B$, there exists $m = m(b)$ such that each Killing generator of order m at b determines a unique local Killing field near b.*

Proof: The subspaces $\mathrm{Kill}^m(b)$ eventually stabilize, so there is $m = m(b)$ such that $\mathrm{Kill}^r(b) = \mathrm{Kill}^\infty(b)$ for all $r \geq m$. Let $A \in \mathrm{Kill}^\infty(b)$, so $D^rK(b)\llcorner A = 0$ for all $r \geq 1$.

Let $\widetilde{A}(b) = \omega_b^{-1}A$. Now define $\widetilde{A}$ near b by flowing along ω-constant vector fields: let

$$\widetilde{A}(\varphi^t_{\widetilde{Y}}b) = \varphi^t_{\widetilde{Y}*}(\widetilde{A}(b))$$

This vector field is well-defined in an exponential neighborhood of b. Further, for all $Y \in \mathfrak{g}$, the bracket $[\widetilde{A},\widetilde{Y}](b) = 0$.

To show that $[\widetilde{A}, \widetilde{Y}] = 0$ in a neighborhood of b for all $Y \in \mathfrak{g}$, it suffices to show

$$(\log_b)_* \left([\widetilde{A}, \widetilde{Y}](\exp(b, tX)) \right) = 0$$

for all $X \in \mathfrak{g}$ and t sufficiently small. Because M is C^ω, it suffices to show

$$\left. \frac{\mathrm{d}^k}{\mathrm{d}t^k} \right|_0 (\log_b)_* \left([\widetilde{A}, \widetilde{Y}](\exp(b, tX)) \right) = 0$$

for all $k \geq 0$.

As in the proof of 3.8 above, this equation follows from the BCH formula and lemma 3.9. The reader is invited to refer to the proof of theorem 6.3 and to complete the present proof. As in 3.8, uniqueness follows from freeness of automorphisms on B. $\diamondsuit$

Theorem 3.11. *Let (M, B, ω) be a compact C^ω Cartan geometry modeled on $(\mathfrak{g}, P)$. There exists $m \in \mathbf{N}$ such that any Killing generator at any $b \in B$ of order m gives rise to a unique local Killing field around b.*

Proof: Recall that $\mathrm{Kill}^m(bp^{-1}) = (\mathrm{Ad}\ p)(\mathrm{Kill}^m(b))$. Then proposition 3.10 above, together with proposition 3.7, implies that for all $x \in M$, there exists $m(x)$ such that any Killing generator of order $m(x)$ at any $b \in \pi^{-1}(x)$ determines a unique local Killing field near x in M.

Let U_x be the neighborhood given by proposition 3.7, on which all local Killing fields near x can be defined. Shrink U_x if necessary so that $k_{m(x)}(y) \leq k_{m(x)}(x)$ for all $y \in U_x$. We wish to show that $m(y) = m(x)$. First,

$$l(x) \leq l(y) \leq k_{m(x)}(y) \leq k_{m(x)}(x)$$

But $l(x) = k_{m(x)}(x)$, so $l(y) = k_{m(x)}(y)$. A local Killing field $\widetilde{A}$ near $b \in \pi^{-1}(y)$ is determined by the value $\omega(\widetilde{A}(b))$, so $\mathrm{Kill}^{loc}(b)$ maps injectively to $\mathrm{Kill}^m(b)$ for any m. If these spaces have the same dimension for $m = m(x)$, then this map is an isomorphism—in other words, every Killing generator of order $m(x)$ at any $b \in \pi^{-1}(y)$ gives rise to a local Killing field near y, and $m(y) = m(x)$.

Now take $U_{x_1}, \ldots, U_{x_n}$ a finite subcover of the covering of M by the neighborhoods U_x. Set $m = \max_i m(x_i)$. $\diamondsuit$

It is well-known that for any C^ω manifold B equipped with a C^ω framing, a local Killing field for the framing near any $b_0 \in B$ can be extended uniquely along curves emanating from b_0 (see [13]). The same is then true in the base of a C^ω Cartan geometry M, because any local Killing field near $x_0 \in M$ has a unique lift to B, and local Killing fields in B project to local Killing fields in M. If two local Killing fields of M have the same germ at a point, then they coincide on their common domain of definition. It follows that if M is simply connected, then extending a local Killing field along curves from some x_0 gives rise to a well-defined global Killing field on M. Then we have the following corollary.

Corollary 3.12. *Let (M, B, ω) be a compact, simply connected C^ω Cartan geometry. There exists $m \in \mathbf{N}$ such that for all $b \in B$, every Killing generator at b of order m gives rise to a unique global Killing field on M, which in turn gives rise to a 1-parameter flow of automorphisms of M.*

4 Stratification theorem in the analytic case

The *Kill^{loc}-relation* is the equivalence relation on M with $x \sim y$ if y can be reached from x by flowing along a finite sequence of local Killing fields. The *Kill^{loc}-orbits* are the equivalence classes for the Kill^{loc}-relation. The next result describes the configuration of these orbits in M; it is a version of Gromov's stratification theorem for compact C^ω Cartan geometries.

The Rosenlicht stratification theorem says that when an algebraic group P acts algebraically on a variety W, then there exists a P-invariant filtration

$$U_0 \cup \cdots \cup U_k = W$$

such that U_i is Zariski open and dense in $\cup_{j \geq i} U_j$ and the quotient $U_i \to U_i/P$ is a submersion onto a smooth algebraic variety (see [14], [2, 2.2]).

