Third In-Class Exam Solutions
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(1) [6] Given that 2 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 2.

Solution. The eigenvectors of $\mathbf{A}$ associated with 2 are all nonzero vectors $\mathbf{v}$ that satisfy $\mathbf{A}\mathbf{v} = 2\mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ that satisfy $(\mathbf{A} - 2\mathbf{I})\mathbf{v} = 0$, which is

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0\,.$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} -v_2 + v_3 &= 0\,, \\ v_1 - v_2 - v_3 &= 0\,, \\ -v_2 + v_3 &= 0\,. \end{aligned}$$

You may solve this system either by elimination or by row reduction. By either method you find that its general solution is

$$v_1 = 2\alpha\,, \quad v_2 = \alpha\,, \quad v_3 = \alpha\,, \qquad \text{for any constant } \alpha\,.$$

The eigenvectors of $\mathbf{A}$ associated with 1 therefore have the form

$$\alpha \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \text{for any nonzero constant } \alpha\,.$$

(2) [11] A 3×3 matrix has eigenpairs

$$\left(1\,, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right), \qquad \left(2\,, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right), \qquad \left(3\,, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right).$$

(a) Give an invertible matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. (You do not have to compute $\mathbf{V}^{-1}$!)

Solution.

$$\mathbf{V} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) Give a general solution to the system $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$.

Solution.

$$\mathbf{x}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} .$$

(3) [6] Transform the equation $\frac{\mathrm{d}^4 w}{\mathrm{d}t^4} - t\frac{\mathrm{d}^3 w}{\mathrm{d}t^3} + \frac{\mathrm{d}w}{\mathrm{d}t} - 3w = \cos(t)$ into a first-order system of ordinary differential equations.

Solution: Because the equation is fourth order, the first order system must have dimension four. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \cos(t) + 3x_1 - x_2 + tx_4 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} w \\ w' \\ w'' \\ w''' \end{pmatrix} .$$

(4) [10] Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^4 \\ 3 + t^6 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} 1 & t^4 \\ t^2 & 3 + t^6 \end{pmatrix} = 3 + t^6 - t^6 = 3 .$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} 1 & t^4 \\ t^2 & 3 + t^6 \end{pmatrix}$. Because $\frac{\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\begin{aligned} \mathbf{A}(t) &= \frac{\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 0 & 4t^3 \\ 2t & 6t^5 \end{pmatrix} \begin{pmatrix} 1 & t^4 \\ t^2 & 3 + t^6 \end{pmatrix}^{-1} \\ &= \frac{1}{3}\begin{pmatrix} 0 & 4t^3 \\ 2t & 6t^5 \end{pmatrix} \begin{pmatrix} 3 + t^6 & -t^4 \\ -t^2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -4t^5 & 4t^3 \\ 6t - 4t^7 & 4t^5 \end{pmatrix} . \end{aligned}$$

(c) Give a general solution to the system you found in part (b).

Solution. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1\begin{pmatrix} 1 \\ t^2 \end{pmatrix} + c_2\begin{pmatrix} t^4 \\ 3 + t^6 \end{pmatrix} .$$

(5) [4] Given that $e^{t\mathbf{A}} = \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix}$, solve the initial-value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Solution. The solution of the initial-value problem is given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2\cosh(2t) + 6\sinh(2t) \\ \sinh(2t) + 3\cosh(2t) \end{pmatrix}. \end{aligned}$$

(6) [9] Consider two interconnected tanks filled with brine (salt water). The first tank contains 60 liters and the second contains 40 liters. Brine with a concentration of 3 grams of salt per liter flows into the first tank at a rate of 4 liters per hour. Well stirred brine flows from the first tank into the second at a rate of 5 liters per hour, from the second into the first at a rate of 2 liters per hour, from the first into a drain at a rate of 1 liter per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 35 grams of salt in the first tank and 7 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution: The rates work out so there will always be 60 liters of brine in the first tank and 40 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 3\cdot 4 + \frac{S_2}{40}\,2 - \frac{S_1}{60}\,5 - \frac{S_1}{60}\,1\,, & S_1(0) &= 35\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{60}\,5 - \frac{S_2}{40}\,2 - \frac{S_2}{40}\,3\,, & S_2(0) &= 7\,. \end{aligned}$$

