HIGHER-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS II: Nonhomogeneous Equations

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Because the presentation of this material in class will differ from that in the book, I felt that notes that closely follow the class presentation might be appreciated.

4. Nonhomogeneous Equations: General Theory

4.1: Particular and General Solutions. We are now ready to study nonhomogeneous linear equations. An n^{th} order nonhomogeneous linear ODE has the normal form

$$\mathrm{L}(t)y = f(t)\,, \tag{4.1}$$

where the differential operator $\mathrm{L}(t)$ has the form

$$\mathrm{L}(t) = \frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{d}{dt} + a_n(t)\,. \tag{4.2}$$

We will assume throughout this section that the coefficients $a_1, a_2, \cdots, a_n$ and the forcing f are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied.

We will exploit the following properties of nonhomogeneous equations.

Theorem 4.1: If $Y_1(t)$ and $Y_2(t)$ are solutions of (4.1) then $Z(t) = Y_1(t) - Y_2(t)$ is a solution of the associated homogeneous equation $\mathrm{L}(t)Z(t) = 0$.

Proof: Because $\mathrm{L}(t)Y_1(t) = f(t)$ and $\mathrm{L}(t)Y_2(t) = f(t)$ one sees that

$$\mathrm{L}(t)Z(t) = \mathrm{L}(t)\big(Y_1(t) - Y_2(t)\big) = \mathrm{L}(t)Y_1(t) - \mathrm{L}(t)Y_2(t) = f(t) - f(t) = 0\,.$$

Theorem 4.2: If $Y_P(t)$ is a solution of (4.1) and $Y_H(t)$ is a solution of the associated homogeneous equation $\mathrm{L}(t)Y_H(t) = 0$ then $Y(t) = Y_H(t) + Y_P(t)$ is also a solution of (4.1).

Proof: Because $\mathrm{L}(t)Y_H(t) = 0$ and $\mathrm{L}(t)Y_P(t) = f(t)$ one sees that

$$\mathrm{L}(t)Y(t) = \mathrm{L}(t)\big(Y_H(t) + Y_P(t)\big) = \mathrm{L}(t)Y_H(t) + \mathrm{L}(t)Y_P(t) = 0 + f(t) = f(t)\,.$$

Theorem 4.2 suggests that we can construct general solutions of the nonhomogeneous equation (4.1) as follows.

(1) Find a general solution $Y_H(t)$ of the *associated homogeneous equation* $\mathrm{L}(t)y = 0$.

(2) Find a *particular solution* $Y_P(t)$ of equation (4.1).

(3) Then $Y_H(t) + Y_P(t)$ is a general solution of (4.1).

Of course, step (1) reduces to finding a fundamental set of solutions of the associated homogeneous equation, $Y_1, Y_2, \cdots, Y_n$. Then

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

If $\mathrm{L}(t)$ has constant coefficients (so that $\mathrm{L}(t) = \mathrm{L}$) then this can be done by the recipe of Section 3.

Example. One can check that $\frac{1}{4}t$ is a particular solution of

$$\mathrm{D}^2 y + 4y = t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $p(z) = z^2 + 4$, which has roots $\pm i2$. A general solution is therefore

$$y = c_1 \cos(2t) + c_2 \sin(2t) + \tfrac{1}{4}t\,.$$

Example. One can check that $-\frac{1}{2}e^t$ is a particular solution of

$$\mathrm{D}^2 y - \mathrm{D}y - 2y = e^t\,.$$

This equation has constant coefficients. Its characteristic polynomial is $p(z) = z^2 - z - 2 = (z-2)(z+1)$, which has roots -1 and 2. A general solution is therefore

$$y = c_1 e^{-t} + c_2 c^{2t} - \tfrac{1}{2}e^t\,.$$

These examples show that when $\mathrm{L}(t)$ has constant coefficients (so that $\mathrm{L}(t) = \mathrm{L}$), finding $Y_P(t)$ becomes the crux of matter. In Section 5 we will study methods for finding $Y_P(t)$ for equations with constant coefficients. If $\mathrm{L}(t)$ has variable coefficients then a fundamental set of solutions of the associated homogeneous equation will generally be given to you. In that case, finding $Y_P(t)$ again becomes the crux of matter. In Section 6 we will study methods for finding $Y_P(t)$ for equations with variable coefficients when a fundamental set of solutions of the associated homogeneous equation is given to you.

4.2: Solutions of Initial-Value Problems. An initial-value problem associated with an n^{th}-order nonhomogeneous linear equation has the form

$$\begin{gathered}\mathrm{D}^n y + a_1(t)\mathrm{D}^{n-1}y + \cdots + a_{n-1}(t)\mathrm{D}y + a_n(t)y = f(t)\,,\\ y(t_I) = y_0\,, \qquad y'(t_I) = y_1\,, \qquad \cdots \qquad y^{(n-1)}(t_I) = y_{n-1}\,,\end{gathered} \tag{4.3}$$

where t_I is the initial time and y_0, y_1, $\cdots$, y_{n-1} are the initial data.

Given a particular solution $Y_P(t)$ of the nonhomogeneous equation and a fundamental set of solutions of the associated homogeneous equation, Y_1, Y_2, $\cdots$, Y_n, you first construct a general solution of the nonhomogeneous equation as

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t) + Y_P(t)\,.$$

You then determine the values of the parameters c_1, $\cdots$, c_n by requiring that this solution satisfy the initial conditions. This leads to a linear algebraic system of the form

$$\begin{aligned} c_1 Y_1(t_I) \quad &+ \quad c_2 Y_2(t_I) \quad + \cdots + \quad c_n Y_n(t_I) \quad &= y_0 - Y_P(t_I)\,,\\ c_1 Y_1'(t_I) \quad &+ \quad c_2 Y_2'(t_I) \quad + \cdots + \quad c_n Y_n'(t_I) \quad &= y_1 - Y_P'(t_I)\,,\\ &\vdots\\ c_1 Y_1^{(n-1)}(t_I) &+ c_2 Y_2^{(n-1)}(t_I) \ + \cdots + \ c_n Y_n^{(n-1)}(t_I) &= y_1 - Y_P^{(n-1)}(t_I)\,.\end{aligned} \tag{4.4}$$

Notice that this differs from the linear algebraic system (2.2) for solving homogeneous initial-value problems only by the terms involving Y_P on the right-hand side. Because Y_1, Y_2, $\cdots$, Y_n, is a fundamental set of solutions of the associated homogeneous equation, their Wronskian $W[Y_1, Y_2, \cdots, Y_n]$ is always nonzero. You can therefore always solve the linear algebraic system (4.4) for any initial time t_I and any initial data y_0, y_1, $\cdots$, y_n, and thereby find the unique solution of the initial value problem (4.3).

Example. Solve the initial-value problem

$$\mathrm{D}^2 y + 4y = t\,, \qquad y(0) = 3\,, \quad y'(0) = -1\,.$$

Solution. We saw earlier that a general solution of the nonhomogeneous equation is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \tfrac{1}{4}t\,.$$

Because

$$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) + \tfrac{1}{4}\,,$$

the initial conditions yield

$$y(0) = c_1 = 3\,, \qquad y'(0) = 2c_2 + \tfrac{1}{4} = -1\,.$$

These can be solved to find $c_1 = 3$ and $c_2 = -\frac{5}{8}$. The solution of the initial-value problem is therefore

$$y(t) = 3\cos(2t) - \tfrac{5}{8}\sin(2t) + \tfrac{1}{4}t\,.$$

Example. Solve the initial-value problem

$$\mathrm{D}^2 y - \mathrm{D}y - 2y = e^t\,, \qquad y(0) = -2\,, \quad y'(0) = 5\,.$$

Solution. We saw earlier that a general solution of the nonhomogeneous equation is

$$y(t) = c_1 e^{-t} + c_2 c^{2t} - \tfrac{1}{2}e^t\,.$$

Because

$$y'(t) = -c_1 e^{-t} + 2c_2 c^{2t} - \tfrac{1}{2}e^t\,,$$

the initial conditions yield

$$y(0) = c_1 + c_2 - \tfrac{1}{2} = -2\,, \qquad y'(0) = -c_1 + 2c_2 - \tfrac{1}{2} = 5\,,$$

which is equivalent to the system

$$c_1 + c_2 = -\tfrac{3}{2}\,, \qquad -c_1 + 2c_2 = \tfrac{11}{2}\,.$$

This can be solved to find $c_1 = -\frac{17}{6}$ and $c_2 = \frac{4}{3}$. The solution of the initial-value problem is therefore

$$y(t) = -\tfrac{17}{6}e^{-t} + \tfrac{4}{3}c^{2t} - \tfrac{1}{2}e^t\,.$$

5. Nonhomogeneous Equations with Constant Coefficients

This section gives three methods by which you can construct particular solutions to an n^{th} order nonhomogeneous linear ordinary differential equation

$$\mathrm{L}y = f(t)\,, \tag{5.1}$$

when the differential operator L has constant coefficients,

$$\mathrm{L} = a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,, \tag{5.2}$$

with $a_0 \neq 0$. The previous section showed that this is the key step in either finding a general solution of (5.1) or solving an initial-value problem associated with (5.1). The first two methods are *Undetermined Coefficients* and *Key Identity Evaluations*. They are related and require the forcing $f(t)$ to have a special form. Because they generally provide the fastest way to find a particular solution whenever $f(t)$ has this special form, it is a good idea to master at least one of these methods. The third method is *Green Functions*. It can be applied to any forcing $f(t)$, but does not yield an explicit particular solution. Rather, it reduces the problem of computing a particular solution to that of evaluating n integrals. Because evaluating such integrals takes time, this method should only be applied when the first two method can not be applied.