Let $W = \mathrm{Hom}(\otimes^m \mathfrak{g}, V)$, and define $\Phi : B \to W$ to be the P-equivariant map sending b to the ω-derivative $D^m K(b)$. When (M, B, ω) is algebraic type, the Rosenlicht stratification of W gives rise to a Kill^{loc}-stratification of M. Recall that a *simple* foliation on a manifold V is one in which the leaves are the fibers of a submersion from V to another manifold U.

Theorem 4.1. *Let (M, B, ω) be a C^ω Cartan geometry of algebraic type modeled on $(\mathfrak{g}, P)$. Suppose that M is compact. Then there exists a stratification by Kill^{loc}-invariant sets*

$$V_0 \cup \cdots \cup V_k = M$$

such that each V_i is open and dense in $\cup_{j \geq i} V_j$, and the $Kill^{loc}$-orbits in V_i are leaves of a simple foliation.

Proof: Let m be given by theorem 3.11, so that every Killing generator of order m on B gives rise to a local Killing field on M. Take $V_i = \pi(\Phi^{-1}(U_i))$, where U_i are the pieces of the Rosenlicht stratification for the P-action on the Zariski closure of $\Phi(B)$ in $W = \mathrm{Hom}(\otimes^m \mathfrak{g}, V)$. Then $\cup V_i = M$ and each V_i is open in $\cup_{j \geq i} V_j$. Since Φ is analytic and each $\cup_{j \geq i} U_j$ is Zariski closed, $\cup_{j > i} V_j$ is an analytic subset of $\cup_{j \geq i} V_j$. Therefore, V_i is also dense in $\cup_{j \geq i} V_j$.

The map Φ descends to $\bar{\Phi} : M \to W/P$. Each quotient $U_i/P = X_i$ is a smooth variety. There is the following commutative diagram.

$$\begin{array}{ccc} B & \stackrel{\Phi}{\to} & W \\ \downarrow & & \downarrow \\ M & \stackrel{\bar{\Phi}}{\to} & W/P \\ \cup & & \cup \\ V_i & \to & X_i \end{array}$$

The fibers of the submersion $V_i \to X_i$ are analytic submanifolds, and the components of the fibers of $\bar{\Phi}$ foliate V_i. Let X_i' be the leaf space of this foliation. The map $X_i' \to X_i$ is a local homeomorphism, so X_i' admits the structure of a smooth manifold for which the quotient map $V_i \to X_i'$ is a submersion.

Now it remains to show that the leaves of these foliations—that is, the components of the fibers of $\bar{\Phi}$—are Kill^{loc}-orbits. Let $\mathcal{F} = \Phi^{-1}(w) \subset B$ for $w \in W$. Note that $\mathcal{F} \mapsto \pi(\mathcal{F})$ is a principal bundle, with fiber $P(w)$, the stabilizer in P of w. For $\bar{w}$ the projection of w in W/P, each component of $\bar{\Phi}^{-1}(\bar{w})$ in M is the image under π of a component of $\mathcal{F}$.

If each component $\mathcal{C}$ of $\mathcal{F}$ is a Kill^{loc}-orbit in B, then each component $\pi(\mathcal{C})$ is a Kill^{loc}-orbit in M. The tangent space $T_b\mathcal{C} = \omega_b^{-1}(\mathrm{Kill}^m(b))$ for all $b \in \mathcal{C}$. On the other hand,

$$\omega_b^{-1}(\mathrm{Kill}^m(b)) = \{X(b) \ : \ X \in \mathrm{Kill}^{loc}(b)\}$$

Thus the Kill^{loc}-orbit of b is contained in $\mathcal{C}$. A point $b \in \mathcal{C}$ has a neighborhood $N_b \subset \mathcal{C}$ such that any $b' \in N_b$ equals $\varphi_Y^1 b$ for some $Y \in \mathrm{Kill}^{loc}(b)$. Then given $a \in \mathcal{C}$, connect b to a by a path and cover this path with finitely many such neighborhoods to reach a from b by flowing along finitely many local Killing fields. $\diamondsuit$

5 Gromov representation

Let (M, B, ω) be a compact C^ω Cartan geometry of algebraic type modeled on $(\mathfrak{g}, P)$. The Frobenius theorem gives local Killing fields from Killing generators of sufficiently high order. A slight extension of the main theorem of [6] gives Killing generators of sufficiently high order in $\mathfrak{p}$ from big groups $H < \text{Aut } M$. This latter theorem is a version of Zimmer's embedding theorem—see [15], [2, 5.2.A]—in the setting of Cartan geometries.

Combining local Killing fields that arise from the embedding theorem with certain Killing fields from H gives rise to local Killing fields that centralize $\mathfrak{h}$ in theorem 5.4 (compare [2, 5.2.A2], [16, 4.3]). Local Killing fields that centralize $\mathfrak{h}$ lift to the universal cover of M and extend to global Killing fields. The fundamental group Γ of M preserves this centralizer $\mathfrak{c}$, and the representation of Γ on $\mathfrak{c}$ is related to the adjoint representation of H in theorem 5.9, a version of Gromov's representation theorem [2, 6.2.D1]. In our centralizer theorem, the group $H < \text{Aut } M$ is not assumed to preserve a finite volume. In neither the centralizer nor the representation theorem is it assumed simple; see [17] for some related statements on existence of Gromov representations for simple H without a finite invariant measure, in the setting of Gromov's rigid geometric structures.

