Sample Problems for Third In-Class Exam
Math 246, Fall 2009, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\, e^{3t}$ from its definition.

(2) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi\,, \\ t - 2\pi & \text{for } t \geq 2\pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

(3) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}\,,$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}\,,$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}\,.$

(4) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}\,, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}\,.$$

Compute the matrices
(a) $\mathbf{A}^T$,
(b) $\overline{\mathbf{A}}$,
(c) $\mathbf{A}^*$,
(d) $5\mathbf{A} - \mathbf{B}$,
(e) $\mathbf{AB}$,
(f) $\mathbf{B}^{-1}$.

(5) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix}\,.$$

(a) Find all the eigenvalues of $\mathbf{A}$.
(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.
(c) Diagonalize $\mathbf{A}$.

(6) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

(7) Transform the equation $\dfrac{\mathrm{d}^3 u}{\mathrm{d}t^3} + t^2 \dfrac{\mathrm{d}u}{\mathrm{d}t} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

(8) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

(9) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4 + 3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x},$$

wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).

(d) For the system found in part (b), solve the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}, \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(10) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

(11) Solve each of the following initial-value problems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(12) Find a general solution for each of the following systems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(c) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(13) Sketch the phase-plane portrait for each of the systems in the previous problem. Indicate typical trajectories. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

(14) Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+1 \\ 4x - x^2 \end{pmatrix}.$$

(a) Find all of its stationary points.
(b) Find a nonconstant function $H(x,y)$ such that every trajectory of the system satisfies $H(x,y) = c$ for some constant c.
(c) Sketch a phase portrait of the system. Indicate its stationary points and some typical trajectories.
(d) Identify each stationary point as being either stable or unstable.

A Short Table of Laplace Transforms

$\mathcal{L}[t^n](s) = \dfrac{n!}{s^{n+1}}$	for $s > 0$.
$\mathcal{L}[\cos(bt)](s) = \dfrac{s}{s^2+b^2}$	for $s > 0$.
$\mathcal{L}[\sin(bt)](s) = \dfrac{b}{s^2+b^2}$	for $s > 0$.
$\mathcal{L}[t^n f(t)](s) = (-1)^n F^{(n)}(s)$	where $F(s) = \mathcal{L}[f(t)](s)$.
$\mathcal{L}[e^{at} f(t)](s) = F(s-a)$	where $F(s) = \mathcal{L}[f(t)](s)$.
$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s)$	where $F(s) = \mathcal{L}[f(t)](s)$ and u is the unit step function.