5.1: Undetermined Coefficients. This method should only be used to find a particular solution of equation (5.1) when the following two conditions are met.

(1) The differential operator L has constant coefficients.

(2) The forcing $f(t)$ has the form

$$\begin{aligned} f(t) = &\big(f_0t^d + f_1t^{d-1} + \cdots + f_d\big)e^{\mu t}\cos(\nu t)\\ &+ \big(g_0t^d + g_1t^{d-1} + \cdots + g_d\big)e^{\mu t}\sin(\nu t)\,, \end{aligned} \tag{5.3}$$

for some nonnegative integer d and real numbers μ and ν. Here we are assuming that $f_0, f_1, \cdots, f_d$ and $g_0, g_1, \cdots, g_d$ are all real and that either $f_0 \neq 0$ or $\nu g_0 \neq 0$. When the forcing $f(t)$ has the form (5.3) it is said to have *characteristic form*. The complex number $\mu + i\nu$ is called its *characteristic* of $f(t)$ while the integer d is called its *degree*.

The first of these conditions is always easy to verify by inspection. Verification of the second usually can also be done by inspection, but sometimes it might require the use of a trigonometric or some other identity. You should be able to identify when a forcing $f(t)$ has the characteristic form (5.3) and, when it does, to read-off its characteristic and degree.

Example: The forcing of the equation $\mathrm{L}y = 2e^{2t}$ has the characteristic form (5.3) with characteristic $\mu + i\nu = 2$ and degree $d = 0$.

Example: The forcing of the equation $\mathrm{L}y = t^2e^{-3t}$ has the characteristic form (5.3) with characteristic $\mu + i\nu = -3$ and degree $d = 2$.

Example: The forcing of the equation $\mathrm{L}y = t\,e^{5t}\sin(3t)$ has the characteristic form (5.3) with characteristic $\mu + i\nu = 5 + i3$ and degree $d = 1$.

Example: The forcing of the equation $\mathrm{L}y = \sin(2t)\cos(2t)$ can be put into the characteristic form (5.3) by using the double-angle identity $\sin(4t) = 2\sin(2t)\cos(2t)$. The equation can thereby be expressed as $\mathrm{L}y = \frac{1}{2}\sin(4t)$. The forcing now has the characteristic form (5.3) with characteristic $\mu + i\nu = i4$ and degree $d = 0$.

The method of Undetermined Coefficients is based on the observation that if the characteristic $\mu + i\nu$ of the forcing $f(t)$ is not a root of the characteristic polynomial $p(z)$ of the operator L then equation (5.1) has a particular solution of the form

$$\begin{aligned} Y_P(t) &= \big(A_0 t^d + A_1 t^{d-1} + \cdots + A_d\big)e^{\mu t}\cos(\nu t) \\ &\quad + \big(B_0 t^d + B_1 t^{d-1} + \cdots + B_d\big)e^{\mu t}\sin(\nu t)\,, \end{aligned} \tag{5.4}$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are real constants. Notice that when $\nu = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish. More generally, if the characteristic $\mu + i\nu$ is a root of $p(z)$ of multiplicity m then equation (5.1) has a particular solution of the form

$$\begin{aligned} Y_P(t) &= \big(A_0 t^{m+d} + A_1 t^{m+d-1} + \cdots + A_d t^m\big)e^{\mu t}\cos(\nu t) \\ &\quad + \big(B_0 t^{m+d} + B_1 t^{m+d-1} + \cdots + B_d t^m\big)e^{\mu t}\sin(\nu t)\,, \end{aligned} \tag{5.5}$$

where A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d are real constants. Notice that when $\nu = 0$ the terms involving B_0, B_1, $\cdots$, B_d all vanish. This case includes the previous one if we understand "$\mu + i\nu$ is a root of $p(z)$ of multiplicity 0" to mean that it is not a root of $p(z)$. When one then sets $m = 0$ in (5.5), it reduces to (5.4).

Given a nonhomogeneous problem $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ has the characteristic form (5.3) with characteristic $\mu + i\nu$, degree d, and multiplicity m, the method of undetermined coefficients will seek a particular solution $Y_P(t)$ in the form (5.5) with A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d as unknowns to be determined. These are the "undetermined coefficients" of the method. There are $2d+2$ unknowns when $\nu \neq 0$, and only $d+1$ unknowns when $\nu = 0$ because in that case the terms involving B_0, B_1, $\cdots$, B_d vanish. These unknowns are determined as follows.

1. Substitute the form (5.5) directly into $\mathrm{L}Y_P$ and collect like terms.
2. Set $\mathrm{L}Y_P = f(t)$ and match the coefficients in front of each of the linearly independent functions that appear on either side.
3. Solve the resulting linear algebraic system to determine the coefficients in the form (5.5).

This linear algebraic system will consist of either $2d+2$ equations for the $2d+2$ coefficients A_0, A_1, $\cdots$, A_d, and B_0, B_1, $\cdots$, B_d (when $\nu \neq 0$) or $d+1$ equations for the $d+1$ coefficients A_0, A_1, $\cdots$, A_d (when $\nu = 0$). Because these coefficients are the parameters of the family (5.5), this method is also sometimes called "Undetermined Parameters".

We now illustrate how the method works.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 6e^{2t}\,.$$

Solution: The characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 9 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = 2$ and degree $d = 0$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $\mu + i\nu = 2$, $d = 0$, and $m = 0$, we see from (5.5) that Y_P has the form

$$Y_P(t) = Ae^{2t}\,.$$

Because

$$Y_P'(t) = 2Ae^{2t}\,, \qquad Y_P''(t) = 4Ae^{2t}\,,$$

we see that

$$\begin{aligned}\mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t)\\ &= 4Ae^{2t} + 4Ae^{2t} + 10Ae^{2t} = 18Ae^{2t}\,.\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 6e^{2t}$ then we see that $18A = 6$, whereby $A = \frac{1}{3}$. Hence,

$$Y_P(t) = \tfrac{1}{3}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 4te^{2t}\,.$$

Solution: As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = 2$ and degree $d = 1$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $\mu + i\nu = 2$, $d = 1$, and $m = 0$, we see from (5.5) that Y_P has the form

$$Y_P(t) = (A_0 t + A_1)e^{2t}\,.$$

Because

$$Y_P'(t) = 2(A_0 t + A_1)e^{2t} + A_0 e^{2t}\,, \qquad Y_P''(t) = 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t}\,,$$

we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\
&= 4(A_0 t + A_1)e^{2t} + 4A_0 e^{2t} + 4(A_0 t + A_1)e^{2t} + 2A_0 e^{2t} + 10(A_0 t + A_1)e^{2t} \\
&= 18(A_0 t + A_1)e^{2t} + 6A_0 e^{2t} \\
&= 18A_0 t e^{2t} + (18A_1 + 6A_0)e^{2t}\,.
\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions te^{2t} and e^{2t} we see that

$$18A_0 = 4\,, \qquad 18A_1 + 6A_0 = 0\,.$$

Upon solving this linear algebraic system for A_0 and A_1 we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence,

$$Y_P(t) = \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9}te^{2t} - \tfrac{2}{27}e^{2t}\,.$$

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = \cos(2t)\,.$$

Solution: As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t} \cos(3t) + c_2 e^{-t} \sin(3t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = i2$ and degree $d = 0$. Because the characteristic $i2$ is not a root of $p(z)$, it has mutplicity $m = 0$.

Because $\mu + i\nu = i2$, $d = 0$, and $m = 0$ we see from (5.5) that Y_P has the form

$$Y_P(t) = A\cos(2t) + B\sin(2t)\,.$$

Because

$$Y_P'(t) = -2A\sin(2t) + 2B\cos(2t)\,, \qquad Y_P''(t) = -4A\cos(2t) - 4B\sin(2t)\,,$$

we see that

$$\begin{aligned} \mathrm{L}Y_P(t) &= Y_P''(t) + 2Y_P'(t) + 10Y_P(t) \\ &= -4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t) + 10A\cos(2t) + 10B\sin(2t) \\ &= (6A + 4B)\cos(2t) + (6B - 4A)\sin(2t)\,. \end{aligned}$$

If we set $\mathrm{L}Y_P(t) = \cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$ and $\sin(2t)$ we see that

$$6A + 4B = 1\,, \qquad -4A + 6B = 0\,.$$

Upon solving this system we find that $A = \frac{3}{26}$ and $B = \frac{1}{13}$, whereby

$$Y_P(t) = \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t)\,.$$

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 4y = t\,\cos(2t)\,.$$

Solution: This problem has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2\,.$$

