Second In-Class Exam Solutions
Math 246, Fall 2009, Professor David Levermore
Thursday, 29 October 2009

(1) [6] Give the interval of definition for the solution of the initial-value problem

$$\sin(t)\,\frac{\mathrm{d}^4x}{\mathrm{d}t^4} + \frac{5}{1-t^2}\,\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{4-t^2}\,, \qquad x(3) = x'(3) = x''(3) = x'''(3) = 0\,.$$

Solution. Put the equation into normal form

$$\frac{\mathrm{d}^4x}{\mathrm{d}t^4} + \frac{5}{\sin(t)(1-t^2)}\,\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3}{\sin(t)(4-t^2)}\,, \qquad x(3) = x'(3) = x''(3) = x'''(3) = 0\,.$$

The coefficient and forcing are both continuous over the interval $(2, \pi)$, which contains the initial time $t = 3$. The coefficient is not defined at $t = n\pi$ for every integer n and at $t = \pm 1$ while the forcing is not defined at $t = n\pi$ for every integer n and at $t = \pm 2$. The interval of definition is therefore $(2, \pi)$.

(2) [14] Solve the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} - 2y = 9e^t\,, \qquad y(0) = 0\,, \quad y'(0) = 2\,.$$

Solution. This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 + z - 2 = (z-1)(z+2)\,.$$

This has the roots 1 and -2, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^t + c_2 e^{-2t}\,.$$

The forcing $9e^t$ has degree $d = 0$ and characteristic $\mu + i\nu = 1$, which is a root of $p(z)$ of multiplicity $m = 1$. A particular solution $y_P(t)$ can be found by the method of either KEY identity evaluations or undetermined coefficients.

KEY Indentity Evaluations. Because $m + d = 1$, you need the KEY identity and its first derivative

$$\mathrm{L}\big(e^{zt}\big) = (z^2 + z - 2)e^{zt}\,, \qquad \mathrm{L}\big(t\,e^{zt}\big) = (z^2 + z - 2)t\,e^{zt} + (2z+1)\,e^{zt}\,.$$

Evaluate these at $z = 1$ to find $\mathrm{L}(e^t) = 0$ and $\mathrm{L}(t\,e^t) = 3e^t$. Multiplying the second of these equations by 3 yields $\mathrm{L}(3t\,e^t) = 9e^t$, which implies $y_P(t) = 3t\,e^t$.

Undetermined Coefficients. Because $m = 1$ and $m + d = 1$, you seek a particular solution of the form

$$y_P(t) = At\,e^t\,.$$

Because $y_P'(t) = At\,e^t + Ae^t$ and $y_P''(t) = At\,e^t + 2Ae^t$, one sees that

$$\mathrm{L}y_P(t) = y_P''(t) + y_P'(t) - 2y_P(t) = \big(At\,e^t + 2Ae^t\big) + \big(At\,e^t + Ae^t\big) - 2At\,e^t = 3Ae^t\,.$$

Setting $\mathrm{L}y_P(t) = 3Ae^t = 9e^t$, we see that $3A = 9$, whereby $A = 3$. Hence, $y_P(t) = 3t\,e^t$.

Imposing the Initial Conditions. By either approach one finds $y_P(t) = 3t\, e^T$, which yields the general solution

$$y(t) = c_1 e^t + c_2 e^{-2t} + 3t\, e^t \,.$$

Because

$$y'(t) = c_1 e^t - 2c_2 e^{-2t} + 3t\, e^t + 3e^t \,,$$

when the initial conditions are imposed, one finds that

$$y(0) = c_1 + c_2 = 0 \,, \qquad y'(0) = c_1 - 2c_2 + 3 = 2 \,.$$

These are solved to find $c_1 = -\frac{1}{3}$ and $c_2 = \frac{1}{3}$. The solution of the initial-value problem is therefore

$$y(t) = \tfrac{1}{3} e^t - \tfrac{1}{3} e^{-2t} + 3t\, e^t \,.$$

(3) [10] Give a general real solution of the equation

$$\mathrm{D}^2 y - 6\mathrm{D}y + 10y = 5\sin(2t) \,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t} \,.$$

