Solutions of Sample Problems for Third In-Class Exam
Math 246, Fall 2009, Professor David Levermore

(1) Compute the Laplace transform of $f(t) = t\,e^{3t}$ from its definition.

Solution. The definition of Laplace transform gives

$$\mathcal{L}[f](s) = \lim_{T\to\infty} \int_0^T e^{-st} t\, e^{3t}\, \mathrm{d}t = \lim_{T\to\infty} \int_0^T t\, e^{(3-s)t}\, \mathrm{d}t\,.$$

This limit diverges to $+\infty$ for $s \leq 3$ because in that case

$$\int_0^T t\, e^{(3-s)t}\, \mathrm{d}t \geq \int_0^T t\, \mathrm{d}t = \frac{T^2}{2}\,,$$

which clearly diverges to $+\infty$ as $T \to \infty$.

For $s > 3$ an integration by parts shows that

$$\begin{aligned}
\int_0^T t\, e^{(3-s)t}\, \mathrm{d}t &= t\, \frac{e^{(3-s)t}}{3-s}\bigg|_0^T - \int_0^T \frac{e^{(3-s)t}}{3-s}\, \mathrm{d}t \\
&= \left(t\, \frac{e^{(3-s)t}}{3-s} - \frac{e^{(3-s)t}}{(3-s)^2} \right)\bigg|_0^T \\
&= \left(T\, \frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) + \frac{1}{(3-s)^2}\,.
\end{aligned}$$

Hence, for $s > 3$ one has that

$$\begin{aligned}
\mathcal{L}[f](s) &= \lim_{T\to\infty} \left[\left(T\, \frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) + \frac{1}{(3-s)^2} \right] \\
&= \frac{1}{(3-s)^2} + \lim_{T\to\infty} \left(T\, \frac{e^{(3-s)T}}{3-s} - \frac{e^{(3-s)T}}{(3-s)^2} \right) \\
&= \frac{1}{(3-s)^2}\,.
\end{aligned}$$

(2) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial-value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 13y = f(t)\,, \qquad y(0) = 4\,, \quad y'(0) = 1\,,$$

where

$$f(t) = \begin{cases} \cos(t) & \text{for } 0 \leq t < 2\pi\,, \\ t - 2\pi & \text{for } t \geq 2\pi\,. \end{cases}$$

You may refer to the table on the last page. DO NOT take the inverse Laplace transform to find $y(t)$, just solve for $Y(s)$!

Solution. The Laplace transform of the initial-value problem is

$$\mathcal{L}[y''](s) + 4\mathcal{L}[y'](s) + 13\mathcal{L}[y](s) = \mathcal{L}[f](s)\,,$$

where

$$\mathcal{L}[y](s) = Y(s)\,,$$
$$\mathcal{L}[y'](s) = sY(s) - y(0) = sY(s) - 4\,,$$
$$\mathcal{L}[y''](s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 4s - 1\,.$$

To compute $\mathcal{L}[f](s)$, first write f as

$$\begin{aligned} f(t) &= \big(1 - u(t-2\pi)\big)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) - u(t-2\pi)\cos(t) + u(t-2\pi)(t-2\pi) \\ &= \cos(t) - u(t-2\pi)\cos(t-2\pi) + u(t-2\pi)(t-2\pi)\,. \end{aligned}$$

Referring to the table on the last page, item 6 with $c = 2\pi$, item 2 with $b = 1$, and item 1 with $n = 1$ then show that

$$\begin{aligned} \mathcal{L}[f](s) &= \mathcal{L}[\cos(t)](s) - \mathcal{L}[u(t-2\pi)\cos(t-2\pi)](s) + \mathcal{L}[u(t-2\pi)(t-2\pi)](s) \\ &= \mathcal{L}[\cos(t)](s) - e^{-2\pi s}\mathcal{L}[\cos(t)](s) + e^{-2\pi s}\mathcal{L}[t](s) \\ &= \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,. \end{aligned}$$

The Laplace transform of the initial-value problem then becomes

$$\big(s^2Y(s) - 4s - 1\big) + 4\big(sY(s) - 4\big) + 13Y(s) = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,,$$

which becomes

$$(s^2 + 4s + 13)Y(s) - 4s - 1 - 16 = \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\,.$$

Hence, $Y(s)$ is given by

$$Y(s) = \frac{1}{s^2 + 4s + 13}\left(4s + 17 + \big(1 - e^{-2\pi s}\big)\frac{s}{s^2+1} + e^{-2\pi s}\frac{1}{s^2}\right).$$

(3) Find the inverse Laplace transforms of the following functions. You may refer to the table on the last page.

