Spring 2009, Math 241 (02) Solution Practice Exam 2

1) a) $\int_0^1 \|\vec{r}\,'(t)\|\,dt$ $\quad \vec{r}\,'(t) = \vec{i} + \sqrt{2t}\,\vec{j} + t\vec{k}$

$\|\vec{r}\,'(t)\| = \sqrt{1+2t+t^2} = \sqrt{(1+t)^2} = t+1$

$\int_0^1 \|\vec{r}\,'(t)\|dt = \int_0^1 (t+1)\,dt = \frac{1}{2}(t+1)^2\Big|_0^1 = 3/2$

b) $a_T = \frac{d}{dt}\|\vec{r}\,'(t)\| = \frac{d}{dt}(t+1) = 1$, $\quad a_N = \|\vec{r}\,'(t)\|\,\|\frac{d\vec{T}}{dt}\|$

$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{t+1}\vec{i} + \frac{\sqrt{2t}}{t+1}\vec{j} + \frac{t}{t+1}\vec{k}$

$\frac{d\vec{T}}{dt}(t) = -\frac{1}{(t+1)^2}\vec{i} + \frac{1-t}{\sqrt{2t}\,(1+t)^2}\vec{j} + \frac{1}{(t+1)^2}\vec{k}$

$a_N = (t+1)\cdot\frac{1}{(t+1)^2}\sqrt{1+\frac{(1-t)^2}{2t}+1} = \frac{1}{\sqrt{2t}}$

c) $K(1) = \frac{\|\frac{d\vec{T}}{dt}(1)\|}{\|\vec{r}\,'(1)\|} = \frac{\frac{1}{2\sqrt{2}}}{2} = \frac{1}{4\sqrt{2}}$

radius of curvature at $\vec{r}(1)$ is $4\sqrt{2}$

2) $z = e^{-y}\cos x$

$\frac{\partial z}{\partial x} = -e^{-y}\sin x \qquad \frac{\partial^2 z}{\partial x^2} = -e^{-y}\cos x$

$\frac{\partial z}{\partial y} = -e^{-y}\cos x \qquad \frac{\partial^2 z}{\partial y^2} = e^{-y}\cos x$

$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

3) a) f increases most rapidly at $(-1,0)$ in the direction of grad $f(-1,0)$ $\quad$ grad $f(x,y) = -2x\vec{i} + (6y+1)\vec{j}$

grad $f(-1,0) = 2\vec{i} + \vec{j}$. The maximum directional derivative at $(-1,0)$ is $\|\text{grad } f(-1,0)\| = \sqrt{5}$

b) grad $f(-1,0) - \vec{k}$ is a normal vector to S at $(-1,0,3)$

$\Rightarrow$ Equation of Tangent to S at $(-1,0,3)$ is

$2(x+1) + y - (z-3) = 0$

(4) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$

$$= (2x)\cos\theta + 2y\sin\theta = 2r\cos^2\theta + 2r\sin^2\theta = \underline{\underline{2r}}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = 2x(-r\sin\theta) + 2y(r\cos\theta)$$

$$\frac{\partial z}{\partial \theta} = -2r^2\sin\theta\cos\theta + 2r^2\sin\theta\cos\theta = \underline{\underline{0}}$$

Spring 2009: Math 241 (Section 02) Solution of Practice Exam 2 (Bis)

(1) a) $\vec{v}(t) = e^t(\cos t - \sin t)\vec{i} + e^t(\sin t + \cos t)\vec{j} + e^t\vec{k}$

$\vec{a}(t) = -2e^t \sin t\, \vec{i} + 2e^t \cos t\, \vec{j} + e^t \vec{k}$

b) $$k(1) = \frac{\|\vec{v}(1) \times \vec{a}(1)\|}{\|\vec{v}(1)\|^3} = \frac{\sqrt{6}\, e^2}{(\sqrt{3}\, e)^3} = \frac{\sqrt{6}\, e^2}{3\sqrt{3}\, e^3} = \frac{\sqrt{2}}{3e}$$

radius of curvature at $\vec{R}(1)$ is $\frac{3e}{\sqrt{2}}$

(2) $z = x^{-1} e^{-y/x} \quad x \neq 0$

$$\frac{\partial z}{\partial x} = -x^{-2} e^{-y/x} + x^{-1} \tfrac{y}{x^2} e^{-y/x} = e^{-y/x} x^{-2} (y/x - 1)$$

$$\frac{\partial z}{\partial y} = -x^{-2} e^{-y/x} \qquad \frac{\partial^2 z}{\partial y^2} = x^{-3} e^{-y/x}$$

$$y \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = y x^{-3} e^{-y/x} - x^{-2} e^{-y/x} = x^{-2} e^{-y/x}(y/x - 1) = \frac{\partial z}{\partial x}$$

(3) a) f increases most rapidly at (2, 3, 0) in the direction of grad f(2, 3, 0)

$\text{grad} f(x,y,z) = y e^z \vec{i} + x e^z \vec{j} + xy e^z \vec{k}$

$\text{grad} f(2,3,0) = 3\vec{i} + 2\vec{j} + 6\vec{k}$

maximum directional derivative at this point is

$\|\text{grad} f(2,3,0)\| = \sqrt{9+4+36} = \sqrt{49} = 7$

b) Normal vector to the tangent plane is $3\vec{i} + 2\vec{j} + 6\vec{k}$, Equation of tangent plane is

$3(x-2) + 2(y-3) + 6z = 0$

(4) $A = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = -\frac{\partial f}{\partial u} - \frac{\partial f}{\partial t}$$

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial w}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial t}$$

$\Rightarrow$ $\underline{\underline{A = 0}}$