5.1 Embedding theorem

If H is a Lie subgroup of Aut M, then the Lie algebra $\mathfrak{h}$ can be viewed as an algebra of global Killing fields on B. If $b \in B$ and X is a nontrivial Killing field on B, the evaluation $X(b) \neq 0$. There are therefore for each $b \in B$ linear injections $\iota_b : \mathfrak{h} \to \mathfrak{g}$ defined by

$$\iota_b(X) = \omega_b(X)$$

The embedding theorem relates the adjoint representation of H on $\mathfrak{h}$ with the representation of a certain subgroup of P on $\iota_b(\mathfrak{h})$. The key ingredient in the proof of the embedding theorem is the Borel density theorem. It essentially says that a finite measure on a variety that is invariant by an algebraic action of a group S is supported on S-fixed points. One must take care, however, that S has no nontrivial compact quotients.

Definition 5.1. *Let H be a Lie group. A Lie subgroup $S < H$ is* discompact *if the Zariski closure $Zar(Ad_{\mathfrak{h}} S)$ has no nontrivial compact algebraic quotients.*

The following statement is a consequence of the Borel density theorem and appears in [6, 3.2].

Theorem 5.2. *(see [18, 2.6] and [19, 3.11]) Let $\psi : S \to Aut\, W$ for S a locally compact group and W an algebraic variety, and assume that $Zar(\psi(S))$ has no nontrivial compact algebraic quotients. Suppose S acts continuously on a topological space M preserving a finite Borel measure μ. Assume $\phi : M \to W$ is an S-equivariant measurable map. Then $\phi(x)$ is fixed by $Zar(\psi(S))$ for μ-almost-every $x \in M$.*

Now we can state the embedding theorem that will be needed.

Theorem 5.3. *Let (M, B, ω) be a Cartan geometry of algebraic type modeled on $(\mathfrak{g}, P)$. Let $H < Aut\ M$ be a Lie group and $S < H$ a discompact subgroup preserving a probability measure μ on M. Denote by $\bar{S}$ the Zariski closure of $Ad_{\mathfrak{h}} S$. For any $m \geq 0$, there exists $\Lambda \subset B$ with $\mu(M \setminus \pi(\Lambda)) = 0$, such that to every $b \in \Lambda$ corresponds an algebraic subgroup $\check{S}_b < Ad_{\mathfrak{g}} P$ with*

1. *$\check{S}_b(\iota_b(\mathfrak{h})) = \iota_b(\mathfrak{h})$*
2. *the representation of $\check{S}_b$ on $\iota_b(\mathfrak{h})$ is equivalent to $\bar{S}$ on $\mathfrak{h}$*
3. *$\check{S}_b$ fixes $D^r K(b)$ for all $0 \leq r \leq m$*

The proof is the same as in [6], except that we apply theorem 5.2 to strata in the P-quotient of the variety

$$\widetilde{W} = \mathrm{Mon}(\mathfrak{h}, \mathfrak{g}) \times V \times \cdots \times \mathrm{Hom}(\otimes^m \mathfrak{g}, V)$$

where $\mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$ consists of the injective linear transformations from $\mathfrak{h}$ to $\mathfrak{g}$. The variety $\widetilde{W}$ is the target of the P-equivariant map

$$\widetilde{\phi}(b) = (\iota_b, K(b), \ldots, D^m K(b))$$

The action of $\bar{S}$ on $\widetilde{W}$ is by

$$g(\rho, \varphi_0, \ldots, \varphi_m) = (\rho \circ g^{-1}, \varphi_0, \ldots, \varphi_m)$$

The action of $p \in P$ on $\mathrm{Mon}(\mathfrak{h}, \mathfrak{g})$ is by post-composition with $\mathrm{Ad}\ p$. Then p acts on the first factor of $\widetilde{W}$ by this action, and on the remaining factors by the actions defined in section 3 above: for $\varphi \in \mathrm{Hom}(\otimes^m \mathfrak{g}, V)$,

$$p.\varphi = p \circ \varphi \circ \mathrm{Ad}\ {}^m p^{-1}$$

Note $\tilde{\phi}$ is P-equivariant.

Let $\phi : M \to \widetilde{W}/P$ be the map induced by $\widetilde{\phi}$; it is S-equivariant. By theorem 5.2, for μ-almost-every x, the point $\phi(x)$ is fixed by $\bar{S}$. Let x be a such a point, and let $b \in \pi^{-1}(x)$. Then define

$$\check{S}_b = \{p \in \mathrm{Ad}_{\mathfrak{g}} P \ : \ p.\widetilde{\phi}(b) = g.\widetilde{\phi}(b) \text{ for some } g \in \bar{S}\}$$

Then $\check{S}_b$ satisfies the conditions (1)-(3) of theorem 5.3.

5.2 Centralizer theorem

The group $\check{S}_b$ gives rise, via the Frobenius theorem, to elements of the stabilizer of $\pi(b)$, which in turn give local Killing fields commuting with $\mathfrak{h}$. Denote by $\widetilde{M}$ the universal cover of M and by q the covering map. Denote by $\mathfrak{c}$ the Lie algebra of global Killing fields on $\widetilde{M}$ commuting with the algebra $\mathfrak{h}$ of Killing fields lifted from the H-action on M. Let $\mathfrak{s}$ be the Lie algebra of S. Given a point y of a manifold N and an algebra $\mathfrak{u}$ of vector fields, denote by $\mathfrak{u}(y)$ the subspace of $T_y N$ consisting of values at y of elements of $\mathfrak{u}$.