Solution. This is a constant coefficient, nonhomogeneous, linear equation. Its characteristic polynomial is

$$p(z) = z^2 - 6z + 10 = (z-3)^2 + 1 = (z-3)^2 + 1^2 \,.$$

This has the conjugate pair of roots $3 \pm i$, which yields a general solution of the associated homogeneous problem

$$y_H(t) = c_1 e^{3t} \cos(t) + c_2 e^{3t} \sin(t) \,.$$

The forcing $5\sin(2t)$ has degree $d = 0$ and characteristic $\mu + i\nu = i2$, which is a root of $p(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of either KEY identity evaluations or undetermined coefficients.

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at $z = i2$, which yields

$$\mathrm{L}\big(e^{i2t}\big) = p(i2) e^{i2t} = \big((i2)^2 - 6(i2) + 10\big) e^{i2t} = 6(1 - i2) e^{i2t} \,.$$

Because the forcing has the form $5\sin(2t) = 5\,\mathrm{Im}(e^{i2t})$, we write

$$\mathrm{L}\left(\frac{5}{6}\frac{e^{i2t}}{1 - i2}\right) = 5e^{i3t} \,,$$

which implies that

$$\begin{aligned} y_P(t) &= \mathrm{Im}\left(\frac{5}{6}\,\frac{e^{i2t}}{1-i2}\right) = \frac{5}{6}\,\mathrm{Im}\left(\frac{e^{i2t}}{1-i2}\,\frac{1+i2}{1+i2}\right) = \frac{5}{6}\,\mathrm{Im}\left(\frac{(1+i2)e^{i2t}}{1^2+2^2}\right) \\ &= \tfrac{1}{6}\,\mathrm{Im}\big((1+i2)e^{i2t}\big) = \tfrac{1}{6}\big(2\cos(2t) + \sin(2t)\big) \,. \end{aligned}$$

A general solution of the equation is therefore

$$y(t) = c_1 e^{3t} \cos(t) + c_2 e^{3t} \sin(t) + \tfrac{1}{3}\cos(2t) + \tfrac{1}{6}\sin(2t) \,.$$

Undetermined Coefficients. Because $m = d = 0$, you seek a particular solution of the form

$$y_P(t) = A\cos(2t) + B\sin(2t) \,.$$

Because

$$y_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad y_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

one sees that

$$\begin{aligned} \mathrm{L}y_P(t) &= y_P''(t) - 2y_P'(t) + 10y_P(t) \\ &= \big[-4A\cos(2t) - 4B\sin(2t)\big] - 6\big[-2A\sin(2t) + 2B\cos(2t)\big] \\ &\quad + 10\big[A\cos(2t) + B\sin(2t)\big] \\ &= (6A - 12B)\cos(3t) + (12A + 6B)\sin(3t)\,. \end{aligned}$$

Setting $\mathrm{L}y_P(t) = 5\sin(2t)$, we see that

$$6A - 12B = 0\,, \qquad 12A + 6B = 5\,,$$

whereby $A = \frac{1}{3}$ and $B = \frac{1}{6}$. Hence, $y_P(t) = \frac{1}{3}\cos(2t) + \frac{1}{6}\sin(2t)$. A general solution of the equation is therefore

$$y(t) = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t) + \tfrac{1}{3}\cos(2t) + \tfrac{1}{6}\sin(2t)\,.$$

(4) [10] What answer will be produced by the following MATLAB commands?

```
>> ode = 'D2y + 8*Dy + 20*y = 4*exp(–2*t)';
>> dsolve(ode, 't')

ans =
```

Solution. The commands ask MATLAB to give a general solution of the equation

$$\mathrm{D}^2 y + 8\mathrm{D}y + 20y = 4e^{-2t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

at least one version of MATLAB will produce the answer

```
exp(–4t)*sin(2*t)*C2 + exp(–4t)*cos(2*t)*C1 + .5*exp(–2*t)
```

Your answer does not have to be given in this MATLAB format. Rather, your answer should be equivalent to it. The fact this is the answer is seen as follows.