Its roots are $\pm i2$. Hence,

$$Y_H(t) = c_1 \cos(2t) + c_2 \sin(2t)\,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = i2$ and degree $d = 1$. Because the characteristic $i2$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because $\mu + i\nu = i2$, $d = 1$, and $m = 1$, we see from (5.5) that Y_P has the form

$$Y_P(t) = (A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\,.$$

Because

$$\begin{aligned}
Y_P'(t) &= -2(A_0t^2 + A_1t)\sin(2t) + (2A_0t + A_1)\cos(2t) \\
&\quad + 2(B_0t^2 + B_1t)\cos(2t) + (2B_0t + B_1)\sin(2t) \\
&= \big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\cos(2t) - \big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\sin(2t)\,, \\
Y_P''(t) &= -2\big(2B_0t^2 + 2(B_1 + A_0)t + A_1\big)\sin(2t) + \big(4B_0t + 2(B_1 + A_0)\big)\cos(2t) \\
&\quad - 2\big(2A_0t^2 + 2(A_1 - B_0)t - B_1\big)\cos(2t) - \big(4A_0t + 2(A_1 - B_0)\big)\sin(2t) \\
&= -\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\quad - \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\,,
\end{aligned}$$

we see that

$$\begin{aligned}
\mathrm{L}Y_P(t) &= Y_P''(t) + 4Y_P(t) \\
&= -\Big[\big(4A_0t^2 + (4A_1 - 8B_0)t - 4B_1 - 2A_0\big)\cos(2t) \\
&\qquad + \big(4B_0t^2 + (4B_1 + 8A_0)t + 4A_1 - 2B_0\big)\sin(2t)\Big] \\
&\quad + 4\Big[(A_0t^2 + A_1t)\cos(2t) + (B_0t^2 + B_1t)\sin(2t)\Big] \\
&= \big(8B_0t + 4B_1 + 2A_0\big)\cos(2t) - \big(8A_0t + 4A_1 - 2B_0\big)\sin(2t)\,.
\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = t\,\cos(2t)$ then by equating the coefficients of the linearly independent functions $\cos(2t)$, $t\,\cos(2t)$, $\sin(2t)$, and $t\,\sin(2t)$, we see that

$$4B_1 + 2A_0 = 0\,, \qquad 8B_0 = 1\,, \qquad 4A_1 - 2B_0 = 0\,, \qquad 8A_0 = 0\,.$$

The solution of this system is $A_0 = 0$, $B_0 = \frac{1}{8}$, $A_1 = \frac{1}{16}$, and $B_1 = 0$, whereby

$$Y_P(t) = \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

A general solution is therefore

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

5.2: Key Identity Evaluations. This method should only be applied to equation (5.1) when the following two conditions are met.

(1) The differential operator L has constant coefficients.

(2) The forcing $f(t)$ has the characteristic form (5.3) for some characteristic $\mu + i\nu$ and degree d.

It is based on the observation that for any forcing of the form (5.3) one can construct explicit formulas for a particular solution of equation (5.1) by evaluating the Key Identity and some of its derivatives with respect to z at $z = \mu + i\nu$. For example, if $p(z)$ is the characteristic polynomial of L then the Key Identity and its first four derivatives with respect to z are

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= p(z)e^{zt} \,, \\
\mathrm{L}\big(t\, e^{zt}\big) &= p(z)\, t\, e^{zt} + p'(z)\, e^{zt} \,, \\
\mathrm{L}\big(t^2 e^{zt}\big) &= p(z)\, t^2 e^{zt} + 2p'(z)\, t\, e^{zt} + p''(z)\, e^{zt} \,, \\
\mathrm{L}\big(t^3 e^{zt}\big) &= p(z)\, t^3 e^{zt} + 3p'(z)\, t^2 e^{zt} + 3p''(z)\, t\, e^{zt} + p'''(z)\, e^{zt} \,, \\
\mathrm{L}\big(t^4 e^{zt}\big) &= p(z)\, t^4 e^{zt} + 4p'(z)\, t^3 e^{zt} + 6p''(z)\, t^2 e^{zt} + 4p'''(z)\, t\, e^{zt} + p^{(4)}(z)\, e^{zt} \,.
\end{aligned} \tag{5.6}$$

Notice that when these are evaluated at $z = \mu + i\nu$ then the terms on the right-hand sides above have the same form as those appearing in the forcing (5.3).

If the characteristic $\mu + i\nu$ is not a root of $p(z)$ then one needs through the d^{th} derivative of the Key Identity with respect to z. These should be evaluated at $z = \mu + i\nu$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $\nu \neq 0$) can then be found so that its right-hand side equals any $f(t)$ given by (5.3). You can then read off Y_P from this linear combination.

More generally, if the characteristic $\mu + i\nu$ is a root of $p(z)$ of multiplicity m then one needs through the $(m + d)^{th}$ derivative of the Key Identity with respect to z. These should be evaluated at $z = \mu + i\nu$. Because $\mu + i\nu$ is a root of multiplicity m, the first m of these will vanish when evaluated at $z = \mu + i\nu$. A linear combination of the resulting $d + 1$ equations (and their conjugates if $\nu \neq 0$) can then be found so that its right-hand side equals any $f(t)$ given by (5.3). You can then read off Y_P from this linear combination. This case includes the previous one if we understand "$\mu + i\nu$ is a root of $p(z)$ of multiplicity 0" to mean that it is not a root of $p(z)$.

Given a nonhomogeneous problem $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ has the characteristic form (5.3) with characteristic $\mu + i\nu$, degree d, and multiplicity m, the method of Key Identity Evaluations will find a particular solution Y_P as follows.

1. Write down the Key Identity through its $(m + d)^{th}$ derivative with respect to z.
2. Evaluate the m^{th} through $(m + d)^{th}$ derivative the of the Key Identity at $z = \mu + i\nu$.
3. Find a linear combination of the resulting $d + 1$ equations (and their conjugates if $\nu \neq 0$) whose right-hand side equals $f(t)$ and read off Y_P.

Remark. The methods of Undetermined Coefficients and Key Identity Evaluations are each fairly painless when m and d are both small and $\nu = 0$. When m is not small then Undetermined Coefficients is usually faster. When m and d are both small and $\nu \neq 0$ then Key Identity Evaluations is usually faster. For the problems you will face both m and d will be small, so $m + d$ will seldom be larger than 3, and more commonly be 0, 1, or 2.

In order to contrast the two methods, we will now illustrate Key Identity Evalutations on the same examples we had previously treated by Undetermined Coefficients.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 6e^{2t}\,.$$

Solution: The characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 9 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the characteristic form (5.3) with characteristic $\mu + i\nu = 2$ and degree $d = 0$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $m + d = 0$, we will only need the Key Identity:

$$\mathrm{L}\big(e^{zt}\big) = (z^2 + 2z + 10)e^{zt}\,.$$

Evaluate this at $z = 2$ to obtain

$$\mathrm{L}\big(e^{2t}\big) = (4 + 4 + 10)e^{2t} = 18e^{2t}\,.$$

Dividing this by 3 gives

$$\mathrm{L}\big(\tfrac{1}{3}e^{2t}\big) = 6e^{2t}\,,$$

from which we read off that

$$Y_P(t) = \tfrac{1}{3}e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{1}{3}e^{2t}\,.$$

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 4te^{2t}\,.$$

Solution: As before the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2 .$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) .$$

To find a particular solution, first notice that the forcing is of the characteristic form (5.3) with characteristic $\mu + i\nu = 2$ and degree $d = 1$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $m + d = 1$, we will only need the Key Identity and its first derivative with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 + 2z + 10)\, e^{zt} , \\ \mathrm{L}\big(t\, e^{zt}\big) &= (z^2 + 2z + 10)\, t\, e^{zt} + (2z+2)\, e^{zt} . \end{aligned}$$

Evaluate these at $z = 2$ to obtain

$$\mathrm{L}\big(e^{2t}\big) = 18e^{2t} , \qquad \mathrm{L}\big(t\, e^{2t}\big) = 18\, t\, e^{2t} + 6\, e^{2t} .$$

Because we want to isolate the $t\, e^{2t}$ term on the right-hand side, subtract one-third the first equation from the second to get

$$\mathrm{L}\big(t\, e^{2t} - \tfrac{1}{3} e^{2t}\big) = \mathrm{L}\big(t\, e^{2t}\big) - \tfrac{1}{3}\mathrm{L}\big(e^{2t}\big) = 18\, t\, e^{2t} .$$

After multiplying this by $\frac{2}{9}$ you can read off that

$$Y_P(t) = \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t} .$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t} .$$

We now illustrate an alternative way to apply the method of Key Identity evaluations approach when you have more than one evaluation of the Key Identity and its derivatives, such as in the previous example.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = 4te^{2t} .$$

Alternative Solution: Proceed as in the last example up to the point

$$\mathrm{L}\big(e^{2t}\big) = 18e^{2t} , \qquad \mathrm{L}\big(t\, e^{2t}\big) = 18\, t\, e^{2t} + 6\, e^{2t} .$$

If we set $Y_P(t) = A_0\, t\, e^{2t} + A_1\, e^{2t}$ then we see that

$$\begin{aligned}\mathrm{L}Y_P(t) = A_0 \mathrm{L}\big(t\, e^{2t}\big) + A_1 \mathrm{L}\big(e^{2t}\big) &= A_0\big(18\, t\, e^{2t} + 6\, e^{2t}\big) + A_1 18 e^{2t} \\ &= 18 A_0\, t\, e^{2t} + (6A_0 + 18A_1) e^{2t}\,.\end{aligned}$$