(a) $F(s) = \dfrac{2}{(s+5)^2}$,

Solution. Referring to the table on the last page, item 1 with $n = 1$ gives $\mathcal{L}[t](s) = 1/s^2$. Item 4 with $a = -5$ and $f(t) = t$ then gives

$$\mathcal{L}[e^{-5t}t](s) = \frac{1}{(s+5)^2}\,.$$

Multiplying this by 2 yields

$$\mathcal{L}[2e^{-5t}t](s) = \frac{2}{(s+5)^2}\,.$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[\frac{2}{(s+5)^2}\right](t) = 2e^{-5t}t\,.$$

(b) $F(s) = \dfrac{3s}{s^2 - s - 6}$,

Solution. The denominator factors as $(s-3)(s+2)$, so the partial fraction decomposition is

$$\frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{\frac{9}{5}}{s-3} + \frac{\frac{6}{5}}{s+2}\,.$$

Referring to the table on the last page, item 1 with $n = 0$ gives $\mathcal{L}[1](s) = 1/s$. Item 5 with $a = 3$ and $f(t) = 1$, and with $a = -2$ and $f(t) = 1$, then gives

$$\mathcal{L}[e^{3t}](s) = \frac{1}{s-3}\,, \qquad \mathcal{L}[e^{-2t}](s) = \frac{1}{s+2}\,,$$

whereby

$$\frac{3s}{s^2 - s - 6} = \tfrac{9}{5}\mathcal{L}[e^{3t}](s) + \tfrac{6}{5}\mathcal{L}[e^{-2t}](s) = \mathcal{L}\big[\tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\big](s)\,.$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[\frac{3s}{s^2 - s - 6}\right](t) = \tfrac{9}{5}e^{3t} + \tfrac{6}{5}e^{-2t}\,.$$

(c) $F(s) = \dfrac{(s-2)e^{-3s}}{s^2 - 4s + 5}$.

Solution. Complete the square in the denominator to get $(s-2)^2 + 1$. Referring to the table on the last page, item 2 with $b = 1$ gives

$$\mathcal{L}[\cos(t)](s) = \frac{s}{s^2 + 1}\,.$$

Item 5 with $a = 2$ and $f(t) = \cos(t)$ then gives

$$\mathcal{L}[e^{2t}\cos(t)](s) = \frac{s-2}{(s-2)^2 + 1}\,.$$

Item 6 with $c = 3$ and $f(t) = e^{2t}\cos(t)$ then gives

$$\mathcal{L}[u(t-3)e^{2(t-3)}\cos(t-3)](s) = e^{-3s}\frac{s-2}{(s-2)^2 + 1}\,.$$

You therefore conclude that

$$\mathcal{L}^{-1}\left[e^{-3s}\frac{s-2}{s^2 - 4s + 5}\right](t) = u(t-3)e^{2(t-3)}\cos(t-3)\,.$$

(4) Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}.$$

Compute the matrices

(a) $\mathbf{A}^T$,

Solution. The transpose of $\mathbf{A}$ is

$$\mathbf{A}^T = \begin{pmatrix} -i2 & 2+i \\ 1+i & -4 \end{pmatrix} .$$

(b) $\overline{\mathbf{A}}$,

Solution. The conjugate of $\mathbf{A}$ is

$$\overline{\mathbf{A}} = \begin{pmatrix} i2 & 1-i \\ 2-i & -4 \end{pmatrix} .$$

(c) $\mathbf{A}^*$,

Solution. The adjoint of $\mathbf{A}$ is

$$\mathbf{A}^* = \begin{pmatrix} i2 & 2-i \\ 1-i & -4 \end{pmatrix} .$$

(d) $5\mathbf{A} - \mathbf{B}$,

Solution. The difference of $5\mathbf{A}$ and $\mathbf{B}$ is given by

$$5\mathbf{A} - \mathbf{B} = \begin{pmatrix} -i10 & 5+i5 \\ 10+i5 & -20 \end{pmatrix} - \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} -7-i10 & -1+i5 \\ 2+i5 & -27 \end{pmatrix} .$$