The problem being solved is a constant coefficient, nonhomogeneous, linear equation. The characteristic polynomial is

$$p(z) = z^2 + 8z + 20 = (z+4)^2 + 4 = (z+4)^2 + 2^2\,.$$

Its roots are the conjugate pair $-4 \pm i2$. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 e^{-4t}\cos(2t) + c_2 e^{-4t}\sin(2t)\,.$$

The forcing $4e^{-2t}$ has degree $d = 0$ and characteristic $\mu + i\nu = -2$, which is a root of $p(z)$ of multiplicity $m = 0$. A particular solution $y_P(t)$ can be found by the method of either KEY identity evaluation or undetermined coefficients.

KEY Indentity Evaluations. Because $m + d = 0$, you only need to evaluate the KEY identity at the characteristic $z = -2$, which yields

$$\mathrm{L}(e^{-2t}) = p(-2)e^{-2t} = \big((-2)^2 + 8(-2) + 20\big)e^{-2t} = 8e^{-2t}\,.$$

Dividing by 2 yields $\mathrm{L}(\frac{1}{2}e^{-2t}) = 4e^{-2t}$, which implies $y_P(t) = \frac{1}{2}e^{-2t}$. A general solution is therefore

$$y(t) = c_1 e^{-4t}\cos(2t) + c_2 e^{-4t}\sin(2t) + \tfrac{1}{2}e^{-2t}\,.$$

Up to notational differences, this is the answer that MATLAB produces.

Direct Substitution. Because $m = d = 0$ and the characteristic is -2, you seek a particular solution of the form

$$y_P(t) = Ae^{-2t}\,.$$

Because

$$y_P'(t) = -2Ae^{-2t}\,, \qquad y_P''(t) = 4Ae^{-2t}\,,$$

one sees that

$$\begin{aligned}\mathrm{L}y_P(t) &= y_P''(t) + 4y_P'(t) + 8y_P(t)\\ &= [4Ae^{-2t}] + 8[-2Ae^{-2t}] + 20[Ae^{-2t}] = 8Ae^{-2t}\,.\end{aligned}$$

Setting $\mathrm{L}y_P(t) = 8Ae^{-2t} = 4e^{-2t}$, we see that $A = \frac{1}{2}$. Hence, $y_P(t) = \frac{1}{2}e^{-2t}$. A general solution is therefore

$$y(t) = c_1 e^{-4t}\cos(2t) + c_2 e^{-4t}\sin(2t) + \tfrac{1}{2}e^{-2t}\,.$$

Up to notational differences, this is the answer that MATLAB produces.

(5) [8] Compute the Green function associated with the differential operator

$$\mathrm{L} = \mathrm{D}^2 + 16\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The Green function $g(t)$ associated with the operator L satisfies the initial-value problem

$$\mathrm{L}g = \mathrm{D}^2 g + 16g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is

$$p(z) = z^2 + 16 = z^2 + 4^2\,,$$

which has roots $\pm i4$. Hence,

$$g(t) = c_1\cos(4t) + c_2\sin(4t)\,.$$

The initial condition $g(0) = 0$ implies $c_1 = 0$. Because

$$g'(t) = 4c_2\cos(4t)\,,$$

the initial condition $g'(0) = 1$ implies $c_2 = \frac{1}{4}$. The Green function is thereby

$$g(t) = \tfrac{1}{4}\sin(4t)\,.$$

(6) [12] Find a particular solution $x_P(t)$ of the equation

$$\mathrm{D}^2 x + 16x = \frac{1}{\cos(4t)}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a constant coefficient nonhomogeneous equation. Its forcing does not have the characteristic form required for the methods of either KEY identity evaluations or undetermined coefficients. You can therefore use either the Green function method or variation of parameters.