If we set $\mathrm{L}Y_P(t) = 4te^{2t}$ then by equating the coefficients of the linearly independent functions $t\, e^{2t}$ and e^{2t} we see that

$$18A_0 - 4\,, \qquad 6A_0 + 18A_1 = 0\,.$$

Upon solving this linear algebraic system for A_0 and A_1 we first find that $A_0 = \frac{2}{9}$ and then that $A_1 = -\frac{1}{3}A_0 = -\frac{2}{27}$. Hence,

$$Y_P(t) = \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t}\,.$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{2}{9} t e^{2t} - \tfrac{2}{27} e^{2t}\,.$$

Remark: Notice that this alternative way to using Key Identity Evaluations led to the same linear algebraic system for A_0 and A_1 that we got from the method of Undetermined Coefficients. This will generally be the case because they are just two different ways to evaluate $\mathrm{L}Y_P(t)$ for the same family of $Y_P(t)$.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 10y = \cos(2t)\,.$$

Solution: As before, the characteristic polynomial is

$$p(z) = z^2 + 2z + 10 = (z+1)^2 + 3^2\,.$$

Its roots are $-1 \pm i3$. Hence,

$$Y_H(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)\,.$$

To find a particular solution, first notice that the forcing is of the characteristic form (5.3) with characteristic $\mu + i\nu = i2$ and degree $d = 0$. Because the characteristic 2 is not a root of $p(z)$, it has multiplicity $m = 0$.

Because $m + d = 0$, we will only need the Key Identity:

$$\mathrm{L}\big(e^{zt}\big) = (z^2 + 2z + 10) e^{zt}\,.$$

Evaluate this at $z = i2$ to obtain

$$\mathrm{L}\big(e^{i2t}\big) = (-4 + i4 + 10)e^{i2t} = (6 + i4)e^{i2t} \,.$$

Dividing this by $6 + i4$ gives

$$\mathrm{L}\bigg(\frac{1}{6 + i4}\, e^{i2t}\bigg) = e^{i2t} = \cos(2t) + i\sin(2t) \,.$$

Taking the real part of each side gives

$$\mathrm{L}\Bigg(\mathrm{Re}\bigg(\frac{1}{6 + i4}\, e^{i2t}\bigg)\Bigg) = \cos(2t) \,,$$

from which we read off that

$$\begin{aligned} Y_P(t) &= \mathrm{Re}\bigg(\frac{1}{6 + i4}\, e^{i2t}\bigg) = \mathrm{Re}\bigg(\frac{6 - i4}{6^2 + 4^2}\, e^{i2t}\bigg) = \tfrac{1}{52}\,\mathrm{Re}\big((6 - i4)e^{i2t}\big) \\ &= \tfrac{1}{52}\,\mathrm{Re}\big((6 - i4)(\cos(2t) + i\sin(2t))\big) = \tfrac{6}{52}\cos(2t) + \tfrac{4}{52}\sin(2t) \,. \end{aligned}$$

A general solution is therefore

$$y = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t) + \tfrac{3}{26}\cos(2t) + \tfrac{1}{13}\sin(2t) \,.$$

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 4y = t\,\cos(2t) \,.$$

Solution: This problem has constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 + 4 = z^2 + 2^2 \,.$$

Its roots are $\pm i2$. Hence,

$$Y_H(t) = c_1\cos(2t) + c_2\sin(2t) \,.$$

To find a particular solution, first notice that the forcing has the characteristic form (5.3) with characteristic $\mu + i\nu = i2$ and degree $d = 1$. Because the characteristic $i2$ is a simple root of $p(z)$, it has multiplicity $m = 1$.

Because $m + d = 2$, we will need the Key Identity and its first two derivatives with respect to z:

$$\begin{aligned} \mathrm{L}\big(e^{zt}\big) &= (z^2 + 4)e^{zt} \,, \\ \mathrm{L}\big(t\,e^{zt}\big) &= (z^2 + 4)t\,e^{zt} + 2z\,e^{zt} \,, \\ \mathrm{L}\big(t^2 e^{zt}\big) &= (z^2 + 4)t^2 e^{zt} + 4z\,t\,e^{zt} + 2e^{zt} \,. \end{aligned}$$

Evaluate these at $z = i2$ to obtain

$$\mathrm{L}\big(e^{i2t}\big) = 0\,, \qquad \mathrm{L}\big(t\,e^{i2t}\big) = i4\,e^{i2t}\,, \qquad \mathrm{L}\big(t^2 e^{i2t}\big) = i8\,t\,e^{i2t} + 2e^{i2t}\,.$$

Because $t\,\cos(2t) = \mathrm{Re}(t\,e^{i2t})$, we want to isolate the $t\,e^{i2t}$ term on the right-hand side. This is done by multiplying the second equation by $i\frac{1}{2}$ and adding it to the third to find

$$\mathrm{L}\big((t^2 + i\tfrac{1}{2}t)e^{i2t}\big) = \mathrm{L}\big(t^2 e^{i2t}\big) + i\tfrac{1}{2}\mathrm{L}\big(t\,e^{i2t}\big) = i8\,t\,e^{i2t}\,.$$

Now divide this by $i8$ to obtain

$$\mathrm{L}\left(\frac{t^2 + i\frac{1}{2}t}{i8}\,e^{i2t}\right) = t\,e^{i2t}\,,$$

from which we read off that

$$Y_P(t) = \mathrm{Re}\left(\frac{t^2 + i\frac{1}{2}t}{i8}\,e^{i2t}\right) = \frac{t}{16}\,\mathrm{Re}\big((1 - i2t)e^{i2t}\big) = \frac{t}{16}\big(\cos(2t) + 2t\sin(2t)\big)\,.$$

A general solution is therefore

$$y = c_1\cos(2t) + c_2\sin(2t) + \tfrac{1}{16}t\,\cos(2t) + \tfrac{1}{8}t^2\sin(2t)\,.$$

Remark: The above example is typical of a case when Key Identity Evaluations is far faster than Undetermined Coefficients. This is because the forcing has a conjugate pair characteristic $\mu \pm i\nu = \pm i2$, positive degree $d = 1$, and small multiplicity $m = 1$. This advantage becomes much more dramatic for larger d, but diminishes for larger m. If you master both methods you will develop a sense about which one is most efficient for any given problem.

5.3: Forcings of Compound Characteristic Form. The methods of Undetermined Coefficients and Key Identity Evaluations can be applied multiple times to construct a particular solution of $\mathrm{L}y = f(t)$ whenever

(1) the differential operator L has constant coefficients,

(2) the forcing $f(t)$ is a sum of terms in the characteristic form (5.3), each with different characteristics.

When the second of these conditions is satisfied the forcing is said to have *compound characteristic form*. The first of these conditions is always easy to verify by inspection. Verification of the second usually can also be done by inspection, but sometimes it might require the use of a trigonometric or some other identity. You should be able to identify when a forcing $f(t)$ can be expressed as a sum of terms that have the characteristic form (5.3), and when it is, to read-off the characteristic and degree of each component.

Example: The forcing of the equation $\mathrm{L}y = \cos(t)^2$ can be written as a sum of terms that have the characteristic form (5.3) by using the identity $\cos(t)^2 = (1 + \cos(2t))/2$. One sees that

$$\mathrm{L}y = \cos(t)^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos(2t) \,.$$

Each term on the right-hand side above has the characteristic form (5.3); the first with characteristic $\mu + i\nu = 0$ and degree $d = 0$, and the second with characteristic $\mu + i\nu = i2$ and degree $d = 0$.

Example: The forcing of the equation $\mathrm{L}y = \sin(2t)\cos(3t)$ can be written as a sum of terms that have the characteristic form (5.3) by using the identity

$$\sin(2t)\cos(3t) = \tfrac{1}{2}\big(\sin(3t + 2t) - \sin(3t - 2t)\big) = \tfrac{1}{2}\big(\sin(5t) - \sin(t)\big) \,.$$

One sees that

$$\mathrm{L}y = \sin(2t)\cos(3t) = \tfrac{1}{2}\sin(5t) - \tfrac{1}{2}\sin(t) \,.$$

Each term on the right-hand side above has the characteristic form (5.3); the first with characteristic $\mu + i\nu = i5$ and degree $d = 0$, and the second with characteristic $\mu + i\nu = i$ and degree $d = 0$.

Example: The forcing of the equation $\mathrm{L}y = \tan(t)$ cannot be written as a sum of terms in characteristic form (5.3) because every such function is smooth (infinitely differentiable) while $\tan(t)$ is not defined at $t = \frac{\pi}{2} + m\pi$ for every integer m.

Given a nonhomogeneous equation $\mathrm{L}y = f(t)$ in which the forcing $f(t)$ is a sum of terms that each have the characteristic form (5.3), you must first identify the characteristic of each term and group all the terms with the same characteristic together. You then decompose $f(t)$ as

$$f(t) = f_1(t) + f_2(t) + \cdots + f_g(t) \,,$$

where each $f_j(t)$ contains all the terms of a given characteristic. Each $f_j(t)$ will then have the characteristic form (5.3) for some degree d and some characteristic $\mu + i\nu$. You then can apply either the method of Undetermined Coefficients or the method of Key Identity Evaluations to find particular solutions Y_{jP} to each of

$$\mathrm{L}Y_{1P}(t) = f_1(t) \,, \quad \mathrm{L}Y_{2P}(t) = f_2(t) \,, \quad \cdots \quad \mathrm{L}Y_{gP}(t) = f_g(t) \,. \tag{5.7}$$

Then $Y_P(t) = Y_{1P}(t) + Y_{2P}(t) + \cdots + Y_{gP}(t)$ is a particular solution of $\mathrm{L}y = f(t)$.