Theorem 5.4. *Let (M, B, ω) be a compact C^ω Cartan geometry of algebraic type. Let $H < Aut\ M$ be a Lie group and $S < H$ a discompact subgroup preserving a probability measure μ on M. Then for μ-almost-every $x \in M$, for every $\tilde{x} \in q^{-1}(x)$, the subspace $\mathfrak{s}(\tilde{x}) \subset \mathfrak{c}(\tilde{x})$.*

Proof: The ideas of the proof are the same as Zimmer's [16]. Let m be given by theorem 3.11, so that any Killing generator of order m at any $b \in B$ gives rise to a unique local Killing field. Let x belong to the full-measure set Λ as in the embedding theorem 5.3, and let $b \in \pi^{-1}(x)$. Denote by $\check{\mathfrak{s}}$ the

Lie algebra of $\check{S}_b$, and by ρ_b the Lie algebra homomorphism of $\check{\mathfrak{s}}$ onto $\bar{\mathfrak{s}}$, the Lie algebra of $\bar{S}$. For any $X \in \check{\mathfrak{s}}$ and $0 \leq r \leq m$,

$$D^r K(b) \llcorner X = (X^{\ddagger}.D^{r-1}K)(b) = -X.(D^{r-1}K(b)) = 0$$

by P-equivariance of $D^{r-1}K$. Therefore $\check{\mathfrak{s}} \subset \mathrm{Kill}^m(b)$. Now the Frobenius theorem guarantees, for each $X \in \check{\mathfrak{s}}$, a local Killing field X^* near b with $\omega_b(X^*) = X$. Because $X^*(b)$ is tangent to the fiber over x, the local Killing field near x induced by X^* fixes x.

Let $Y \in \mathfrak{h}$, viewed as a Killing field on B. Compute

$$\begin{aligned} \mathrm{d}\omega(X^*, Y) &= X^*.\omega(Y) - (L_Y\omega)(X^*) \\ &= X^*.\omega(Y) \\ &= (L_{X^*}\omega)(Y) + \omega[X^*, Y] \\ &= \omega[X^*, Y] \end{aligned}$$

On the other hand, since $\omega_b(X^*) \in \mathfrak{p}$, the curvature $\Omega_b(X^*, Y) = 0$, so

$$\begin{aligned} \omega_b[X^*, Y] &= \mathrm{d}\omega_b(X^*, Y) \\ &= [\omega_b(X^*), \omega_b(Y)] = [X, \iota_b(Y)] \\ &= \iota_b((\rho_b X)(Y)) = \omega_b((\rho_b X)(Y)) \end{aligned}$$

Both $[X^*, Y]$ and $(\rho_b X)(Y)$ are local Killing fields. They are determined by their values at any point of B, so they must be equal. We conclude that for all $Y \in \mathfrak{h}$,

$$[X^*, Y] = (\rho_b X)(Y)$$

Now, given $X \in \mathfrak{s}$ and $x \in M$ satisfying the conclusion of the embedding theorem, choose any $b \in \pi^{-1}(x)$ and let Y^* be the local Killing field on M fixing x with $(\rho_b \circ \omega_b)(Y^*) = \mathrm{ad}_{\mathfrak{h}} X$. Then define $X^c = X - Y^*$. It is a local Killing field near x satisfying

- $X^c(x) = X(x) - Y^*(x) = X(x)$
- for all $W \in \mathfrak{h}$,

$$[X^c, W] = [X - Y^*, W] = [X, W] - ((\rho_b \circ \omega_b)(Y^*))(W) = 0$$

Now X^c lifts to a local Killing field near any $\tilde{x} \in \widetilde{M}$. Because $\widetilde{M}$ is real-analytic and simply connected, there is a unique global extension of X^c to $\widetilde{M}$, which will also be denoted X^c. Now $X^c \in \mathfrak{c}$, and $X^c(\tilde{x}) = X(\tilde{x})$. Such an X^c exists for any $X \in \mathfrak{s}$, so the theorem is proved. $\diamondsuit$

5.3 Gromov representation

We first review Zimmer's notion of the algebraic hull of a measurable cocycle. Two references on this subject are [20] and [21].

Definition 5.5. *Let S be a locally compact group acting on a topological space M preserving an ergodic probability measure μ. Let L be a topological group. An L-valued* measurable cocycle *for the S-action on M is a measurable map $\alpha : S \times M \to L$ satisfying*

$$\alpha(gh, x) = \alpha(g, hx)\alpha(h, x)$$

for all $g, h \in S$ and almost-every $x \in M$.

Definition 5.6. *Let S be a locally compact group acting by automorphisms of a V-vector bundle E over a topological space M. Suppose that S preserves an ergodic probability measure μ on M. A* measurable trivialization *of E is a measurable map $t : E \to M \times V$ of the form $t(x, v) = (x, t_x v)$, where t_x is a linear isomorphism $E_x \to V$ for almost-every x.*

A measurable trivialization t gives rise to a $\mathrm{GL}(V)$-valued measurable cocycle α_t where

$$t(g(x, v)) = (gx, \alpha(g, x)(t_x v))$$

Definition 5.7. *Let S, M, μ, V, and E be as in the previous definition. The* algebraic hull *of the S-action is the minimal algebraic subgroup $L < GL(V)$ for which there exists a measurable trivialization t of E with $\alpha_t(S \times M) \subseteq L$.*

The algebraic hull is well defined up to conjugacy in $\mathrm{GL}(V)$; this is a consequence of the Borel density theorem. See [20].

We will need the following fundamental facts about the algebraic hull. A *virtual epimorphism* of algebraic groups is a homomorphism $\sigma : L_1 \to L_2$ for which $\sigma(L_1)$ is a Zariski dense subgroup of L_2 of finite index.