Green Function. By the previous problem, the associated Green function is given by $g(t) = \frac{1}{4}\sin(4t)$. A particular solution is therefore

$$\begin{aligned}
x_P(t) &= \tfrac{1}{4}\int_0^t \sin(4t-4s)\,\frac{1}{\cos(4s)}\,\mathrm{d}s \\
&= \tfrac{1}{4}\int_0^t \big[\sin(4t)\cos(4s) - \cos(4t)\sin(4s)\big]\,\frac{1}{\cos(4s)}\,\mathrm{d}s \\
&= \tfrac{1}{4}\sin(4t)\int_0^t \frac{\cos(4s)}{\cos(4s)}\,\mathrm{d}s - \tfrac{1}{4}\cos(4t)\int_0^t \frac{\sin(4s)}{\cos(4s)}\,\mathrm{d}s \\
&= \tfrac{1}{4}\sin(4t)\,t + \tfrac{1}{16}\cos(4t)\,\log(|\cos(4t)|)\,.
\end{aligned}$$

Variation of Parameters. By the previous problem, a general solution of the associated homogeneous problem is

$$x_H(t) = c_1\cos(4t) + c_2\sin(4t)\,.$$

We therefore seek a solution of the nonhomogeneous problem in the form

$$x = u_1(t)\cos(4t) + u_2(t)\sin(4t)\,,$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$\begin{aligned}
u_1'(t)\cos(4t) + u_2'(t)\sin(4t) &= 0\,, \\
-u_1'(t)4\sin(4t) + u_2'(t)4\cos(4t) &= \frac{1}{\cos(4t)}\,.
\end{aligned}$$

The solution of this system is

$$u_1'(t) = -\tfrac{1}{4}\frac{\sin(4t)}{\cos(4t)}\,, \qquad u_2'(t) = \tfrac{1}{4}\,.$$

Integrate these equations to obtain

$$u_1(t) = c_1 + \tfrac{1}{16}\log(|\cos(4t)|)\,, \qquad u_2(t) = c_2 + \tfrac{1}{4}t\,.$$

A particular solution is therefore

$$x_P(t) = \tfrac{1}{16}\cos(4t)\,\log(|\cos(4t)|) + \tfrac{1}{4}\sin(4t)\,t\,.$$

Remark: You can use the formulas for $u_1'(t)$ and $u_2'(t)$ given in the book, but this becomes more involved than simply setting up and solving the linear algebraic system as was done above.

Remark: It is clear that the Green function method gets to the definite integrals quicker.

(7) [12] Let L be a linear ordinary differential operator with constant coefficients. Suppose that all the roots of its characteristic polynomial (listed with their multiplicities) are $3 + i2$, $3 + i2$, $3 - i2$, $3 - i2$, $i5$, $-i5$, -4, -4, -4, 0, 0.

(a) Give the order of L.

Solution. There are 11 roots listed above, so the degree of the characteristic polynomial is 11, whereby the order of L is 11.

(b) Give a general real solution of the homogeneous equation $\mathrm{L}y = 0$.

Solution. A general solution is

$$\begin{aligned} y(t) &= c_1 e^{3t}\cos(2t) + c_2 e^{3t}\sin(2t) + c_3 t\, e^{3t}\cos(2t) + c_4 t\, e^{3t}\sin(2t) \\ &= c_5\cos(5t) + c_6\sin(5t) + c_7 e^{-4t} + c_8 t e^{-4t} + c_9 t^2 e^{-4t} + c_{10} + c_{11} t\,. \end{aligned}$$

The reasoning is as follows:

- the double conjugate pair $3 \pm i2$ yields

$$e^{3t}\cos(2t)\,, \quad e^{3t}\sin(2t)\,, \quad t\, e^{3t}\cos(2t)\,, \quad \text{and} \quad t\, e^{3t}\sin(2t)\,;$$

- the single conjugate pair $\pm i5$ yields $\cos(5t)$ and $\sin(5t)$;
- the triple real root -4 yields e^{-4t}, $t\, e^{-4t}$, and $t^2 e^{-4t}$;
- the double real root 0 yields 1 and t.

(8) [12] The functions t and t^3 are solutions of the homogeneous equation

$$t^2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 3t\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = 0 \qquad \text{over } t > 0\,.$$

(You do not have to check that this is true!)

(a) Compute their Wronskian.