Example: If $\mathrm{L}y = \mathrm{D}^4 y + 25\mathrm{D}^2 y = f(t)$ with

$$f(t) = e^{2t} + 9\cos(5t) + 4t^2 e^{2t} - 7t\sin(5t) + 8 - 6t \,,$$

you decompose $f(t)$ as $f(t) = f_1(t) + f_2(t) + f_3(t)$, where

$$f_1(t) = 8 - 6t \,, \quad f_2(t) = \big(1 + 4t^2\big)e^{2t} \,, \quad f_3(t) = 9\cos(5t) - 7t\sin(5t) \,.$$

Here $f_1(t)$, $f_2(t)$, and $f_3(t)$ contain all the terms of $f(t)$ with characteristic 0, 2, and $i5$, respectively. They each have the characteristic form (5.3) with degree 1, 2, and 1 respectively. The characteristic polynomial is $p(z) = z^4 + 25z^2 = z^2(z^2 + 5^2)$, which has roots 0, 0, $-i5$, $i5$. We thereby see that the characteristics 0, 2, and $i5$ have multiplicities 2, 0, and 1 respectively.

The method of Undetermined Coefficients seeks particular solutions of the problems in (5.7) that by (5.5) have the forms

$$\begin{aligned}
Y_{1P}(t) &= A_0t^3 + A_1t^2\,, \\
Y_{2P}(t) &= \big(A_0t^2 + A_1t + A_2\big)e^{2t}\,, \\
Y_{3P}(t) &= \big(A_0t^2 + A_1t\big)\cos(5t) + \big(B_0t^2 + B_1t\big)\sin(5t)\,.
\end{aligned}$$

The method leads to three systems of linear algebraic equations to solve — systems of two equations, three equations, and four equations. We will not solve them here.

Key Identity Evalutaions is often the fastest way to solve nonhomogeneous equations whose forcings have compound characteristic form because the Key Identity and its derivatives only have to be computed once. In the problem at hand, $m+d$ for the characteristics 0, 2, and $i5$ are 3, 2, and 2. We therefore need the Key Identity and its first three derivatives with respect to z:

$$\begin{aligned}
\mathrm{L}\big(e^{zt}\big) &= \big(z^4 + 25z^2\big)\,e^{zt}\,, \\
\mathrm{L}\big(t\,e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t\,e^{zt} + \big(4z^3 + 50z\big)\,e^{zt}\,, \\
\mathrm{L}\big(t^2e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^2e^{zt} + 2\big(4z^3 + 50z\big)\,t\,e^{zt} + \big(12z^2 + 50\big)\,e^{zt}\,, \\
\mathrm{L}\big(t^3e^{zt}\big) &= \big(z^4 + 25z^2\big)\,t^3e^{zt} + 3\big(4z^3 + 50z\big)\,t^2e^{zt} + 3\big(12z^2 + 50\big)\,t\,e^{zt} + 24z\,e^{zt}\,,
\end{aligned}$$

For the characteristic 0 one has $m = 2$ and $m + d = 3$, so we evaluate the second through third derivative of the Key Identity at $z = 0$ to obtain

$$\mathrm{L}\big(t^2\big) = 50\,, \qquad \mathrm{L}\big(t^3\big) = 150t\,.$$

It follows that $\mathrm{L}\big(\frac{4}{25}t^2 - \frac{1}{25}t^3\big) = 8 - 6t$, whereby $Y_{1P}(t) = \frac{4}{25}t^2 - \frac{1}{25}t^3$.

For the characteristic 2 one has $m = 0$ and $m + d = 2$, so we evaluate the zeroth through second derivative of the Key Identity at $z = 2$ to obtain

$$\begin{aligned}
\mathrm{L}\big(e^{2t}\big) &= 116\,e^{2t}\,, \\
\mathrm{L}\big(t\,e^{2t}\big) &= 116\,t\,e^{2t} + 132e^{2t}\,, \\
\mathrm{L}\big(t^2e^{2t}\big) &= 116\,t^2e^{2t} + 264\,t\,e^{2t} + 98\,e^{2t}\,.
\end{aligned}$$

You eliminate $t\,e^{2t}$ from the right-hand sides by multiplying the second equation by $\frac{264}{116}$ and subtracting it from the third equation, thereby obtaining

$$\mathrm{L}\big(t^2e^{2t} - \tfrac{264}{116}t\,e^{2t}\big) = 116\,t^2e^{2t} + \big(98 - \tfrac{264}{116}132\big)\,e^{2t}\,.$$

Dividing this by 29 gives

$$\mathrm{L}\big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t}\big) = 4\,t^2e^{2t} + \big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\,e^{2t}\,.$$

You eliminate e^{2t} from the right-hand side above by multiplying the first equation by $\frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2}\big)$ and subtracting it from the above equation, thereby obtaining

$$\mathrm{L}\Big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2}\big)\,e^{2t}\Big) = 4\,t^2e^{2t}\,.$$

Next, by multiplying the first equation by $\frac{1}{116}$ and adding it to the above equation you obtain

$$\mathrm{L}\Big(\tfrac{1}{29}t^2e^{2t} - \tfrac{66}{29^2}t\,e^{2t} - \tfrac{1}{116}\big(\tfrac{98}{29} - \tfrac{66\cdot 132}{29^2} - 1\big)\,e^{2t}\Big) = \big(1 + 4\,t^2\big)\,e^{2t}\,,$$

whereby $Y_{2P}(t) = \frac{1}{29}t^2e^{2t} - \frac{66}{29^2}t\,e^{2t} - \frac{1}{116}\big(\frac{98}{29} - \frac{66\cdot 132}{29^2} - 1\big)\,e^{2t}$.

For the characteristic $i5$ one has $m = 1$ and $m+d = 2$, so we evaluate the first through second derivative of the Key Identity at $z = i5$ to obtain

$$\mathrm{L}\big(t\,e^{i5t}\big) = -i250\,e^{i5t}\,, \qquad \mathrm{L}\big(t^2e^{i5t}\big) = -i2\cdot 250\,t\,e^{i5t} - 250\,e^{i5t}\,.$$

Upon multiplying the first equation by i and adding it to the second we find that

$$\mathrm{L}\big(t^2e^{i5t} + it\,e^{i5t}\big) = -i2\cdot 250\,t\,e^{i5t}\,.$$

The first equation and the above equation imply

$$\mathrm{L}\big(i\tfrac{1}{250}t\,e^{i5t}\big) = e^{i5t}\,, \qquad \mathrm{L}\big(\tfrac{1}{500}t^2e^{i5t} + i\tfrac{1}{500}t\,e^{i5t}\big) = -i\,t\,e^{i5t}\,.$$

The real parts of the above equations are

$$\mathrm{L}\big(-\tfrac{1}{250}t\,\sin(5t)\big) = \cos(5t)\,, \qquad \mathrm{L}\big(\tfrac{1}{500}t^2\cos(5t) - \tfrac{1}{500}t\,\sin(5t)\big) = t\,\sin(5t)\,.$$

This implies that

$$\mathrm{L}\big(-\tfrac{9}{250}t\,\sin(5t) - \tfrac{7}{500}t^2\cos(5t) + \tfrac{7}{500}t\,\sin(5t)\big) = 9\cos(5t) - 7t\,\sin(5t)\,,$$

whereby $Y_{3P}(t) = -\frac{11}{500}t\,\sin(5t) - \frac{7}{500}t^2\cos(5t)$.

5.4: Green Functions: Constant Coefficient Case. This method can be used to construct a particular solution of an n^{th} order nonhomogeneous linear ODE in the normal form

$$\mathrm{L}y = f(t) \tag{5.8}$$

whenever the differential operator L has constant coefficients and is in normal form,

$$\mathrm{L} = \mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n\,. \tag{5.9}$$

Specifically, a particular solution of (5.8) is given by

$$Y_P(t) = \int_{t_I}^{t} g(t-s)f(s)\,\mathrm{d}s\,, \tag{5.10}$$

where t_I is any initial time and $g(t)$ is the solution of the homogeneous initial-value problem

$$\mathrm{L}g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 0\,, \quad \cdots \quad g^{(n-2)}(0) = 0\,, \quad g^{(n-1)}(0) = 1\,. \tag{5.11}$$

The function g is called the *Green function* associated with the operator L. Solving the initial-value problem (5.11) for the Green function is never difficult. The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (5.10). However, evaluating this integral explicitly can be quite difficult or impossible. At worst, you can leave your answer in terms of a definite integral.

