Math 241 (Section 02) Solution Practice Exam 4

1) a) $\vec{h}(x,y) = (2xy - y^2)\vec{i} + (x^2 - 2xy)\vec{j} = \text{grad} f(x,y)$?

$f_x = 2xy - y^2 \Rightarrow f(x,y) = x^2y - y^2x + g(y)$

$f_y = x^2 - 2xy \Rightarrow f_y = x^2 - 2xy + g'(y) = x^2 - 2xy \Rightarrow g'(y) = 0 \Rightarrow g(y) = c$

$f(x,y) = x^2y - xy^2$ is a potential for h.

b) $\vec{r}(0) = (1,0)$. $\vec{r}(\sqrt{\tfrac{\pi}{2}}) = (0,1)$

$\int_C \vec{h}\, d\vec{r} = f(0,1) - f(1,0) = 0$

c) $\int_C (2xy - y^2)\,dx + (x^2 - 2xy)\,dy = \cancel{\int_0^1 (2(1-t)t - t^2)}$

$= \int_0^1 \left[-[2(1-t)t - t^2] + [(1-t)^2 - 2(1-t)t] \right] dt = 0$

$= \int_0^1 (-2t + 3t^2 + 3t^2 - 4t + 1)\,dt = \int_0^1 (6t^2 - 6t + 1)\,dt = 2t^3 - 3t^2 + t \Big|_0^1 = 0$

d) Fundamental theorem of line integral. Since h is a conservative vector field, the integral is path independent.

2) $x = u \quad y = v + x^2 = u^2 + v$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 2u & 1 \end{vmatrix} = 1 \qquad \iint_R (xy - x^3)\,dA = \int_1^2 \int_0^3 [u(u^2+v) - u^3]\,dv\,du$

$= \int_1^2 u \int_0^3 v\,dv\,du = \frac{4-1}{2} \cdot \frac{9}{2} = \frac{27}{4}$

a) $f(x,y,z) = xyz^2 + \sin 2z$

) $\operatorname{curl} \vec{F} = 0$

) $\vec{r}(-1) = \cos(\pi)\,\vec{i} + \vec{j} - \frac{\pi}{4}\vec{k} = -\vec{i} + \vec{j} - \frac{\pi}{4}\vec{k}$ $\quad \vec{r}(1) = \cos(\pi)\,\vec{i} + \vec{j} + \frac{\pi}{4}\vec{k} = -\vec{i} + \vec{j} + \frac{\pi}{4}\vec{k}$

$$\oint_C \vec{F} \cdot d\vec{r} = f\left(-1, 1, \tfrac{\pi}{4}\right) - f\left(-1, 1, -\tfrac{\pi}{4}\right)$$

$$= -\frac{\pi^2}{16} + 1 - \left(-\frac{\pi^2}{16} - 1\right) = 2$$

2)

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{2v}{(u+v)^2} & \frac{2u}{(u+v)^2} \\ 1 & 1 \end{vmatrix} = -\frac{2(u+v)}{(u+v)^2} = \frac{-2}{u+v}$$

$$\iint_R y\cos(xy)\,dA = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_1^2 (u+v)\cos(2v)\left|\left(\frac{-2}{u+v}\right)\right| du\,dv$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_1^2 \cos(2v)\,du\,dv$$

$$\vec{r}_\rho \times \vec{r}_\theta = -\rho^2 \sin\theta\, \vec{i} - \rho^2 \cos\theta\, \vec{j} + \rho\, \vec{k}$$

$$\|\vec{r}_\rho \times \vec{r}_\theta\| = \sqrt{\rho^4 + \rho^2} = \rho\sqrt{1+\rho^2}$$

$$\text{Area} = \int_0^1 \int_0^{2\pi} \rho\sqrt{1+\rho^2}\, d\theta\, d\rho = (2\pi)\, \frac{2}{3}\left(\frac{1}{2}\right)(1+\rho^2)^{3/2} \Big|_0^1$$

$$= (2\pi)\left(\frac{1}{3}\right)\left(2^{3/2} - 1\right) = \frac{2\pi\,(2\sqrt{2}-1)}{3} \,//$$

(5)

3) a) $\vec{r}(\rho, \theta) = \rho\cos\theta\, \vec{i} + \rho\sin\theta\, \vec{j} + \rho^2 \sin\theta\cos\theta\, \vec{k}$ $\quad \rho \in [0,1]$, $\theta \in [0, 2\pi]$

$\vec{r}_\rho = \cos\theta\, \vec{i} + \sin\theta\, \vec{j} + 2\rho\sin\theta\cos\theta\, \vec{k}$

$\vec{r}_\theta = -\rho\sin\theta\, \vec{i} + \rho\cos\theta\, \vec{j} + \rho^2 \cos 2\theta\, \vec{k}$

$$\vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \rho\sin 2\theta \\ -\rho\sin\theta & \rho\cos\theta & \rho^2\cos 2\theta \end{vmatrix}$$

$$\text{area} = \iint \|\vec{r}_\rho \times \vec{r}_\theta\|\, dA$$

$$\vec{r}_\rho \times \vec{r}_\theta = -\rho^2 \sin\theta\, \vec{i} - \rho^2 \cos\theta\, \vec{j} + \rho\, \vec{k}$$

$$\|\vec{r}_\rho \times \vec{r}_\theta\| = \sqrt{\rho^4 + \rho^2} = \rho\sqrt{1+\rho^2}$$

$$\text{Area} = \int_0^1 \int_0^{2\pi} \rho\sqrt{1+\rho^2}\, d\theta\, d\rho = (2\pi)\, \tfrac{2}{3}\left(\tfrac{1}{2}\right)(1+\rho^2)^{3/2}\Big|_0^1$$

$$= (2\pi)\left(\tfrac{1}{3}\right)\left(2^{3/2}-1\right) = \frac{2\pi\,(2\sqrt{2}-1)}{3} \quad //$$

(4)

$$\iiint_D (z^2+1)\, dV = \int_0^{2\pi}\int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} (z^2+1)\, r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^1 r \; \tfrac{1}{3}z^3 + z \Big|_{\sqrt{3}r}^{\sqrt{4-r^2}} dr$$

$$= 2\pi \int_0^1 r \left(\tfrac{1}{3}(4-r^2)^{3/2} + (4-r^2)^{1/2} - \frac{3\sqrt{3}r^3}{3} - \sqrt{3}r\right) dr$$

$$= 2\pi \int_0^1 \left[\tfrac{1}{3} r(4-r^2)^{3/2} + r(4-r^2)^{1/2} - \sqrt{3}r^4 - \sqrt{3}r^2\right] dr$$

$$= 2\pi \left[-\tfrac{1}{15}(4-r^2)^{5/2} - \tfrac{1}{3}(4-r^2)^{3/2} - \frac{\sqrt{3}}{5}r^5 - \frac{\sqrt{3}}{3}r^3 \Big|_0^1 \right]$$

$$= 2\pi \left[-\frac{3^{5/2}}{15} - \frac{3^{3/2}}{3} - \frac{\sqrt{3}}{5} - \frac{\sqrt{3}}{3} + \frac{1}{15}(4)^{5/2} + \frac{(4)^{3/2}}{3} \right]$$