You could leave the answer in the above form. It can however be simplified to

$$\begin{aligned} \frac{\mathrm{d}S_1}{\mathrm{d}t} &= 12 + \frac{S_2}{20} - \frac{S_1}{10}\,, & S_1(0) &= 35\,, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} &= \frac{S_1}{12} - \frac{S_2}{8}\,, & S_2(0) &= 7\,. \end{aligned}$$

(7) [12] Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 5 & -3 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 4z + 13 = (z+2)^2 + 9\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are the conjugate pair $-2 \pm i3$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{-2t}\left[\mathbf{I}\cos(3t) + \big(\mathbf{A} + 2\mathbf{I}\big)\frac{\sin(3t)}{3}\right] \\ &= e^{-2t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(3t) + \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}\frac{\sin(3t)}{3}\right] \\ &= e^{-2t}\begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}\,. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{-2t}\begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) & -\frac{2}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) & \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{-2t}\begin{pmatrix} \cos(3t) + \frac{1}{3}\sin(3t) \\ \frac{5}{3}\sin(3t) \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} -\frac{2}{3}\sin(3t) \\ \cos(3t) - \frac{1}{3}\sin(3t) \end{pmatrix}\,. \end{aligned}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 9 = (z-3)^2\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is the double real root 3. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{3t}\big[\mathbf{I} + (\mathbf{A} - 3\mathbf{I})t\big] = e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}t\right] \\ &= e^{3t}\begin{pmatrix} 1-2t & 2t \\ -2t & 1+2t \end{pmatrix}\,. \end{aligned}$$

A general solution is therefore given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t}\begin{pmatrix} 1-2t & 2t \\ -2t & 1+2t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^{3t}\begin{pmatrix} 1-2t \\ -2t \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 2t \\ 1+2t \end{pmatrix}\,. \end{aligned}$$

(8) [8] Sketch a phase-plane portrait for each of the two systems in the previous problem. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

(a) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z+2)^2+9$, one sees that $\mu = -2$ and $\delta = -9$. There are no real eigenpairs. Because $\mu = -2 < 0$, $\delta = -9 < 0$, and $a_{21} = 5 > 0$ the phase portrait is a *counterclockwise spiral sink*. The origin is thereby *attracting* (and also *stable*). The phase portrait should indicate a family of counterclockwise spiral trajectories that approach the origin.

(b) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-3)^2$, one sees that $\mu = 3$ and $\delta = 0$. Because

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix},$$

we see that the eigenvectors associated with 1 have the form

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{for some } \alpha \neq 0\,.$$

The associated solutions will therefore lie on the line $y = x$. Because $\mu = 3 > 0$, $\delta = 0$, and $a_{21} = -2 < 0$ the phase portrait is a *clockwise twist source*. The origin is thereby *repelling* (and also *unstable*). The phase portrait should show that on the line $y = x$ there is one trajectory that emerges from each side of the origin. Every other trajectory emerges from the origin with a clockwise twist.

(9) [9] Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -4x + x^3 \end{pmatrix}.$$

(a) Find all of its stationary points.

Solution. Stationary points satisfy

$$\begin{aligned} 0 &= y\,, \\ 0 &= -4x + x^3 = x(x^2-4) = x(x+2)(x-2)\,. \end{aligned}$$

The top equation shows that $y = 0$ while the bottom equation shows that either $x = 0$ or $x = -2$ or $x = 2$. The stationary points of the system are therefore

$$(0,0)\,, \qquad (-2,0)\,, \qquad (2,0)\,.$$

(b) Find a nonconstant function $h(x,y)$ such that every trajectory of the system satisfies $h(x,y) = c$ for some constant c.