Solution. The Wronskian is

$$W[t, t^3](t) = \det\begin{pmatrix} t & t^3 \\ 1 & 3t^2 \end{pmatrix} = 3t^3 - t^3 = 2t^3\,.$$

(b) Give a general solution of the equation

$$t^2\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 3t\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = \frac{t^4}{1+t^2} \qquad \text{over } t > 0\,.$$

Solution. Because $W[t, t^3](x) = 2t^3 > 0$ over $t > 0$, the functions t and t^3 are linearly independent. A general solution of the associated homogeneous problem is

$$y_H(t) = c_1 t + c_2 t^3\,.$$

Because this problem has variable coefficients, you must use the method of general Green functions or variation of parameters to find a particular solution $y_P(t)$. In either case you should first divide by t^2 to bring the equation into its normal form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{3}{t}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{3}{t^2}\, y = \frac{t^2}{1+t^2} \qquad \text{over } t > 0\,.$$

General Green Function. The Green function $G(t,s)$ is given by

$$G(t,s) = \frac{\det\begin{pmatrix} s & s^3 \\ t & t^3 \end{pmatrix}}{W[s,s^3](s)} = \frac{st^3 - ts^3}{2s^3} = \frac{t^3 - ts^2}{2s^2} \,.$$

The Green function formula then yields the solution

$$\begin{aligned}
y(t) &= \int_0^t G(t,s)\,\frac{s^2}{1+s^2}\,\mathrm{d}s = \tfrac{1}{2}\int_0^t \frac{t^3 - ts^2}{1+s^2}\,\mathrm{d}s \\
&= \tfrac{1}{2}t^3 \int_0^t \frac{1}{1+s^2}\,\mathrm{d}s - \tfrac{1}{2}t\int_0^t \frac{s^2}{1+s^2}\,\mathrm{d}s \\
&= \tfrac{1}{2}t^3 \int_0^t \frac{1}{1+s^2}\,\mathrm{d}s - \tfrac{1}{2}t\int_0^t 1 - \frac{1}{1+s^2}\,\mathrm{d}s \\
&= \tfrac{1}{2}t^3\tan^{-1}(t) - \tfrac{1}{2}t\big[t - \tan^{-1}(t)\big] \,.
\end{aligned}$$

Variation of Parameters. Seek a solution in the form

$$y = u_1(t)t + u_2(t)t^3 \,.$$

where $u_1'(t)$ and $u_2'(t)$ satisfy the linear algebraic system

$$\begin{aligned}
u_1'(t)t + u_2'(t)t^3 &= 0 \,, \\
u_1'(t)1 + u_2'(t)3t^2 &= \frac{t^2}{1+t^2} \,.
\end{aligned}$$

The solution of this system is

$$u_1'(t) = -\tfrac{1}{2}\frac{t^2}{1+t^2} \,, \qquad u_2'(t) = \tfrac{1}{2}\frac{1}{1+t^2} \,.$$

Alternatively, because you know that $W[t,t^3](t) = 2t^3$, you can directly use the formulas from the book to obtain

$$u_1'(t) = \frac{-t^3\cdot\frac{t^2}{1+t^2}}{2t^3} = -\tfrac{1}{2}\frac{t^2}{1+t^2} \,, \qquad u_2'(t) = \frac{t\cdot\frac{t^2}{1+t^2}}{2t^3} = \tfrac{1}{2}\frac{1}{1+t^2} \,.$$

No matter how they are obtained, you integrate these equations to find

$$u_1(t) = c_1 - \tfrac{1}{2}t + \tfrac{1}{2}\tan^{-1}(t) \,, \qquad u_2(t) = c_2 + \tfrac{1}{2}\tan^{-1}(t) \,.$$

A general solution is therefore

$$y = c_1 t + c_2 t^3 - \tfrac{1}{2}t^2 + \tfrac{1}{2}t\tan^{-1}(t) + \tfrac{1}{2}t^3\tan^{-1}(t) \,.$$

(9) [8] When a 1.5 kilogram mass is hung vertically from a spring, at rest it stretches the spring .2 m. (Gravitational acceleration is $g = 9.8$ m/sec^2.) At $t = 0$ the mass is displaced .3 m above its rest position and is released with a downward initial velocity of .4 m/sec. Assume that the spring force is proportional to displacement, that there is no drag force, and that the mass is driven by an external force of $F_{ext}(t) = 3\cos(\omega t)$ Newtons (1 Newton = 1 kg m/sec^2), where up is taken to be positive.