Proposition 5.8. *Let S, M, μ, V, and E be as above.*

1. *Let $\widetilde{M}$ be the universal cover of M, $\Gamma \cong \pi_1(M)$, and $\widetilde{S}$ the connected group of lifts of S to $\widetilde{M}$. Let $\rho : \Gamma \to GL(V)$ be a representation and let $E = \widetilde{M} \times_\rho V$. Then $\widetilde{S}$ acts by automorphisms of E, and the algebraic hull is contained in $Zar(\rho(\Gamma))$.*

2. *Let E_0 be an S-invariant subbundle of E. There is a virtual epimorphism from the algebraic hull of S on E to the algebraic hull of S on E_0.*

3. *Let E_0 be as above, and let $E' = E/E_0$. There is a virtual epimorphism from the algebraic hull of S on E to the algebraic hull of S on E'.*

4. *Suppose there is a trivialization t of E in which $\alpha_t(g, x) = \rho(g)$ for $\rho : S \to GL(V)$ a homomorphism. Then the algebraic hull of the S-action is $Zar(\rho(S))$.*

Proof: For (1), note that $\widetilde{S}$ commutes with Γ, so the $\widetilde{S}$-action on $\widetilde{M} \times V$ by $g(\tilde{x}, v) = (g\tilde{x}, v)$ commutes with the Γ-action on the product. Then the $\widetilde{S}$-action on $\widetilde{M} \times V$ descends to E. The rest is proposition 3.4 of [16], or exercise 6.5.3 of [21]; it involves straightforward arguments with measurable cocycles.

Items (2), (3), and (4) are straightforward; they appear as propositions 3.3 and 3.5 of [16]. See also lemma 6.5.4 of [21]. $\diamondsuit$

Let $S < \mathrm{Aut}\ M$ be as above. Denote by $\mathfrak{s}_x$ the Lie algebra of the stabilizer in S of $x \in M$. Suppose that $\mathfrak{s}_x$ is an ideal $\mathfrak{s}_0 \lhd \mathfrak{s}$. Then denote by $J(S, x) = \mathrm{Zar}(\bar{\mathrm{Ad}}\ S)$, where $\bar{\mathrm{Ad}}$ is the representation of S on $\mathfrak{s}/\mathfrak{s}_0$ obtained as a quotient of the adjoint representation.

Theorem 5.9. *Let (M, B, ω) be a compact C^ω Cartan geometry of algebraic type. Let $S < Aut\ M$ be discompact, and suppose that S preserves a probability measure μ on M. Then for μ-almost-every $x \in M$, $\mathfrak{s}_x \lhd \mathfrak{s}$, and there is a representation ρ of $\pi_1(M) \cong \Gamma$ for which $Zar(\rho(\Gamma))$ contains a subgroup with a virtual epimorphism to $J(S, x)$.*

Proof: By decomposing μ into ergodic components if necessary, we may assume that μ is ergodic.

We first present the standard argument due to Zimmer that almost every stabilizer is an ideal. Let $\mathrm{Gr}\ ^k(\mathfrak{s})$ denote the Grassmannian of k-dimensional subspaces of $\mathfrak{s}$, and define

$$
\begin{aligned}
\psi \quad : \quad & M \to \mathrm{Gr}\ \mathfrak{s} = \bigcup_{k=0}^{\dim \mathfrak{s}} \mathrm{Gr}^k \mathfrak{s} \\
& x \mapsto \mathfrak{s}_x
\end{aligned}
$$

The group S acts on $W = \text{Gr } \mathfrak{s}$ via Ad S, and Zar(Ad S) has no compact algebraic quotients by the discompactness assumption. The map ψ is S-equivariant. The Borel density theorem 5.2 thus applies, and for μ-almost-every x, the stabilizer $\mathfrak{s}_x$ is Ad S-fixed—in other words, it is an ideal $\mathfrak{s}_0$.

Now suppose $\mathfrak{s}_x = \mathfrak{s}_0 \triangleleft \mathfrak{s}$ and in addition that x satisfies the conclusion of the centralizer theorem 5.4, so $\mathfrak{s}(\tilde{x}) \subset \mathfrak{c}(\tilde{x})$ for every $\tilde{x} \in q^{-1}(x)$. Because the Killing fields of $\mathfrak{s}$ on $\widetilde{M}$ are lifted from M, they commute with Γ. Therefore the centralizer $\mathfrak{c}$ is normalized by Γ. Let ρ be the representation of Γ on $\mathfrak{c}$. By proposition 5.8 (1), the algebraic hull of $\widetilde{S}$ on $E = \widetilde{M} \times_\rho \mathfrak{c}$ is contained in $\text{Zar}(\rho(\Gamma))$. Note that in fact the $\widetilde{S}$-action on E factors through S, because any element of $\widetilde{S} \cap \Gamma = \ker(\widetilde{S} \to S)$ centralizes $\mathfrak{c}$.

Denote by $T\mathcal{O}$ the tangent bundle to S-orbits in M

$$T\mathcal{O} = \{(x, Y(x)) \ : \ x \in M, \ Y \in \mathfrak{s}\}$$

There is an obvious measurable trivialization $t : T\mathcal{O} \to M \times \mathfrak{s}/\mathfrak{s}_0$ in which the cocycle for the S-action is $\alpha(g, x) = \bar{\text{Ad}}\ g$. Then by proposition 5.8 (4), the algebraic hull of S on $T\mathcal{O}$ equals $J(S, x)$.