Solution. The associated first-order equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4x + x^3}{y}\,.$$

This equation is separable, so can be integrated as

$$\int y\,\mathrm{d}y = \int -4x + x^3\,\mathrm{d}x\,,$$

whereby you find that

$$\tfrac{1}{2}y^2 = -2x^2 + \tfrac{1}{4}x^4 + c\,.$$

You can thereby set

$$h(x,y) = \tfrac{1}{2}y^2 + 2x^2 - \tfrac{1}{4}x^4\,.$$

Alternative Solution. An alternative approach is to notice that

$$\partial_x f(x,y) + \partial_y g(x,y) = \partial_x y + \partial_y(-4x + x^3) = 0\,.$$

The system is therefore Hamiltonian with $h(x,y)$ such that

$$\partial_y h(x,y) = y\,, \qquad -\partial_x h(x,y) = -4x + x^3\,.$$

Integrating the first equation above yields $h(x,y) = \tfrac{1}{2}y^2 + c(x)$. Substituting this into the second equation gives

$$-c'(x) = -4x + x^3\,.$$

Integrating this equation yields $c(x) = 2x^2 - \tfrac{1}{4}x^4$, whereby

$$h(x,y) = \tfrac{1}{2}y^2 + 2x^2 - \tfrac{1}{4}x^4\,.$$

(10) [8] Compute the Laplace transform of $f(t) = u(t-4)\,e^{-3t}$ from its definition. (Here u is the unit step function.)

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} u(t-4)\,e^{-3t}\,\mathrm{d}t = \lim_{T\to\infty} \int_4^T e^{-(s+3)t}\,\mathrm{d}t\,.$$

When $s \leq -3$ this limit diverges to $+\infty$ because in that case one has for every $T > 4$ that

$$\int_4^T e^{-(s+3)t}\,\mathrm{d}t \geq \int_4^T \mathrm{d}t = T - 4\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

When $s > -3$ one has for every $T > 4$ that

$$\int_4^T e^{-(s+3)t}\,\mathrm{d}t = -\frac{e^{-(s+3)t}}{s+3}\bigg|_4^T = -\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)4}}{s+3}\,,$$

whereby

$$\mathcal{L}[f](s) = \lim_{T\to\infty}\left[-\frac{e^{-(s+3)T}}{s+3} + \frac{e^{-(s+3)4}}{s+3}\right] = \frac{e^{-(s+3)4}}{s+3}\,.$$

(11) [9] Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 6\frac{\mathrm{d}y}{\mathrm{d}t} + 18y = f(t)\,, \qquad y(0) = 2\,, \quad y'(0) = -4\,,$$

where

$$f(t) = \begin{cases} \frac{1}{3}t & \text{for } 0 \le t < 3\,, \\ e^{3-t} & \text{for } t \ge 3\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 6\mathcal{L}[y'](s) + 18\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\begin{aligned} \mathcal{L}[y](s) &= Y(s)\,, \\ \mathcal{L}[y'](s) &= sY(s) - y(0) = sY(s) - 2\,, \\ \mathcal{L}[y''](s) &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s + 4\,. \end{aligned}$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(u(t) - u(t-3)\big)\tfrac{1}{3}t + u(t-3)e^{3-t} \\ &= u(t)\tfrac{1}{3}t + u(t-3)\big(e^{3-t} - \tfrac{1}{3}t\big) \\ &= \tfrac{1}{3}t + u(t-3)\big(e^{3-t} - \tfrac{1}{3}(t-3) - 1\big)\,. \end{aligned}$$