Before we verify that $Y_P(t)$ given by (5.10) is a solution of (5.8), let us work a few examples to show how the method works.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y - y = \frac{2}{e^t + e^{-t}}\,.$$

Solution: The operator L has constant coefficients and is already in normal form. Its characteristic polynomial is given by $p(z) = z^2 - 1 = (z-1)(z+1)$, which has roots ± 1. A general solution of the associated homogeneous equation is therefore

$$Y_H(t) = c_1 e^t + c_2 e^{-t}\,.$$

By (5.11) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g - g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^t + c_2 e^{-t}$. The first initial condition implies $g(0) = c_1 + c_2 = 0$. Because $g'(t) = c_1 e^t - c_2 e^{-t}$, the second condition implies $g'(0) = c_1 - c_2 = 1$. Upon solving these equations you find that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. The Green function is therefore $g(t) = \frac{1}{2}(e^t - e^{-t}) = \sinh(t)$.

The particular solution given by (5.10) with $t_I = 0$ is then

$$Y_P(t) = \int_0^t \frac{e^{t-s} - e^{-t+s}}{e^s + e^{-s}}\,\mathrm{d}s = e^t \int_0^t \frac{e^{-s}}{e^s + e^{-s}}\,\mathrm{d}s - e^{-t} \int_0^t \frac{e^s}{e^s + e^{-s}}\,\mathrm{d}s\,.$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned}
\int_0^t \frac{e^{-s}}{e^s + e^{-s}}\,\mathrm{d}s &= \int_0^t \frac{e^{-2s}}{1 + e^{-2s}}\,\mathrm{d}s = -\tfrac{1}{2}\log\bigl(1 + e^{-2s}\bigr)\Big|_{s=0}^{t} = -\tfrac{1}{2}\log\left(\frac{1 + e^{-2t}}{2}\right), \\
\int_0^t \frac{e^{s}}{e^s + e^{-s}}\,\mathrm{d}s &= \int_0^t \frac{e^{2s}}{e^{2s} + 1}\,\mathrm{d}s = \tfrac{1}{2}\log\bigl(e^{2s} + 1\bigr)\Big|_{s=0}^{t} = \tfrac{1}{2}\log\left(\frac{e^{2t} + 1}{2}\right).
\end{aligned}$$

The above expression for $Y_P(t)$ thereby becomes

$$Y_P(t) = -\tfrac{1}{2}e^t \log\left(\frac{1+e^{-2t}}{2}\right) - \tfrac{1}{2}e^{-t}\log\left(\frac{e^{2t}+1}{2}\right).$$

A general solution is therefore $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark: Notice that in the above example the definite integral in the expression for $Y_P(t)$ given by (5.10) splits into two definite integrals over s whose integrands do not involve t. This kind of splitting always happens. In general, if L is an n^{th} order operator then the expression for $Y_P(t)$ given by (5.10) always splits into n definite integrals over s whose integrands do not involve t. To do this when the Green function involves terms like $e^{\mu t}\cos(\nu t)$ or $e^{\mu t}\sin(\nu t)$ requires the use of the trigonometric identities

$$\begin{aligned} \cos(\phi-\psi) &= \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\,, \\ \sin(\phi-\psi) &= \sin(\phi)\cos(\psi) - \cos(\phi)\sin(\psi)\,. \end{aligned} \tag{5.12}$$

You should know these identities by now.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 9y = \frac{27}{16 + 9\sin(3t)^2}\,.$$

Solution: The operator L has constant coefficients and is already in normal form. Its characteristic polynomial is given by $p(z) = z^2 + 9 = z^2 + 3^2$, which has roots $\pm i3$. A general solution of the associated homogeneous equation is therefore

$$Y_H(t) = c_1\cos(3t) + c_2\sin(3t)\,.$$

By (5.11) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g + 9g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1\cos(3t) + c_2\sin(3t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2\sin(3t)$. Because $g'(t) = 3c_2\cos(3t)$, the second condition implies $g'(0) = 3c_2 = 1$, whereby $c_2 = \frac{1}{3}$. The Green function is therefore $g(t) = \frac{1}{3}\sin(3t)$.

The particular solution given by (5.10) with $t_I = 0$ is then

$$Y_P(t) = \int_0^t \sin\big(3(t-s)\big)\frac{9}{16 + 9\sin(3s)^2}\,\mathrm{d}s\,.$$

By (5.12) with $\phi = 3t$ and $\psi = 3s$, you see $\sin\big(3(t-s)\big) = \sin(3t)\cos(3s) - \cos(3t)\sin(3s)$. You can use this to express $Y_P(t)$ as

$$Y_P(t) = \sin(3t)\int_0^t \frac{9\cos(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s - \cos(3t)\int_0^t \frac{9\sin(3s)}{16 + 9\sin(3s)^2}\,\mathrm{d}s\,.$$

The definite integrals in the above expression can be evaluated as

$$\begin{aligned}
\int_0^t \frac{9\cos(3s)}{16+9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{\frac{9}{16}\cos(3s)}{1+\frac{9}{16}\sin(3s)^2}\,\mathrm{d}s \\
&= \tfrac{1}{4}\tan^{-1}\!\big(\tfrac{3}{4}\sin(3s)\big)\Big|_{s=0}^{t} = \tfrac{1}{4}\tan^{-1}\!\big(\tfrac{3}{4}\sin(3t)\big)\,. \\
\int_0^t \frac{9\sin(3s)}{16+9\sin(3s)^2}\,\mathrm{d}s &= \int_0^t \frac{9\sin(3s)}{25-9\cos(3s)^2}\,\mathrm{d}s = \int_0^t \frac{\frac{9}{25}\sin(3s)}{1-\frac{9}{25}\cos(3s)^2}\,\mathrm{d}s \\
&= -\tfrac{1}{10}\log\!\left(\frac{1+\frac{3}{5}\cos(3s)}{1-\frac{3}{5}\cos(3s)}\right)\Bigg|_{s=0}^{t} = -\tfrac{1}{10}\log\!\left(\frac{1+\frac{3}{5}\cos(3t)}{1-\frac{3}{5}\cos(3t)}\,\frac{\frac{2}{5}}{\frac{8}{5}}\right).
\end{aligned}$$

Here the first integral has the form

$$\tfrac{1}{4}\int \frac{\mathrm{d}u}{1+u^2} = \tfrac{1}{4}\tan^{-1}(u) + C\,, \qquad \text{where } u = \tfrac{3}{4}\sin(3s)\,,$$

while by using partial fractions you see that the second has the form

$$-\tfrac{1}{5}\int \frac{\mathrm{d}u}{1-u^2} = -\tfrac{1}{10}\log\!\left(\frac{1+u}{1-u}\right) + C\,, \qquad \text{where } u = \tfrac{3}{5}\cos(3s)\,.$$

The above expression for $Y_P(t)$ thereby becomes

$$Y_P(t) = \tfrac{1}{4}\sin(3t)\tan^{-1}\!\big(\tfrac{3}{4}\sin(3t)\big) + \tfrac{1}{10}\cos(3t)\log\!\left(\frac{5+3\cos(3t)}{5-3\cos(3t)}\,\frac{1}{4}\right).$$

A general solution is therefore $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark: One can evaluate any integral whose integrand is a rational function of sine and cosine. The integrals in the above example are of this type. The next example illustrates what happens in most instances when the Green function method is applied — namely, the integrals that arise cannot be evaluated analytically.

Example: Find a general solution of

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 5y = \frac{1}{1+t^2}\,.$$

Solution: The operator L has constant coefficients and is already in normal form. Its characteristic polynomial is given by $p(z) = z^2 + 2z + 5 = (z+1)^2 + 2^2$, which has roots $-1 \pm i2$. A general solution of the associated homogeneous equation is therefore

$$Y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)\,.$$

By (5.11) the Green function g associated with L is the solution of the initial-value problem

$$\mathrm{D}^2 g + 2\mathrm{D}g + 5g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

Set $g(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$. The first initial condition implies $g(0) = c_1 = 0$, whereby $g(t) = c_2 e^{-t}\sin(2t)$. Because $g'(t) = 2c_2 e^{-t}\cos(2t) - c_2 e^{-t}\sin(2t)$, the second condition implies $g'(0) = 2c_2 = 1$, whereby $c_2 = \frac{1}{2}$. The Green function is therefore $g(t) = \frac{1}{2}e^{-t}\sin(2t)$.

The particular solution given by (5.10) with $t_I = \pi$ is then

$$Y_P(t) = \int_\pi^t \tfrac{1}{2}e^{-t+s}\sin\big(2(t-s)\big)\frac{1}{1+s^2}\,\mathrm{d}s\,.$$

By (5.12) with $\phi = 2t$ and $\psi = 2s$, you see $\sin\big(2(t-s)\big) = \sin(2t)\cos(2s) - \cos(2t)\sin(2s)$. You can use this to express $Y_P(t)$ as

$$Y_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_\pi^t \frac{e^s\cos(2s)}{1+s^2}\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_\pi^t \frac{e^s\sin(2s)}{1+s^2}\,\mathrm{d}s\,.$$

The above definite integrals cannot be evaluated analytically. You can therefore leave the answer in terms of these integrals. A general solution is therefore $y = Y_H(t) + Y_P(t)$ where $Y_H(t)$ and $Y_P(t)$ are given above.

