First In-Class Exam Solutions
Math 246, Fall 2009, Professor David Levermore

(1) [6] Suppose you have used the Runge-Kutta method to approximate the solution of an initial-value problem over the time interval $[0, 4]$ with 1000 uniform time steps. About how many uniform time steps would you need to reduce the global error of your approximation by a factor of 16?

Solution. The Runge-Kutta is fourth order, so its global error scales like h^4. To reduce the error by a factor of 16, you must reduce h by a factor of $16^{\frac{1}{4}} = 2$. You must therefore double the number of time steps, which means you need 2000 uniform time steps.

(2) [8] Sketch the graph that you expect would be produced by the following MATLAB commands.

```
[x, y] = meshgrid(−2:0.25:2,−2:0.25:2)
contour(x, y, y − x.ˆ2, [−2, −2])
axis square
```

Solution. Your sketch should show both x and y axes marked from -2 to 2 and the parabola $y = x^2 - 2$. The tick marks on the axes should mark intervals of length .25.

(3) [20] Find the explicit solution for each of the following initial-value problems and identify its interval of definition.

(a) $\dfrac{dy}{dt} = 5t^4 e^{-y}\,, \quad y(0) = 10.$

Solution. This equation is separable. Its separated differential form is

$$e^y \, dy = 5t^4 \, dt\,, \qquad \Longrightarrow \qquad e^y = t^5 + c\,.$$

The initial condition $y(0) = 10$ implies that $c = e^{10} - 0^5 = e^{10}$. Therefore $e^y = t^5 + e^{10}$, which can be solved as

$$y = \log(t^5 + e^{10})\,, \quad \text{with interval of definition } t > -e^2\,.$$

Here we need $t^5 > -e^{10}$ for the log to be defined. The interval of definition is obtained by taking the fifth root of both sides of this inequality.

(b) $\dfrac{dw}{dt} = \dfrac{t^2 - 2tw}{1 + t^2}\,, \quad w(3) = 2.$

Solution. This equation is linear. Its linear normal form is

$$\frac{dw}{dt} + \frac{2t}{1 + t^2} w = \frac{t^2}{1 + t^2}\,.$$

An integrating factor is $\exp\left(\int_0^t \frac{2s}{1+s^2}\, ds\right) = \exp(\log(1 + t^2)) = 1 + t^2$, so that

$$\frac{d}{dt}\big((1 + t^2)w\big) = (1 + t^2) \cdot \frac{t^2}{1 + t^2} = t^2\,, \qquad \Longrightarrow \qquad (1 + t^2)w = \tfrac{1}{3}t^3 + c\,.$$

The initial condition $w(3) = 2$ implies that $c = (1 + 3^2) \cdot 2 - \frac{1}{3}3^3 = 20 - 9 = 11$. Therefore

$$w = \frac{\frac{1}{3}t^3 + 11}{1 + t^2}\,, \quad \text{with interval of definition } -\infty < t < \infty\,.$$

(4) [16] Consider the differential equation $\dfrac{\mathrm{d}p}{\mathrm{d}t} = p^2(4-p)(8-p)$.

(a) Sketch its phase-line. Indicate all of the stationary (equilibrium) solutions and classify each as being either stable, unstable, or semistable.
(b) If $p(0) = 10$, how does the solution $p(t)$ behave as $t \to \infty$?
(c) If $p(0) = 6$, how does the solution $p(t)$ behave as $t \to \infty$?
(d) If $p(0) = 2$, how does the solution $p(t)$ behave as $t \to \infty$?
(e) If $p(0) = -2$, how does the solution $p(t)$ behave as $t \to \infty$?

Solution (a). The stationary solutions are $p = 0$, $p = 4$, and $p = 8$. A sign analysis of $p^2(4-p)(8-p)$ shows that the phase-line for this equation is therefore

$$\begin{array}{ccccccc} + & & + & & - & & + \\ \to\to\to\to & \bullet & \to\to\to\to & \bullet & \leftarrow\leftarrow\leftarrow\leftarrow & \bullet & \to\to\to\to\ p \\ & 0 & & 4 & & 8 & \\ & \text{semistable} & & \text{stable} & & \text{unstable} & \end{array}$$

Solution (b). The phase-line shows that if $p(0) = 10$ then $p(t) \to \infty$ as $t \to \infty$.
Solution (c). The phase-line shows that if $p(0) = 6$ then $p(t) \to 4$ as $t \to \infty$.
Solution (d). The phase-line shows that if $p(0) = 2$ then $p(t) \to 4$ as $t \to \infty$.
Solution (e). The phase-line shows that if $p(0) = -2$ then $p(t) \to 0$ as $t \to \infty$.

(5) [16] Consider the following MATLAB function M-file.

```
function [t,y] = solveit(ti, yi, tf, n)

h = (tf - ti)/n;
t = zeros(n + 1, 1);
y = zeros(n + 1, 1);
t(1) = ti;
y(1) = yi;
for k = 1:n
thalf = t(k) + h/2;
yhalf = y(k) + (h/2)*(4*t(k) - (y(k))^2);
t(k + 1) = t(k) + h;
y(k + 1) = y(k) + h*(4*thalf - (yhalf)^2);
end
```

(a) What is the initial-value problem being approximated numerically?
(b) What is the numerical method being used?
(c) What are the output values of t(2) and y(2) that you would expect for input values of ti = 1, yi = 2, tf = 9, n = 40?