Referring to the table on the last page, item 6 with $c = 3$ and $h(t) = e^{-t} - \frac{1}{3}t - 1$ shows that

$$\mathcal{L}\big[u(t-3)\big(e^{3-t} - \tfrac{1}{3}(t-3) - 1\big)\big](s) = \mathcal{L}[u(t-3)h(t-3)](s) = e^{-3s}\mathcal{L}[h](s)\,,$$

while item 1 with $a = -1$ and $n = 0$, with $a = 0$ and $n = 1$, and with $a = 0$ and $n = 0$ shows that

$$\mathcal{L}[e^{-t}](s) = \frac{1}{s+1}\,, \qquad \mathcal{L}[t](s) = \frac{1}{s^2}\,, \qquad \mathcal{L}[1](s) = \frac{1}{s}\,,$$

whereby

$$\mathcal{L}[\tfrac{1}{3}t](s) = \frac{1}{3s^3}\,, \qquad \mathcal{L}[h](s) = \frac{1}{s+1} - \frac{1}{3s^2} - \frac{1}{s}\,.$$

Therefore

$$\mathcal{L}[f](s) = \frac{1}{3s^2} + e^{-3s}\left(\frac{1}{s+1} - \frac{1}{3s^2} - \frac{1}{s}\right).$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 2s + 4\big) + 6\big(sY(s) - 2\big) + 18Y(s) = \frac{1 - e^{-3s}}{3s^2} + \frac{e^{-3s}}{s+1} - \frac{e^{-3s}}{s}\,,$$

which becomes

$$(s^2 + 6s + 18)Y(s) - 2s + 4 - 12 = \frac{1 - e^{-3s}}{3s^2} + \frac{e^{-3s}}{s+1} - \frac{e^{-3s}}{s}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 6s + 18}\left(2s + 8 + \frac{1 - e^{-3s}}{3s^2} + \frac{e^{-3s}}{s+1} - \frac{e^{-3s}}{s}\right).$$

(12) [8] Find the inverse Laplace transforms of the function

$$F(s) = e^{-4s}\,\frac{s+5}{s^2-5s+6}\,.$$

You may refer to the table on the last page.

Solution. The denominator factors as $(s-3)(s-2)$, so the partial fraction decomposition is

$$\frac{s+5}{s^2-5s+6} = \frac{s+5}{(s-3)(s-2)} = \frac{8}{s-3} + \frac{-7}{s-2}\,.$$

Referring to the table on the last page, item 1 with $a=3$ and $n=0$, and with $a=2$ and $n=0$ shows that

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}\,, \qquad \mathcal{L}[e^{2t}](s) = \frac{1}{s-2}\,.$$

You therefore obtain

$$\mathcal{L}^{-1}\left[\frac{s+5}{s^2-5s+6}\right](t) = 8\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - 7\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = 8e^{3t} - 7e^{2t}\,.$$

Then item 6 with $c=4$ and $h(t) = 8e^{3t} - 7e^{2t}$ shows that

$$\begin{aligned}\mathcal{L}^{-1}\left[e^{-4s}\frac{s+5}{s^2-5s+6}\right](t) &= u(t-4)\,\mathcal{L}^{-1}\left[\frac{s+5}{s^2-5s+6}\right](t-4)\\ &= u(t-4)\left(8e^{3(t-4)} - 7e^{2(t-4)}\right).\end{aligned}$$

A Short Table of Laplace Transforms

$$\begin{aligned}
\mathcal{L}[e^{at}t^n](s) &= \frac{n!}{(s-a)^{n+1}} && \text{for } s>a\,,\\
\mathcal{L}[e^{at}\cos(bt)](s) &= \frac{s-a}{(s-a)^2+b^2} && \text{for } s>a\,,\\
\mathcal{L}[e^{at}\sin(bt)](s) &= \frac{b}{(s-a)^2+b^2} && \text{for } s>a\,,\\
\mathcal{L}[e^{at}h(t)](s) &= H(s-a) && \text{where } H(s)=\mathcal{L}[h(t)](s)\,,\\
\mathcal{L}[t^n h(t)](s) &= (-1)^n H^{(n)}(s) && \text{where } H(s)=\mathcal{L}[h(t)](s)\,,\\
\mathcal{L}[u(t-c)h(t-c)](s) &= e^{-cs}H(s) && \text{where } H(s)=\mathcal{L}[h(t)](s)\\
& && \text{and } u \text{ is the step function}\,.
\end{aligned}$$