The evaluation map $\epsilon : \widetilde{M} \times \mathfrak{c} \to T\widetilde{M}$ with $\epsilon(\tilde{x}, Y) = Y(\tilde{x})$ descends to an S-equivariant map $\bar{\epsilon} : E \to TM$. The kernel E_0 is an S-invariant subset of E, in which each fiber $(E_0)_x$ is a vector subspace of E_x. The dimension of $(E_0)_x$ is S-invariant, so we may consider E_0 a subbundle of E. The algebraic hull of S on E virtually surjects onto the algebraic hull of S on $E' = E/E_0$ by proposition 5.8 (3).

The map ϵ factors through an isomorphism almost-everywhere from $E' = E/E_0$ to an S-invariant subbundle $\bar{\epsilon}(E)$ of TM, so the algebraic hulls on these two are isomorphic. But $\bar{\epsilon}(E)$ also contains the S-invariant subbundle $T\mathcal{O}$, so the algebraic hull of S on $\bar{\epsilon}(E)$ virtually surjects onto the algebraic hull of S on $T\mathcal{O}$ by proposition 5.8 (2).

We conclude that the algebraic hull of S on E, which is contained in $\text{Zar}(\rho(\Gamma))$, virtually surjects onto $J(S, x)$, as desired. $\diamondsuit$

Corollary 5.10. *Let $S <$ Aut M be semisimple with no compact local factors. Suppose that S preserves a finite volume form on M. Then there is a representation ρ of $\pi_1(M) \cong \Gamma$ for which Zar$(\rho(\Gamma))$ contains a subgroup with a virtual epimorphism to Zar(Ad S).*

Proof: Let μ be the finite measure determined by the S-invariant volume form on M. There are only finitely-many nontrivial ideals of $\mathfrak{s}$. For each nonzero ideal $\mathfrak{s}_x$, the fixed set has empty interior (see [6, 7.1]). The S-action is thus locally free—that is, $\mathfrak{s}_x = 0$—almost everywhere. Then $J(S,x) = \mathrm{Zar}(\mathrm{Ad}\ S)$. Since also S is discompact, the corollary follows from theorem 5.9. $\diamondsuit$

6 Frobenius and open-dense results in smooth case

The analytic Frobenius theorem says that a Killing generator at any $x \in M$ gives rise to a local Killing field. In this section we show that Killing generators of smooth Cartan geometries still give rise to local Killing fields on an open dense subset of M, consisting of the *regular* points. Recall that $k(x)$, for $x \in M$ is the dimension of $\mathrm{Kill}^\infty(b)$ for any $b \in \pi^{-1}(x)$.

Definition 6.1. *Let (M,B,ω) be a C^∞ Cartan geometry. The* regular *points of M are those $x \in M$ for which $k(x)$ is locally constant.*

Because $k(x)$ is lower semicontinuous, the regular points are an open, dense subset of M.

Proposition 6.2. *Suppose that, for $X \in \mathfrak{g}$, the curve $\gamma(t) = \exp(b, tX)$ consists of regular points for all $t \in [-1,1]$. Then there exists m such that $Kill^m(\gamma(t)) = Kill^\infty(\gamma(t))$ for all $t \in [-1,1]$. Moreover, for any $b \in B$ and $A \in Kill^\infty(b)$,*

$$\omega_{\gamma(t)}(\varphi^t_{\widetilde{X}*} A) \in Kill^\infty(\gamma(t))$$

for all $t \in [-1,1]$.

Proof: Let $m(b)$ be such that $k_r(b) = k(b)$ for all $r \geq m(b)$. For all t sufficiently small,

$$k(\gamma(t)) \leq k_{m(b)}(\gamma(t)) \leq k_{m(b)}(b) = k(b)$$

The regularity assumption means $k(\gamma(t)) = k(b)$, so $k_{m(b)}(\gamma(t)) = k(\gamma(t))$ for all t sufficiently small. Now repeating the argument along the compact curve γ shows that $k_{m(b)}(\gamma(t)) = k(\gamma(t))$ for all $t \in [-1,1]$, and $\mathrm{Kill}^r(\gamma(t)) = \mathrm{Kill}^\infty(\gamma(t))$ for all $r \geq m(b)$.

Let as above $V = \wedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, and for $r \in \mathbf{N}$, let

$$\mathcal{W}^r = \bigoplus_{i=0}^{r} \mathrm{Hom}(\otimes^i \mathfrak{g}, V)$$

where we set $\otimes^0 \mathfrak{g} = \mathbf{R}$. For $X \in \mathfrak{g}$ and $(K_0, \ldots, K_r) \in \mathcal{W}^r$, write

$$(K_0, \ldots, K_r) \llcorner X = (K_1 \llcorner X, \ldots, K_r \llcorner X) \in \mathcal{W}^{r-1}$$

Now denote as usual by K the curvature function $B \to V$. For $b \in B$, let

$$\mathcal{D}^r K(b) = (K(b), \ldots, D^r K(b)) \in \mathcal{W}^r$$

and

$$\begin{aligned} \mathcal{C}_b^r \quad : \quad & \mathfrak{g} \to \mathcal{W}^{r-1} \\ & X \mapsto \mathcal{D}^r K(b) \llcorner X \end{aligned}$$

The kernel of $\mathcal{C}_b^r$ is $\mathrm{Kill}^r(b)$. By the discussion in the previous paragraph, $\ker \mathcal{C}_{\gamma(t)}^{m(b)} = \ker \mathcal{C}_{\gamma(t)}^{m(b)+1}$ for all t. Therefore, the functionals on $\mathfrak{g}$ appearing in the decomposition of $\mathcal{C}_{\gamma(t)}^{m(b)+1}$ in terms of a basis of $\mathcal{W}^{m(b)}$ are linear combinations of the functionals appearing in any decomposition of $\mathcal{C}_{\gamma(t)}^{m(b)}$ in terms of any basis of $\mathcal{W}^{m(b)-1}$.