(e) $\mathbf{AB}$,

Solution. The product of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} -i2 & 1+i \\ 2+i & -4 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} -i2{\cdot}7 + (1+i){\cdot}8 & -i2{\cdot}6 + (1+i){\cdot}7 \\ (2+i){\cdot}7 - 4{\cdot}8 & (2+i){\cdot}6 - 4{\cdot}7 \end{pmatrix} \\ &= \begin{pmatrix} 8-i6 & 7-i5 \\ -18+i7 & -16+i6 \end{pmatrix} . \end{aligned}$$

(f) $\mathbf{B}^{-1}$.

Solution. Observe that it is clear that $\mathbf{B}$ has an inverse because

$$\det(\mathbf{B}) = \det\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix} = 7 \cdot 7 - 6 \cdot 8 = 49 - 48 = 1 .$$

The inverse of $\mathbf{B}$ is given by

$$\mathbf{B}^{-1} = \frac{1}{\det(\mathbf{B})} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} .$$

(5) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 4 & -1 \end{pmatrix} .$$

(a) Find all the eigenvalues of $\mathbf{A}$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 15 = (z-1)^2 - 16 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 4, or simply -3 and 5.

(b) For each eigenvalue of $\mathbf{A}$ find all of its eigenvectors.

Solution (using the Cayley-Hamilton method from notes). One has

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix} , \qquad \mathbf{A} - 5\mathbf{I} = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix} .$$

Every nonzero column of $\mathbf{A} - 5\mathbf{I}$ has the form

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } \alpha_1 \neq 0 .$$

These are all the eigenvectors associated with -3. Similarly, every nonzero column of $\mathbf{A} + 3\mathbf{I}$ has the form

$$\alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for some } \alpha_2 \neq 0 .$$

These are all the eigenvectors associated with 5.

(c) Diagonalize $\mathbf{A}$.

Solution. If you use the eigenpairs

$$\left(-3 , \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right) , \qquad \left(5 , \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right) ,$$

then set

$$\mathbf{V} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} , \qquad \mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} .$$

Because $\det(\mathbf{V}) = 1 \cdot 2 - (-2) \cdot 3 = 2 + 6 = 8$, you see that

$$\mathbf{V}^{-1} = \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} .$$

You conclude that $\mathbf{A}$ has the diagonalization

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \frac{1}{8} \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} .$$

You do not have to multiply these matrices out. Had you started with different eigenpairs, the steps would be the same as above but you would obtain a different diagonalization.

(6) Given that 1 is an eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix} ,$$

find all the eigenvectors of $\mathbf{A}$ associated with 1.

Solution. The eigenvectors of $\mathbf{A}$ associated with 1 are all nonzero vectors $\mathbf{v}$ that satisfy $\mathbf{A}\mathbf{v} = \mathbf{v}$. Equivalently, they are all nonzero vectors $\mathbf{v}$ that satisfy $(\mathbf{A}-\mathbf{I})\mathbf{v} = 0$, which is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 .$$

The entries of $\mathbf{v}$ thereby satisfy the homogeneous linear algebraic system

$$\begin{aligned} v_1 - v_2 + \ \ v_3 &= 0 , \\ v_1 \qquad - \ \ v_3 &= 0 , \\ - v_2 + 2v_3 &= 0 . \end{aligned}$$

You may solve this system either by elimination or by row reduction. By either method you find that its general solution is

$$v_1 = \alpha , \quad v_2 = 2\alpha , \quad v_3 = \alpha , \qquad \text{for any constant } \alpha .$$

The eigenvectors of $\mathbf{A}$ associated with 1 therefore have the form

$$\alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{for any nonzero constant } \alpha .$$

(7) Transform the equation $\dfrac{\mathrm{d}^3u}{\mathrm{d}t^3} + t^2\dfrac{\mathrm{d}u}{\mathrm{d}t} - 3u = \sinh(2t)$ into a first-order system of ordinary differential equations.