Remark: You should never use the Green function method whenever the methods of Undetermined Coefficients and Key Identity Evaluations can be applied. For example, for the equation

$$\mathrm{L}y = \mathrm{D}^2 y + 2\mathrm{D}y + 5y = t\,,$$

the Green function method leads to the expression

$$Y_P(t) = \tfrac{1}{2}e^{-t}\sin(2t)\int_0^t e^s\cos(2s)s\,\mathrm{d}s - \tfrac{1}{2}e^{-t}\cos(2t)\int_0^t e^s\sin(2s)s\,\mathrm{d}s\,.$$

The evaluation of these integrals requires several integration by parts. The time it would take you to do these integrals is much longer than the time it would take you to carry out either of the other two methods!

Now let us verify that $Y_P(t)$ given by (5.10) indeed always gives a solution of (5.8) when $g(t)$ is the solution of the initial-value problem (5.11). We will use the fact from multivariable calculus that for any continuously differentiable $K(t,s)$ one has

$$\mathrm{D}\int_{t_I}^t K(t,s)\,\mathrm{d}s = K(t,t) + \int_{t_I}^t \partial_t K(t,s)\,\mathrm{d}s\,, \qquad \text{where} \quad \mathrm{D} = \frac{d}{dt}\,.$$

Because $g(0) = 0$, you see from (5.10) that

$$\mathrm{D}Y_P(t) = g(0)f(t) + \int_{t_I}^{t} \mathrm{D}g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}g(t-s)f(s)\,\mathrm{d}s\,.$$

If $2 < n$ then because $\mathrm{D}g(0) = g'(0) = 0$, you see from the above that

$$\mathrm{D}^2 Y_P(t) = g'(0)f(t) + \int_{t_I}^{t} \mathrm{D}^2 g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}^2 g(t-s)f(s)\,\mathrm{d}s\,.$$

If you continue to argue this way then because $\mathrm{D}^{k-1}g(0) = g^{(k-1)}(0) = 0$ for $k < n$, you see that for every $k < n$

$$\mathrm{D}^k Y_P(t) = g^{(k-1)}(0)f(t) + \int_{t_I}^{t} \mathrm{D}^k g(t-s)f(s)\,\mathrm{d}s = \int_{t_I}^{t} \mathrm{D}^k g(t-s)f(s)\,\mathrm{d}s\,.$$

Similarly, because $\mathrm{D}^{n-1}g(0) = g^{(n-1)}(0) = 1$ then you see that

$$\mathrm{D}^n Y_P(t) = g^{(n-1)}(0)f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s)f(s)\,\mathrm{d}s = f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s)f(s)\,\mathrm{d}s\,.$$

Because $\mathrm{L}g(t) = 0$, it follows that $\mathrm{L}g(t-s) = 0$. Then by the above formulas for $\mathrm{D}^k Y_P(t)$ you see that

$$\begin{aligned}
\mathrm{L}Y_P(t) = p(\mathrm{D})Y_P(t) &= \mathrm{D}^n Y_P(t) + a_1 \mathrm{D}^{n-1} Y_P(t) + \cdots + a_{n-1}\mathrm{D}Y_P(t) + a_n Y_P(t) \\
&= f(t) + \int_{t_I}^{t} \mathrm{D}^n g(t-s)f(s)\,\mathrm{d}s + \int_{t_I}^{t} a_1 \mathrm{D}^{n-1} g(t-s)f(s)\,\mathrm{d}s \\
&\quad + \cdots + \int_{t_I}^{t} a_{n-1}\mathrm{D}g(t-s)f(s)\,\mathrm{d}s + \int_{t_I}^{t} a_n g(t-s)f(s)\,\mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} p(\mathrm{D})g(t-s)f(s)\,\mathrm{d}s \\
&= f(t) + \int_{t_I}^{t} \mathrm{L}g(t-s)f(s)\,\mathrm{d}s = f(t)\,.
\end{aligned}$$

Therefore, $Y_P(t)$ given by (5.10) is a solution of (5.8). Moreover, one sees from the above calculations that it is the unique solution of (5.8) that satisfies the initial conditions

$$Y_P(t_I) = 0\,, \quad Y_P'(t_I) = 0\,, \quad \cdots \quad Y_P^{(n-1)}(t_I) = 0\,.$$

6. Nonhomogeneous Equations with Variable Coefficients

6.1: Introduction. We now return to study nonhomogeneous linear equations for the general case of with variable coefficients that was begun in Section 4.1. An n^{th} order nonhomogeneous linear ODE has the normal form

$$\mathrm{L}(t)y = f(t)\,, \tag{6.1}$$

where the differential operator $\mathrm{L}(t)$ has the normal form

$$\mathrm{L}(t) = \frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{d}{dt} + a_n(t)\,. \tag{6.2}$$

We will assume throughout this section that the coefficients $a_1, a_2, \cdots, a_n$ and the forcing f are continuous over an interval (t_L, t_R), so that Therorem 1.1 can be applied.

Recall the following strategy for constructing general solutions of the nonhomogeneous equation (6.1) that we developed in Section 4.1.

(1) Find a general solution $Y_H(t)$ of the associated homogeneous equation $\mathrm{L}(t)y = 0$.

(2) Find a particular solution $Y_P(t)$ of equation (6.1).

(3) Then $Y_H(t) + Y_P(t)$ is a general solution of (6.1).

If you can find a fundamental set $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$ of solutions to the associated homogeneous equation $\mathrm{L}(t)y = 0$ then a general solution of that equation is given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

In the ensuing sections we will explore two methods to construct a particular solution $Y_P(t)$ of equation (6.1) from $f(t)$ and the fundamental set $Y_1(t)$, $Y_2(t)$, $\cdots$, $Y_n(t)$. One method is called *variation of parameters*, while the other is called the *general Green function* method, which is an extension of the Green function method presented in Section 5.4 for constant coefficient equations to the case of variable coefficient equations. We will see that these methods are essentially equivalent. What lies behind them is the following.

> **Important Fact:** If you know a general solution of the associated homogeneous equation $\mathrm{L}(t)y = 0$ then you can *always* reduce the construction of a general solution of (6.1) to the problem of finding n primitives.

Because at this point you only know how to find general solutions of homogeneous equations with constant coefficients, problems you will be given will generally fall into one of two categories. Either (1) the operator $\mathrm{L}(t)$ will have variable coefficients and you will be given a fundamental set of solutions for the associated homogeneous equation, or (2) the operator $\mathrm{L}(t)$ will have constant coefficients (i.e. $\mathrm{L}(t) = \mathrm{L}$) and you will be expected to find a fundamental set of solutions for the associated homogeneous equation. In the later case the general Green function method reduces to the method presented in Section 5.4.

6.2: Variation of Parameters: Second Order Case. We begin by deriving the method of variation of parameters for second order equations that are in the normal form

$$\mathrm{L}(t)y = \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = f(t)\,. \tag{6.3}$$

Suppose you know that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$. A general solution of the associated homogeneous equation is therefore given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,. \tag{6.4}$$

The idea of the method of variation of parameters is to seek solutions of (6.3) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,. \tag{6.5}$$

In other words you simply replace the parameters c_1 and c_2 in (6.5) with unknown functions $u_1(t)$ and $u_2(t)$. These functions are the varying parameters referred to in the title of the method. These two functions will be governed by a system of two equations, one of which is derived by requiring that (6.3) is satisfied, and the other of which is chosen to simplify the resulting system.

Let us see how this is done. Differentiating (6.5) yields

$$\frac{dy}{dt} = u_1(t)Y_1'(t) + u_2(t)Y_2'(t) + u_1'(t)Y_1(t) + u_2'(t)Y_2(t)\,. \tag{6.6}$$

We now choose to impose the condition

$$u_1'(t)Y_1(t) + u_2'(t)Y_2(t) = 0\,, \tag{6.7}$$

whereby (6.6) simplifies to

$$\frac{dy}{dt} = u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\,. \tag{6.8}$$

Differentiating (6.8) then yields

$$\frac{d^2y}{dt^2} = u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,. \tag{6.9}$$

Now substituting (6.5), (6.8), and (6.9) into (6.3), grouping the terms that multiply $u_1(t)$, $u_1'(t)$, $u_2(t)$, and $u_2'(t)$, and using the fact that $\mathrm{L}(t)Y_1(t) = 0$ and $\mathrm{L}(t)Y_2(t) = 0$, we obtain

$$\begin{aligned}
f(t) = \mathrm{L}(t)y &= \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y \\
&= \big[u_1(t)Y_1''(t) + u_2(t)Y_2''(t) + u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\big] \\
&\quad + a_1(t)\big[u_1(t)Y_1'(t) + u_2(t)Y_2'(t)\big] + a_2(t)\big[u_1(t)Y_1(t) + u_2(t)Y_2(t)\big] \\
&= u_1(t)\big[Y_1''(t) + a_1(t)Y_1'(t) + a_2(t)Y_1(t)\big] + u_1'(t)Y_1'(t) \\
&\quad + u_2(t)\big[Y_2''(t) + a_1(t)Y_2'(t) + a_2(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1(t)\big[\mathrm{L}(t)Y_1(t)\big] + u_1'(t)Y_1'(t) + u_2(t)\big[\mathrm{L}(t)Y_2(t)\big] + u_2'(t)Y_2'(t) \\
&= u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t)\,.
\end{aligned} \tag{6.10}$$

