Math 464

Homework: Due on 5/7

1) For $j \in \mathbb{Z}$, let $V_j \subset L^2(\mathbb{R})$ be the space of all bandlimited functions for which the Fourier transform $\hat{f} = 0$ outside the interval $[-2^j, 2^j]$, that is, the support $\hat{f}$ is included in $[-2^j, 2^j]$.

(a) Carefully justify why V_j satisfies the first four conditions in the definition of a multiresolution analysis.

(b) Let $\phi(x) = \frac{\sin \pi x}{\pi x}$, show that ϕ satisfies the last condition in the definition of a MRA.

(c) Find the two scales equation corresponding to this MRA, that is find the coefficients p_k such that

$$\phi(x) = \sum_{k \in \mathbb{Z}} p_k \phi(2x - k).$$

(d) Find an expansion for the wavelet ψ associated with ϕ.

2) Let $h(x) = \max(0, 1 - |x|)$.

(a) Show that $\hat{h}(\gamma) = \frac{\sin^2 \pi\gamma}{\pi^2 \gamma^2}$.

(b) Show that

$$\frac{\pi^2}{\sin^2 \pi\gamma} = \sum_{k=-\infty}^{\infty} \frac{1}{(\gamma+k)^2}.$$

[Hint: Use the Fourier transform domain of the orthonormality condition, applied to the Haar scaling function.]

(c) Deduce that

$$\frac{3 - 2\sin^2 \pi\gamma}{\sin^4 \pi\gamma} = \frac{3}{\pi^4} \sum_{k=-\infty}^{\infty} \frac{1}{(\gamma+k)^4}.$$

[Hint: differentiate twice both sides of the equation obtained in (b.).]

(d) Prove that $\{\phi(x-k), k \in \mathbb{Z}\}$ form an orthonormal system, where ϕ is given by

$$\hat{\phi}(\gamma) = \frac{\sin^2 \pi\gamma}{\pi^2\gamma^2 \sqrt{1 - \frac{2}{3}\sin^2 \pi\gamma}}.$$

3) Let $\{V_j, j \in \mathbb{Z}\}$ be a MRA with scaling function ϕ satisfying the two-scales equation

$$\phi(x) = \sum_k p_k \phi(2x - k) \qquad with \qquad p_k = 2\int_{-\infty}^{\infty} \phi(x)\overline{\phi(2x-k)}\, dx.$$

Recall that the corresponding wavelet is given by

$$\psi(x) = \sum_k (-1)^k \overline{p}_{1-k} \phi(2x - k).$$

Let

$$f_j(x) = \sum_k a_k^j \phi(2^j x - k) = f_{j-1} + w_{j-1} = \sum_k a_k^{j-1} \phi(2^{j-1}x - k) + \sum_k b_k^{j-1} \psi(2^{j-1}x - k)$$

where $f_{j-1} \in V_{j-1}$, and $w_{j-1} \in W_{j-1}$. Prove that

$$a_l^{j-1} = 2^{-1} \sum_k \overline{p}_{k-2l} a_k^j$$

and

$$b_l^{j-1} = 2^{-1} \sum_k (-1)^k p_{1-k+2l} a_k^j.$$