Solution: Because the equation is third order, the first order system must have dimension three. The simplest such first order system is

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \sinh(2t) + 3x_1 - t^2 x_2 \end{pmatrix} , \quad \text{where} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix} .$$

(8) Consider two interconnected tanks filled with brine (salt water). The first tank contains 100 liters and the second contains 50 liters. Brine flows with a concentration of 2 grams of salt per liter flows into the first tank at a rate of 3 liters per hour. Well stirred brine flows from the first tank to the second at a rate of 5 liters per hour, from the second to the first at a rate of 2 liters per hour, and from the second into a drain at a rate of 3 liters per hour. At $t = 0$ there are 5 grams of salt in the first tank and 20 grams in the second. Give an initial-value problem that governs the amount of salt in each tank as a function of time.

Solution: The rates work out so there will always be 100 liters of brine in the first tank and 50 liters in the second. Let $S_1(t)$ be the grams of salt in the first tank and $S_2(t)$ be the grams of salt in the second tank. These are governed by the initial-value problem

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 2\cdot 3 + \frac{S_2}{50}\,2 - \frac{S_1}{100}\,5\,, \qquad S_1(0) = 2\,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{S_1}{100}\,5 - \frac{S_2}{50}\,2 - \frac{S_2}{50}\,3\,, \qquad S_2(0) = 20\,.$$

You could leave the answer in the above form. It can however be simplified to

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} = 6 + \frac{S_2}{25} - \frac{S_1}{20}\,, \qquad S_1(0) = 2\,,$$
$$\frac{\mathrm{d}S_2}{\mathrm{d}t} = \frac{S_1}{20} - \frac{S_2}{10}\,, \qquad S_2(0) = 20\,.$$

(9) Consider the vector-valued functions $\mathbf{x}_1(t) = \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix}$, $\mathbf{x}_2(t) = \begin{pmatrix} t^2 \\ 3 \end{pmatrix}$.

(a) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2](t)$.

Solution.

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix} = 3t^4 + 9 - 2t^4 = t^4 + 9\,.$$

(b) Find $\mathbf{A}(t)$ such that $\mathbf{x}_1$, $\mathbf{x}_2$ is a fundamental set of solutions to $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}$ wherever $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$.

Solution. Let $\mathbf{\Psi}(t) = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}$. Because $\dfrac{\mathbf{\Psi}(t)}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{\Psi}(t)$, one has

$$\mathbf{A}(t) = \frac{\mathbf{\Psi}(t)}{\mathrm{d}t}\mathbf{\Psi}(t)^{-1} = \begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}^{-1}$$
$$= \frac{1}{t^4+9}\begin{pmatrix} 4t^3 & 2t \\ 4t & 0 \end{pmatrix}\begin{pmatrix} 3 & -t^2 \\ -2t^2 & t^4+3 \end{pmatrix} = \frac{1}{t^4+9}\begin{pmatrix} 8t^3 & 6t-2t^5 \\ 12t & -4t^3 \end{pmatrix}\,.$$

(c) Give a fundamental matrix $\mathbf{\Psi}(t)$ for the system found in part (b).

Solution. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a fundamental matrix for the system found in part (b) is simply given by

$$\mathbf{\Psi}(t) = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} t^4+3 & t^2 \\ 2t^2 & 3 \end{pmatrix}\,.$$

(d) For the system found in part (b), solve the initial-value problem

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}(t)\mathbf{x}\,, \qquad \mathbf{x}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,.$$

Solution. Because $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ is a fundamental set of solutions to the system found in part (b), a general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = c_1 \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ 3 \end{pmatrix} .$$

The initial condition then implies that

$$\mathbf{x}(1) = c_1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4c_1 + c_2 \\ 2c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

from which we see that $c_1 = \frac{3}{10}$ and $c_2 = -\frac{1}{5}$. The solution of the initial-value problem is thereby

$$\mathbf{x}(t) = \tfrac{3}{10} \begin{pmatrix} t^4+3 \\ 2t^2 \end{pmatrix} - \tfrac{1}{5} \begin{pmatrix} t^2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}t^4 - \frac{1}{5}t^2 + \frac{9}{10} \\ \frac{3}{5}t^2 - \frac{3}{5} \end{pmatrix} .$$

(10) Compute $e^{t\mathbf{A}}$ for the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z - 3 = (z-1)^2 - 4 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are 1 ± 2. One then has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t \left[\mathbf{I}\cosh(2t) + (\mathbf{A} - \mathbf{I})\frac{\sinh(2t)}{2} \right] \\ &= e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(2t) + \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix} \frac{\sinh(2t)}{2} \right] \\ &= e^t \begin{pmatrix} \cosh(2t) & 2\sinh(2t) \\ \frac{1}{2}\sinh(2t) & \cosh(2t) \end{pmatrix} . \end{aligned}$$