Solution (a). The initial-value problem being approximated numerically is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 4t - y^2\,, \qquad y(\mathrm{ti}) = \mathrm{yi}\,.$$

Solution (b). The Heun-midpoint method is being used. (This is clear from the lines defining thalf and yhalf.)

Solution (c). When ti = 1, yi = 2, tf = 9, n = 40 one has h = (tf − ti)/n = (9 − 1)/40 = .2, t(1) = ti = 1, and y(1) = yi = 2.
Setting k = 1 inside the "for" loop then yields

$$\text{thalf} = \text{t}(1) + \text{h}/2 = 1 + .1 = 1.1$$

$$\text{yhalf} = \text{y}(1) + (\text{h}/2)\,(4\,\text{t}(1) - \text{y}(1)^2) = 2 + .1\,(4\cdot 1 - 2^2) = 2\,,$$

$$\text{t}(2) = \text{t}(1) + \text{h} = 1 + .2 = 1.2\,,$$

$$\text{y}(2) = \text{y}(1) + \text{h}\,(4\,\text{thalf} - \text{yhalf}^2) = 2 + .2\,(4\cdot 1.1 - 2^2)\,.$$

The above answer got full credit, but y(2) = 2.08 if you worked out the arithmetic.

(6) [14] What is the maximum amount a student can borrow with a five-year loan at an interest rate of 5% per year compounded continuously assuming that she can make payments continuously at a constant rate of 2400 dollars per year? Hint: Write down an initial-value problem that governs $B(t)$, the balance of the loan at t years.

Solution. Because the loan get paid-off in five years, the balance $B(t)$ satisfies the initial-value problem

$$\frac{\mathrm{d}B}{\mathrm{d}t} = .05B - 2400\,, \qquad B(5) = 0\,.$$

The equation is linear and can be put into the integrating factor form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-.05t}B\big) = -2400e^{-.05t}\,,$$

which implies that

$$e^{-.05t}B = 48000e^{-.05t} + c\,.$$

The initial condition $B(5) = 0$ implies that $c = -48000e^{-.25}$. Therefore

$$B(t) = 48000 - 48000e^{.05(t-5)}\,.$$

The maximum amount she can borrow is $B(0) = 48000(1 - e^{-.25})$.

(7) [20] Give an implicit general solution to each of the following differential equations.

(a) $(3x^2\sin(y) + e^x)\,\mathrm{d}x + (x^3\cos(y) + 2y)\,\mathrm{d}y = 0\,.$

Solution: This differential form is *exact* because

$$\partial_y(3x^2\sin(y) + e^x) = 3x^2\cos(y) \quad = \quad \partial_x(x^3\cos(y) + 2y) = 3x^2\cos(y)\,.$$

We can therefore find $H(x,y)$ such that

$$\partial_x H(x,y) = 3x^2\sin(y) + e^x\,, \qquad \partial_y H(x,y) = x^3\cos(y) + 2y\,.$$

Integrating the first equation with respect to x yields

$$H(x,y) = x^3\sin(y) + e^x + h(y)\,.$$

Plugging this expression for $H(x,y)$ into the second equation gives

$$x^3\cos(y) + h'(y) = \partial_y H(x,y) = x^3\cos(y) + 2y\,,$$

which yields $h'(y) = 2y$. Taking $h(y) = y^2$, a general solution is therefore given implicitly by

$$x^3\sin(y) + e^x + y^2 = c\,.$$

(b) $(3x^2y + 2xy + y^3)\,\mathrm{d}x + (x^2 + y^2)\,\mathrm{d}y = 0\,.$

Solution. This differential form is *not exact* because

$$\partial_y(3x^2y + 2xy + y^3) = 3x^2 + 2x + 3y^2 \quad \neq \quad \partial_x(x^2 + y^2) = 2x\,.$$

You therefore seek an *integrating factor* μ such that

$$\partial_y[(3x^2y + 2xy + y^3)\mu] = \partial_x[(x^2 + y^2)\mu]\,.$$

Expanding the partial derivatives yields

$$(3x^2y + 2xy + y^3)\partial_y\mu + (3x^2 + 2x + 3y^2)\mu = (x^2 + y^2)\partial_x\mu + 2x\mu\,.$$

If you set $\partial_y\mu = 0$ then this becomes

$$(3x^2 + 2x + 3y^2)\mu = (x^2 + y^2)\partial_x\mu + 2x\mu\,,$$

which reduces to $\partial_y\mu = 3\mu$. This yields the integrating factor $\mu = e^{3x}$.

Because e^{3x} is an integrating factor, the differential form

$$e^{3x}(3x^2y + 2xy + y^3)\,\mathrm{d}x + e^{3x}(x^2 + y^2)\,\mathrm{d}y = 0 \quad \text{is exact}\,.$$

You can therefore find $H(x, y)$ such that

$$\partial_x H(x, y) = e^{3x}(3x^2y + 2xy + y^3)\,, \qquad \partial_y H(x, y) = e^{3x}(x^2 + y^2)\,.$$

Integrating the second equation with respect to y yields

$$H(x, y) = e^{3x}(x^2y + \tfrac{1}{3}y^3) + h(x)\,.$$

Plugging this expression for $H(x, y)$ into the first equation gives

$$3e^{3x}(x^2y + \tfrac{1}{3}y^3) + e^{3x}2xy + h'(x) = \partial_x H(x, y) = e^{3x}(3x^2y + 2xy + y^3)\,,$$

which yields $h'(x) = 0$. Taking $h(x) = 0$, a general solution is therefore given implicitly by

$$e^{3x}(x^2y + \tfrac{1}{3}y^3) = c\,.$$