Math 620: Homework 1

1. (a) Show that if I is an ideal in a Dedekind domain such that both I^2 and I^3 are principal, then I is principal.
(b) It is known that every ideal I of $\mathbf{Z}[\sqrt{-5}]$ is such that I^2 is principal. Find all solutions in $\mathbf{Z}$ of

$$y^2 = x^3 - 5.$$

2. Let R be a Dedekind domain.
(a) Let P be a nonzero prime ideal of R and let n be an integer. Show that $P^n \supset P^{n+1}$ with $P^n \neq P^{n+1}$.
(b) Let $P_1, \ldots, P_m$ be nonzero prime ideals of R and let $n_1, \ldots, n_m$ be non-negative integers. Let $x_1, \ldots, x_m \in R$ be such that $x_i \in P_i^{n_i}$ but $x_i \notin P_i^{n_i+1}$. Suppose that $x \in R$ is such that $x \equiv x_i \mod P_i^{n_i+1}$ for $1 \leq i \leq m$. Show that

$$(x) = P_1^{n_1} \cdots P_m^{n_m} I,$$

where I is an ideal that is relatively prime to $P_1, \ldots, P_m$.
(c) Show that a Dedekind domain with only finitely many prime ideals is a PID.

3. Let R be a Dedekind domain and let $S \subset R$ be a subset closed under multiplication, with $0 \notin S$. Show that $S^{-1}R$ is a Dedekind domain.

4. Let R be a Dedekind domain and let $P_1, \ldots, P_m$ be nonzero prime ideals of R. Let $S = R \setminus (P_1 \cup \cdots \cup P_m)$.
(a) Show that S is a multiplicatively closed subset of R.
(b) Show that $S^{-1}R$ is a PID.