(b) $\mathbf{A} = \begin{pmatrix} 6 & 4 \\ -1 & 2 \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 8z + 16 = (z-4)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which has the double root 4. One then has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{4t} \left[\mathbf{I} + (\mathbf{A} - 4\mathbf{I})t \right] \\ &= e^{4t} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} t \right] \\ &= e^{4t} \begin{pmatrix} 1+2t & 4t \\ -t & 1-2t \end{pmatrix} . \end{aligned}$$

(11) Solve each of the following initial-value problems.

(a) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - z - 12 = (z+3)(z-4)\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are -3 and 4. These have the form $\frac{1}{2} \pm \frac{7}{2}$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{\frac{1}{2}t}\left[\mathbf{I}\cosh\left(\tfrac{7}{2}t\right) + \left(\mathbf{A} - \tfrac{1}{2}\mathbf{I}\right)\frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\right] \\ &= e^{\frac{1}{2}t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh\left(\tfrac{7}{2}t\right) + \begin{pmatrix} \frac{3}{2} & 2 \\ 5 & -\frac{3}{2} \end{pmatrix}\frac{\sinh\left(\frac{7}{2}t\right)}{\frac{7}{2}}\right] \\ &= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

The solution of the initial-value problem is therefore

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) + \frac{3}{7}\sinh\left(\frac{7}{2}t\right) & \frac{4}{7}\sinh\left(\frac{7}{2}t\right) \\ \frac{10}{7}\sinh\left(\frac{7}{2}t\right) & \cosh\left(\frac{7}{2}t\right) - \frac{3}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= e^{\frac{1}{2}t}\begin{pmatrix} \cosh\left(\frac{7}{2}t\right) - \frac{1}{7}\sinh\left(\frac{7}{2}t\right) \\ -\cosh\left(\frac{7}{2}t\right) + \frac{13}{7}\sinh\left(\frac{7}{2}t\right) \end{pmatrix}. \end{aligned}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 5 = (z-1)^2 + 4\,.$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $1 \pm i2$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^{t}\left[\mathbf{I}\cos(2t) + \left(\mathbf{A} - \mathbf{I}\right)\frac{\sin(2t)}{2}\right] \\ &= e^{t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(2t) + \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\frac{\sin(2t)}{2}\right] \\ &= e^{t}\begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix}. \end{aligned}$$

The solution of the initial-value problem is therefore

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos(2t) + \frac{1}{2}\sin(2t) \\ -2\sin(2t) + \cos(2t) \end{pmatrix} .\end{aligned}$$

(12) Find a general solution for each of the following systems.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2z + 1 = (z-1)^2 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which is 1, a double root. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= e^t\big[\mathbf{I} + (\mathbf{A} - \mathbf{I})t\big] = e^t \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} t \right] \\ &= e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} .\end{aligned}$$

A general solution is therefore given by

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^t \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= c_1 e^t \begin{pmatrix} 1+2t \\ t \end{pmatrix} + c_2 e^t \begin{pmatrix} -4t \\ 1-2t \end{pmatrix} .\end{aligned}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 + 16 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $\pm i4$. One therefore has

$$\begin{aligned} e^{t\mathbf{A}} &= \left[\mathbf{I}\cos(4t) + \mathbf{A}\frac{\sin(4t)}{4} \right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(4t) + \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix} \frac{\sin(4t)}{4} \right] \\ &= \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} .\end{aligned}$$

A general solution is therefore given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & -\frac{5}{4}\sin(4t) \\ \sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= c_1 \begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ \sin(4t) \end{pmatrix} + c_2 \begin{pmatrix} -\frac{5}{4}\sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}.$$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Solution. The characteristic polynomial of $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -5 & 1 \end{pmatrix}$ is given by

$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 6z + 25 = (z-3)^2 + 16 .$$