The resulting system that governs $u_1(t)$ and $u_2(t)$ is thereby given by (6.7) and (6.10):

$$\begin{aligned} u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,, \\ u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= f(t)\,. \end{aligned} \tag{6.11}$$

This is a linear system of two algebraic equations for $u_1'(t)$ and $u_2'(t)$. Because

$$\big(Y_1(t)Y_2'(t) - Y_2(t)Y_1'(t)\big) = W[Y_1, Y_2](t) \neq 0\,,$$

one can always solve this system to find

$$u_1'(t) = -\frac{Y_2(t)f(t)}{W[Y_1, Y_2](t)}\,, \qquad u_2'(t) = \frac{Y_1(t)f(t)}{W[Y_1, Y_2](t)}\,,$$

or equivalently

$$u_1(t) = -\int \frac{Y_2(t)f(t)}{W[Y_1, Y_2](t)}\,\mathrm{d}t\,, \qquad u_2(t) = \int \frac{Y_1(t)f(t)}{W[Y_1, Y_2](t)}\,\mathrm{d}t\,. \tag{6.12}$$

Letting $u_{1P}(t)$ and $u_{2P}(t)$ be any primitives of the respective right-hand sides above, one sees that

$$u_1(t) = c_1 + u_{1P}(t)\,, \qquad u_2(t) = c_2 + u_{2P}(t)\,,$$

whereby (6.5) yields the general solution

$$y = c_1 Y_1(t) + u_{1P}(t)Y_1(t) + c_2 Y_2(t) + u_{2P}(t)Y_2(t)\,.$$

Notice that this decomposes as $y = Y_H(t) + Y_P(t)$ where

$$Y_H(t) = c_1 Y_1(t) + c_2 Y_2(t)\,, \qquad Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,. \tag{6.13}$$

There are two approaches to applying variation of parameters. One mentioned in the book is to memorize the formulas (6.12). I am not a fan of this approach for a couple of reasons. First, students often confuse which of the two formulas gets the minus sign. Second, and more importantly, these formulas do not cleanly generalize to the higher order case. The other approach is to construct the linear system (6.11), which can then be rather easily solved for $u_1'(t)$ and $u_2'(t)$. The work it takes to solve this system is about the same work as it takes to generate the integrands in (6.12). The linear system (6.11) is symmetric in $u_1'(t)$ and $u_2'(t)$, so is less subject to sign errors. Moreover, it also has a clean generalization to the higher order case. Whichever approach you take, you will be led to the same two integrals.

Given $Y_1(t)$ and $Y_2(t)$, a fundamental set of solutions to the associated homogeneous equation, you proceed as follows.

1) Write the form of the solution you seek:

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t)\,.$$

2) Write the linear algebraic system for $u_1'(t)$ and $u_2'(t)$:

$$\begin{aligned} u_1'(t)Y_1(t) + u_2'(t)Y_2(t) &= 0\,,\\ u_1'(t)Y_1'(t) + u_2'(t)Y_2'(t) &= f(t)\,. \end{aligned}$$

The form of the left-hand sides of this system mimics the form of the solution you seek. The first equation simply replaces $u_1(t)$ and $u_2(t)$ with $u_1'(t)$ and $u_2'(t)$, while the second also replaces $Y_1(t)$ and $Y_2(t)$ with $Y_1'(t)$ and $Y_2'(t)$. The $f(t)$ on the right-hand side will be correct only if you have written the equation $\mathrm{L}(t)y = f(t)$ in normal form!

3) Solve the linear algebraic system to find explicit expressions for $u_1'(t)$ and $u_2'(t)$. This is always very easy to do, especially if you start with the first equation.

4) Find primitives $u_{1P}(t)$ and $u_{2P}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. One then has

$$u_1(t) = c_1 + u_{1P}(t)\,, \qquad u_2(t) = c_2 + u_{2P}(t)\,,$$

where c_1 and c_2 are the arbitrary constants of integration.

5) Upon placing this result into the form of the solution that you wrote down in step 1, you will obtain the general solution $y = Y_H(t) + Y_P(t)$, where

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t)\,, \qquad Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,.$$

For initial-value problems you must determine c_1 and c_2 from the initial conditions.

Example: Find a general solution of

$$\frac{d^2y}{dt^2} + y = \sec(t)\,.$$

Before presenting the solution, notice that while this equation has constant coefficients, the forcing is not of the form that would allow you to use the method of Undetermined Coefficients. You should be able to recognize this right away. While you can use the Green function method to solve this problem, here we will solve it using variation of parameters.

Solution: Because this problem has constant coefficients, it is easily found that

$$Y_H(t) = c_1\cos(t) + c_2\sin(t)\,.$$

Hence, we will seek a solution of the form

$$y = u_1(t)\cos(t) + u_2(t)\sin(t)\,,$$

where

$$u_1'(t)\cos(t) + u_2'(t)\sin(t) = 0\,,$$
$$-u_1'(t)\sin(t) + u_2'(t)\cos(t) = \sec(t)\,.$$

Solving this system by any means you choose yields

$$u_1'(t) = -\frac{\sin(t)}{\cos(t)}\,, \qquad u_2'(t) = 1\,.$$

These can be integrated analytically to obtain

$$u_1(t) = c_1 + \log(|\cos(t)|)\,, \qquad u_2(t) = c_2 + t\,.$$

Therefore a general solution is

$$y = c_1\cos(t) + c_2\sin(t) + \log(|\cos(t)|)\cos(t) + t\sin(t)\,.$$

Remark: The primitives $u_1(t)$ and $u_2(t)$ that we had to find above are the same ones needed to evaluate the integrals that arise when you solve this problem with the Green function method. This will always be the case.

Example: Given that t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, find a general solution of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = (1+t^2)^2 e^t\,.$$

Before presenting the solution, you should be able to recognize that this equation has variable coefficients, and thereby see that you must use either variation of parameters or a general Green function to solve this problem. You should also notice that this equation is not in normal form, so you should bring it into the normal form

$$\frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\,\frac{dy}{dt} + \frac{2}{1+t^2}\,y = (1+t^2)e^t\,.$$

Solution: Because t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, we have

$$Y_H(t) = c_1 t + c_2(t^2 - 1)\,.$$

Hence, we will seek a solution of the form

$$y = u_1(t)t + u_2(t)(t^2 - 1)\,,$$

where

$$\begin{aligned} u_1'(t)t + u_2'(t)(t^2-1) &= 0\,, \\ u_1'(t)1 + u_2'(t)2t &= (1+t^2)e^t\,. \end{aligned}$$

Solving this system by any means you choose yields

$$u_1'(t) = -(t^2-1)e^t\,, \qquad u_2'(t) = te^t\,.$$

These can be integrated analytically "by parts" to obtain

$$u_1(t) = c_1 - (t-1)^2e^t\,, \qquad u_2(t) = c_2 + (t-1)e^t\,.$$

Therefore a general solution is

$$\begin{aligned} y &= c_1t + c_2(t^2-1) - (t-1)^2e^tt + (t-1)e^t(t^2-1) \\ &= c_1t + c_2(t^2-1) + (t-1)^2e^t\,. \end{aligned}$$

6.3: Variation of Parameters: Higher Order Case. The method of variation of parameters extends to higher order linear equations in the normal form

$$\mathrm{L}(t)y = \frac{d^ny}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = f(t)\,. \tag{6.14}$$

While this material was not covered in class and you will not be tested on it, a summary is given here for the sake of completeness.

Suppose you know that $Y_1(t), Y_2(t), \cdots, Y_n(t)$ are linearly independent solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$. A general solution of the associated homogeneous equation is therefore given by

$$Y_H(t) = c_1Y_1(t) + c_2Y_2(t) + \cdots + c_nY_n(t)\,.$$

The idea of the method of variation of parameters is to seek solutions of (6.14) in the form

$$y = u_1(t)Y_1(t) + u_2(t)Y_2(t) + \cdots + u_n(t)Y_n(t)\,, \tag{6.15}$$

where $u_1'(t), u_2'(t), \cdots, u_n'(t)$ satisfy the linear algebraic system

$$\begin{array}{ccccccccl} u_1'(t)Y_1(t) & + & u_2'(t)Y_2(t) & + \cdots + & u_n'(t)Y_n(t) & = 0\,, \\ u_1'(t)Y_1'(t) & + & u_2'(t)Y_2'(t) & + \cdots + & u_n'(t)Y_n'(t) & = 0\,, \\ & & & & & \vdots \\ u_1'(t)Y_1^{(n-2)}(t) & + & u_2'(t)Y_2^{(n-2)}(t) & + \cdots + & u_n'(t)Y_n^{(n-2)}(t) & = 0\,, \\ u_1'(t)Y_1^{(n-1)}(t) & + & u_2'(t)Y_2^{(n-1)}(t) & + \cdots + & u_n'(t)Y_n^{(n-1)}(t) & = f(t)\,. \end{array} \tag{6.16}$$

Because

$$\det\begin{pmatrix} Y_1(t) & Y_2(t) & \cdots & Y_n(t) \\ Y_1'(t) & Y_2'(t) & \cdots & Y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ Y_1^{(n-1)}(t) & Y_2^{(n-1)}(t) & \cdots & Y_n^{(n-1)}(t) \end{pmatrix} = W[Y_1, