

Math 464

Homework: Due on 3/26

1) The goal of this problem is to prove the convergence of the Fourier series of a continuous 2π -periodic real-valued function F at a point where F is differentiable. Note that the Fourier series of F is given by

$$S(F)(\gamma) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\gamma) + b_k \sin(k\gamma),$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} F(\gamma) d\gamma, \quad a_k = \frac{1}{\pi} \int_0^{2\pi} F(\gamma) \cos(k\gamma) d\gamma, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} F(\gamma) \sin(k\gamma) d\gamma,$$

for $k = 1, 2, \dots$. For a positive integer N let

$$S_N(\gamma) = a_0 + \sum_k^N a_k \cos(k\gamma) + b_k \sin(k\gamma).$$

a) By substituting a_k and b_k in S_N prove that

$$S_N(\gamma) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(\lambda) \left(\frac{1}{2} + \sum_{k=1}^N \cos(k(\lambda - \gamma)) \right) d\lambda.$$

b) Prove that for any number $u \in [-\pi, \pi]$,

$$\frac{1}{2} \cos(u) + \cos(2u) + \dots + \cos(Nu) = \begin{cases} \frac{\sin((N+1/2)u)}{2 \sin(u/2)} & : u \neq 0 \\ N + 1/2 & : u = 0 \end{cases}$$

Conclude that

$$S_N(\gamma) = \int_{-\pi}^{\pi} F(\lambda) P_N(\lambda - \gamma) d\lambda,$$

where

$$P_N(u) = \frac{1}{2\pi} \frac{\sin((N+1/2)u)}{\sin(u/2)}.$$

c) Prove that $\int_{-\pi}^{\pi} P_N(\gamma) d\gamma = 1$.

d) Conclude that

$$\int_{-\pi}^{\pi} F(\gamma + \lambda) P_N(\lambda) d\lambda \rightarrow F(\gamma).$$

[This requires some work.]

2) In parts a) and b) below, graph the given Function F , and the partial sums $S_N(\gamma) = a_0 + \sum_k^N a_k \cos(k\gamma) + b_k \sin(k\gamma)$ of its Fourier series for $N = 1, 2, 5, 7, 10$. Comment on the quality of the approximation of F by S_N .

a) $F(\gamma) = \gamma^2$ on $[-\pi, \pi]$, where F is 2π -periodic.

b) $F(\gamma) = \begin{cases} \pi - \gamma & : 0 \leq \gamma \leq \pi \\ -\pi - \gamma & : -\pi \leq \gamma < 0 \end{cases}$ where F is 2π -periodic.