The eigenvalues of $\mathbf{A}$ are the roots of this polynomial, which are $3 \pm i4$. One therefore has

$$e^{t\mathbf{A}} = e^{3t}\left[\mathbf{I}\cos(4t) + (\mathbf{A} - 3\mathbf{I})\frac{\sin(4t)}{4}\right]$$

$$= e^{3t}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(4t) + \begin{pmatrix} 2 & 4 \\ -5 & -2 \end{pmatrix}\frac{\sin(4t)}{4}\right]$$

$$= e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}.$$

A general solution is therefore given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) & \sin(4t) \\ -\frac{5}{4}\sin(4t) & \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= c_1 e^{3t}\begin{pmatrix} \cos(4t) + \frac{1}{2}\sin(4t) \\ -\frac{5}{4}\sin(4t) \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \sin(4t) \\ \cos(4t) - \frac{1}{2}\sin(4t) \end{pmatrix}.$$

(13) Sketch the phase-plane portrait for each of the systems in the previous problem. Indicate typical trajectories. For each portrait identify its type and give a reason why the origin is either attracting, stable, unstable, or repelling.

(a) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-1)^2$, one sees that $\mu = 1$ and $\delta = 0$. Because

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix},$$

we see that the eigenvectors associated with 1 have the form

$$\alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{for some } \alpha \neq 0 .$$

Because $\mu = 1 > 0$, $\delta = 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise twist source*. The origin is thereby *unstable*. The phase portrait should show there is one trajectory that emerges from the origin on each side of the line $y = x/2$. Every other trajectory emerges from the origin with a counterclockwise twist.

(b) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = z^2 + 16$, one sees that $\mu = 0$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 0$, $\delta = -16 < 0$, and $a_{21} > 0$ the phase portrait is a *counterclockwise center*. The origin is thereby *stable*. The phase portrait should indicate a family of counterclockwise elliptical trajectories that go around the origin.

(c) **Solution.** Because the characteristic polynomial of $\mathbf{A}$ is $p(z) = (z-3)^2 + 16$, one sees that $\mu = 3$ and $\delta = -16$. There are no real eigenpairs. Because $\mu = 3$, $\delta = -16 < 0$, and $a_{21} < 0$ the phase portrait is a *clockwise spiral source*. The origin is thereby *unstable*. The phase portrait should indicate a family of clockwise spiral trajectories that emerge from the origin.

(14) Consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y+1 \\ 4x - x^2 \end{pmatrix}.$$

(a) Find all of its stationary points.

Solution. Stationary points satisfy

$$\begin{aligned} 0 &= y + 1\,, \\ 0 &= 4x - x^2 = x(4-x)\,. \end{aligned}$$

The top equation shows that $y = -1$ while the bottom equation shows that either $x = 0$ or $x = 4$. The stationary points of the system are therefore

$$(0,-1)\,, \qquad (4,-1)\,.$$

(b) Find a nonconstant function $H(x,y)$ such that every trajectory of the system satisfies $H(x,y) = c$ for some constant c.

Solution. The associated first-order equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x - x^2}{y+1}\,.$$

This equation is separable, so can be integrated as

$$\int y + 1 \,\mathrm{d}y = \int 4x - x^2 \,\mathrm{d}x\,,$$

whereby you find that

$$\tfrac{1}{2}(y+1)^2 = 2x^2 - \tfrac{1}{3}x^3 + c\,.$$

You can thereby set

$$H(x,y) = \tfrac{1}{2}(y+1)^2 - 2x^2 + \tfrac{1}{3}x^3\,.$$

Alternative Solution. An alternative approach is to notice that

$$\partial_x f(x,y) + \partial_y g(x,y) = \partial_x(y+1) + \partial_y(4x - x^2) = 0\,.$$

The system is therefore Hamiltonian with $H(x,y)$ such that

$$\partial_y H(x,y) = y + 1\,, \qquad -\partial_x H(x,y) = 4x - x^2\,.$$

Integrating the first equation above yields $H(x,y) = \frac{1}{2}(y+1)^2 + h(x)$. Substituting this into the second equation gives

$$-h'(x) = 4x - x^2 \,.$$

Integrating this equation yields $h(x) = -2x^2 + \frac{1}{3}x^3$, whereby

$$H(x,y) = \tfrac{1}{2}(y+1)^2 - 2x^2 + \tfrac{1}{3}x^3 \,.$$

(c) Sketch a phase portrait of the system. Indicate its stationary points and some typical trajectories.

Solution. Solving $H(x,y) = c$ for y, you see that trajectories lie on the curves

$$y = -1 \pm \sqrt{2\big(c + 2x^2 - \tfrac{1}{3}x^3\big)}\,,$$

wherever $c + 2x^2 - \frac{1}{3}x^3 \geq 0$. Each cubic in the family $p_c(x) = c + 2x^2 - \frac{1}{3}x^3$ has a local minimum at $x = 0$ with value $p_c(0) = c$ and a local maximum at $x = 4$ with value $p_c(4) = c + 2\cdot 4^2 - \frac{1}{3}\cdot 4^3 = c + (2 - \frac{4}{3})4^2 = c + \frac{2}{3}\cdot 16 = c + \frac{32}{3}$. On the side, sketch five of these cubics for $c < -\frac{32}{3}$, $c = -\frac{32}{3}$, $-\frac{32}{3} < c < 0$, $c = 0$, and $c > 0$. You can see those points x for which each of these cubics $p_c(x)$ is nonnegative. A phase portrait is obtained by first sketching $y = -1 \pm \sqrt{2p_c(x)}$ over those points x for which each $p_c(x)$ is nonnegative, and then adding arrows to indicate the direction of the trajectories. The arrows go to the "right" for $y > -1$ and to the "left" for $y < -1$. This will be illustrated during the review.

The curves $y = -1 \pm \sqrt{2p_c(x)}$ will hit the stationary point $(0,-1)$ when $c = 0$. This point will be a saddle, and therefore unstable. The stationary point $(4,-1)$ is a isolated point on $y = -1 \pm \sqrt{2p_c(x)}$ with $c = -\frac{32}{3}$. This point will be a center point, and therefore stable.

Remark. You can sketch a phase portrait with MATLAB as follows. The values of $H(x,y)$ at the stationary points $(0,-1)$ and $(4,-1)$ are

$$H(0,-1) = \tfrac{1}{2}(-1+1)^2 - 2\cdot 0^2 + \tfrac{1}{3}0^3 = 0\,,$$
$$H(4,-1) = \tfrac{1}{2}(-1+1)^2 - 2\cdot 4^2 + \tfrac{1}{3}4^3 = (-2 + \tfrac{4}{3})4^2 = \tfrac{2}{3}\cdot 16 = -\tfrac{32}{3}\,,$$

You should then pick three values c_1, c_3, and c_5 such that $c_1 < -\frac{32}{3} < c_3 < 0 < c_5$ and use "contour" to plot the five level sets

$$H(x,y) = -\tfrac{32}{3}\,, \qquad H(x,y) = 0\,,$$
$$H(x,y) = c_1\,, \quad H(x,y) = c_3\,, \quad H(x,y) = c_5\,.$$

(d) Identify each stationary point as being either stable or unstable.

Solution. As indicated above, a correct phase portrait will give you the answer to this part. However, you can also get the answer without the phase portait as follows. The Hessian matrix $\mathbf{H}(x,y)$ of second partial derivatives is

$$\mathbf{H}(x,y) = \begin{pmatrix} \partial_{xx}H(x,y) & \partial_{xy}H(x,y) \\ \partial_{yx}H(x,y) & \partial_{yy}H(x,y) \end{pmatrix} = \begin{pmatrix} -4+2x & 0 \\ 0 & 1 \end{pmatrix}.$$

Evaluating this at the stationary points yields

$$\mathbf{H}(0,-1) = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{H}(4,-1) = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}.$$

Because the matrix $\mathbf{H}(0,-1)$ is diagonal, you can easily see that its eigenvalues are -4 and 1. Because these have different signs, the stationary point $(0,-1)$ is a saddle and is therefore unstable. Similarly, because the matrix $\mathbf{H}(4,-1)$ is diagonal, you can easily see that its eigenvalues are 4 and 1. Because these have the same sign, the stationary point $(4,-1)$ is a center an is therefore stable.

A Short Table of Laplace Transforms

$$\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}[\cos(bt)](s) = \frac{s}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2 + b^2} \qquad \text{for } s > 0\,.$$

$$\mathcal{L}[t^n f(t)](s) = (-1)^n F^{(n)}(s) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)\,.$$

$$\mathcal{L}[e^{at} f(t)](s) = F(s-a) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)\,.$$

$$\mathcal{L}[u(t-c)f(t-c)](s) = e^{-cs}F(s) \qquad \text{where } F(s) = \mathcal{L}[f(t)](s)$$

and u is the unit step function.