Denote $\omega_{\gamma(t)}(\varphi_{\widetilde{X}*}^t A) = A(t)$ and by $\widetilde{A}$ the corresponding vector field along γ. Then, for each $1 \le r \le m(b)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left(D^r K(\gamma(t)) \llcorner A(t)\right) &= \left(\widetilde{X}.\widetilde{A}.D^{r-1} K\right)(\gamma(t)) \\ &= \left(\widetilde{A}.\widetilde{X}.D^{r-1} K\right)(\gamma(t)) \\ &= \left(D^{r+1} K(\gamma(t)) \llcorner A(t)\right) \llcorner X \end{aligned}$$

using that $[\widetilde{X}, \widetilde{A}] = 0$. There results a system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}_{\gamma(t)}^r (A(t)) = \mathcal{C}_{\gamma(t)}^{r+1}(A(t)) \llcorner X$$

as r ranges from 1 to $m(b)$. At $t = 0$, all $\mathcal{C}_{\gamma(0)}^r(A(0)) = 0$. Then

$$\mathcal{C}_{\gamma(t)}^r(A(t)) = \mathcal{D}^r K(\gamma(t)) \llcorner A(t) \equiv 0$$

is the unique solution for all $1 \le r \le m(b) + 1$, and $A(t) \in \mathrm{Kill}^{m(b)}(\gamma(t)) = \mathrm{Kill}^\infty(\gamma(t))$ for all t. $\diamondsuit$

Theorem 6.3. *Let (M, B, ω) be a C^∞ Cartan geometry and let $U \subseteq M$ be the set of regular points. For each component $U_0 \subseteq U$, there exists $m = m(U_0)$ such that every Killing generator of order m at any $b \in \pi^{-1}(U_0)$ gives rise to a unique local Killing field near $\pi(b)$.*

Proof: Let $b \in U_0$, and let m be such that $\mathrm{Kill}^m(b) = \mathrm{Kill}^\infty(b)$. Then by proposition 6.2, for all $b' \in U_0$, there is also $\mathrm{Kill}^m(b') = \mathrm{Kill}^\infty(b')$. So it suffices to show that any Killing generator at a point lying over the regular set determines a local Killing field.

Let $A \in \mathrm{Kill}^\infty(b)$ for $b \in \pi^{-1}(U)$. As in the proof of proposition 3.10, we define a vector field $\widetilde{A}$ in an exponential neighborhood of b by $\widetilde{A}(\exp(b, tX)) = \varphi^t_{\widetilde{X}*} A$. By proposition 6.2, $\omega(\widetilde{A})$ is a Killing generator everywhere it is defined.

To show that $\widetilde{A}$ descends to a local Killing field near $\pi(b)$, it suffices to show it is a local Killing field near b. Then we must show that for any $Y, X \in \mathfrak{g}$ and sufficiently small T, the bracket

$$[\widetilde{A}, \widetilde{Y}](\exp(b, TX)) = 0$$

We will show that, in the chart $\log_b$, this field satisfies the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} \log_{b*} \left([\widetilde{A}, \widetilde{Y}](\exp(b, tX)) \right) = 0$$

Because the initial value at $t = 0$ is zero, this will imply vanishing for all t.

Let $b(T) = \exp(b, TX)$ and $\Psi_T = (\log_b \circ \exp_{b(T)})_*$. Then

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_T \log_{b*} \left([\widetilde{A}, \widetilde{Y}](\exp(b, tX)) \right) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \log_{b*} \left([\widetilde{A}, \widetilde{Y}](\exp(b, (T+t)X)) \right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left[(\Psi_T \circ \log_{b(T)*}) \left([\widetilde{A}, \widetilde{Y}](\exp(b(T), tX)) \right) \right] \end{aligned}$$

So it suffices to show that for each T,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \log_{b(T)*} \left([\widetilde{A}, \widetilde{Y}](\exp(b(T), tX)) \right) = 0$$

Now

$$\begin{aligned} \log_{b(T)*} \left([\widetilde{A}, \widetilde{Y}](\exp(b(T), tX)) \right) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \log_{b(T)*} \left(\varphi^s_{\widetilde{Y}*} (\widetilde{A}(\varphi^t_{\widetilde{X}} b(T))) - \widetilde{A}(\varphi^s_{\widetilde{Y}} \varphi^t_{\widetilde{X}} b(T)) \right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \log_{b(T)*} \left(\varphi^s_{\widetilde{Y}*} \varphi^t_{\widetilde{X}*} (\widetilde{A}(b(T))) - \varphi^1_{\widetilde{Z}(t,s)*} (\widetilde{A}(b(T))) \right) \end{aligned}$$

where

$$Z(t,s) = (\log_{b(T)} \circ \varphi^s_{\widetilde{Y}} \circ \varphi^t_{\widetilde{X}})(b(T)) = \zeta_{b(T)}(tX, sY)$$

as in the BCH formula. Write $\widetilde{A}_T = \widetilde{A}(b(T))$. Now

$$\log_{b(T)*}\left(\varphi^s_{\widetilde{Y}*}\varphi^t_{\widetilde{X}*}(\widetilde{A}_T) - \varphi^1_{\widetilde{Z}(t,s)*}(\widetilde{A}_T)\right) = \tag{2}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 \left[(\log_{b(T)} \circ \varphi^s_{\widetilde{Y}} \circ \varphi^t_{\widetilde{X}})(\varphi^u_{\widetilde{A}} b(T))\right] - (\log_{b(T)} \circ \varphi^1_{\widetilde{Z}(t,s)})_*(\widetilde{A}_T) \tag{3}$$