(a) Formulate an initial-value problem that governs the motion of the mass for $t > 0$. (DO NOT solve this initial-value problem, just write it down!)

Solution. Let $h(t)$ be the displacement (in meters) of the mass from its rest position at time t (in seconds), with upward displacements being positive. The governing initial-value problem then has the form

$$m\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + kh = F_{ext}(t)\,, \qquad h(0) = .3\,, \quad h'(0) = -.4\,,$$

where m is the mass and k is the spring constant. The problem says that $m = 1.5$ kilograms. The spring constant is obtained by balancing the weight of the mass ($mg = 1.5 \cdot 9.8$ Newtons) with the force applied by the spring when it is stetched .2 m. This gives $k\,.2 = 1.5 \cdot 9.8$, or

$$k = \frac{1.5 \cdot 9.8}{.2} = \frac{1.5 \cdot 98}{2} \quad \text{Newtons/m}\,.$$

Because $F_{ext}(t) = 3\cos(\omega t)$, the governing initial-value problem is therefore

$$1.5\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + \frac{1.5 \cdot 98}{2}\,h = 3\cos(\omega t)\,, \qquad h(0) = .3\,, \quad h'(0) = -.4\,.$$

While the above answer was sufficient, had you put the initial-value problem into normal form you should have obtained

$$\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 49\,h = 2\cos(\omega t)\,, \qquad h(0) = .3\,, \quad h'(0) = -.4\,.$$

(b) What is the natural frequency of this spring?

Solution. The natural frequency of the spring is given by

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.5 \cdot 98}{1.5 \cdot 2}} = \sqrt{49} = 7 \quad 1/\text{sec}\,.$$

(c) At what value of the driving frequency ω does resonance occur?

Solution. Resonance occurs when the driving frequency ω equals the natural fequency of the spring ω_o. Given the answer to part (b), resonance occurs when

$$\omega = \omega_o = 7 \quad 1/\text{sec}\,.$$

(10) [8] The vertical displacement of a mass on a spring is given by

$$h(t) = 3e^{-t}\cos(\pi t) - 4e^{-t}\sin(\pi t)\,.$$

(a) Why is this system under damped?

Solution. The system is under damped because the given displacement corresponds to the underlying characteristic polynomial having the complex conjugate pair of roots $-1 \pm i\pi$.

(b) Express $h(t)$ in the form $h(t) = Ae^{-t}\cos(\omega t - \delta)$ with $A > 0$ and $0 \le \delta < 2\pi$, identifying the quasiperiod and phase of the oscillation. (The phase may be expressed in terms of an inverse trig function.)

Solution. By compairing

$$Ae^{-t}\cos(\omega t - \delta) = Ae^{-t}\cos(\delta)\cos(\omega t) + Ae^{-t}\sin(\delta)\sin(\omega t)\,,$$

with $h(t) = 3e^{-t}\cos(\pi t) - 4e^{-t}\sin(\pi t)$, we see that $\omega = \pi$ and that

$$A\cos(\delta) = 3\,, \qquad A\sin(\delta) = -4\,.$$

This shows that (A, δ) are the polar coordinates of the point in the plane whose Cartesian coordinates are $(3, -4)$. Clearly A is given by

$$A = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5\,.$$

Because $(3, -4)$ lies in the fourth quadrant, the phase δ satisfies $\frac{3\pi}{2} < \delta < 2\pi$. Because

$$\sin(\delta) = -\tfrac{4}{5}\,, \qquad \tan(\delta) = -\tfrac{4}{3}\,, \qquad \cos(\delta) = \tfrac{3}{5}\,,$$

you can express the phase by any one of the formulas

$$\delta = 2\pi - \sin^{-1}\left(\tfrac{4}{5}\right), \qquad \delta = 2\pi - \tan^{-1}\left(\tfrac{4}{3}\right), \qquad \delta = 2\pi - \cos^{-1}\left(\tfrac{3}{5}\right).$$

Finally, the quasiperiod T is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\,.$$