Let $c(u) = \varphi^u_{\widetilde{A}} b(T)$. The first term of (3) can be written

$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 \left[\left((\log_{b(T)} \circ \exp_{c(u)}) \circ (\log_{c(u)} \circ \varphi^s_{\widetilde{Y}} \circ \varphi^t_{\widetilde{X}})\right)(c(u))\right] \tag{4}$$

$$= \left[\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 (\log_{b(T)} \circ \exp_{c(u)})\right](Z_0(t,s)) + \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 Z_u(t,s) \tag{5}$$

where $Z_0(t,s) = Z(t,s)$, and

$$Z_u(t,s) = (\log_{c(u)} \circ \varphi^s_{\widetilde{Y}} \circ \varphi^t_{\widetilde{X}})(c(u)) = \zeta_{c(u)}(tX, sY)$$

Now the first term of (5) is

$$\begin{aligned}
\left[\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 (\log_{b(T)} \circ \exp_{c(u)})\right](Z_0(t,s)) &= \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 (\log_{b(T)} \circ \varphi^1_{\widetilde{Z}_0(t,s)} \circ \varphi^u_{\widetilde{A}})(b(T)) \\
&= (\log_{b(T)} \circ \varphi^1_{\widetilde{Z}_0(t,s)})_*(\widetilde{A}_T)
\end{aligned}$$

Thus the first term of (5) cancels with the second term of (3), and we are left to show

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 Z_u(t,s) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 \zeta_{c(u)}(tX, sY) = 0$$

We have

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 \zeta_{c(u)}(tX, sY) &= \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \zeta_{\varphi^u_{\widetilde{A}} b(T)}(tX, sY)\right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 \left(\frac{1}{2}\omega_{\varphi^u_{\widetilde{A}} b(T)}[\widetilde{X}, \widetilde{Y}]\right) \\
&= \frac{1}{2}\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_0 \left([X,Y] - K_{\varphi^u_{\widetilde{A}} b(T)}(X,Y)\right) \\
&= -\frac{1}{2}(\widetilde{A}.K)_{b(T)}(X,Y) \\
&= 0
\end{aligned}$$

because $\widetilde{A}(b(T))$ is a Killing generator. $\diamondsuit$

Theorem 6.4. *Let (M, B, ω) be a C^∞ Cartan geometry of algebraic type. Suppose that M contains a dense Kill^{loc}-orbit. Then M contains an open, dense, locally homogeneous subset.*

Proof: Let $\mathcal{O} \subset M$ be a dense Kill^{loc}-orbit. Because the regular set U is open and Kill^{loc}-invariant, it contains $\mathcal{O}$. Because $\mathcal{O}$ is connected, U has only one component. Let m be such that for all $b \in \pi^{-1}(U)$, any Killing generator of order m at b gives rise to a local Killing field near $\pi(b)$ (such m exists by 6.3).

The map $\Phi : B \to \mathrm{Hom}(\otimes^m \mathfrak{g}, V)$ gives rise to a stratification as in theorem 4.1

$$V_1 \cup \cdots \cup V_k = M$$

such that $\bar{\Phi}$ is a smooth map of each V_i onto a smooth variety. Because V_1 is open and Kill^{loc}-invariant, it contains $\mathcal{O}$. Therefore, $V_1 \cap U$ is open and dense. The same argument as for theorem 4.1 shows that components of fibers of $\bar{\Phi}$ in $V_1 \cap U$ are Kill^{loc}-orbits, and they are closed in V_1. Then

$$\mathcal{O} = \bar{\mathcal{O}} \cap V_1 \cap U = V_1 \cap U$$

so $\mathcal{O}$ is an open, dense, locally homogeneous subset of M. $\diamondsuit$

Question 6.5. *This question is asked in [3] section 7.3: Can the conclusion of theorem 6.4 above be strengthened to say that M is locally homogeneous?*

The forthcoming corollary gives a positive answer in a very special case. For (M, B, ω) a Cartan geometry modeled on $(\mathfrak{g}, P)$, the tangent bundle TM can be identified with $B \times_P (\mathfrak{g}/\mathfrak{p})$ (see [12, 4.5.1]). The Cartan geometry will be called *unimodular* when the representation of P on $\mathfrak{g}/\mathfrak{p}$ has image in $\mathrm{SL}(\mathfrak{g}/\mathfrak{p})$. In this case, there is a volume form on (M, B, ω) preserved by Aut M.

Corollary 6.6. *(see [6, 1.8]) Let (M, B, ω) be a compact, simply connected, unimodular, C^ω Cartan geometry of algebraic type. Let $H <$ Aut M be a connected Lie subgroup. If H has a dense orbit in M, then M is homogeneous: there exists $H' <$ Aut M acting transitively.*

Proof: If H has a dense orbit in M, then there is a dense Kill^{loc}-orbit in M. By theorem 6.4, there is an open dense Kill^{loc}-orbit $U \subseteq M$. But all local Killing fields on M extend to global ones because M is C^ω and simply connected (see [13]), and they are complete because M is compact. Then the volume-preserving automorphism group of M has an open orbit. The conclusion then follows from theorem 1.7 of [6]. $\diamondsuit$

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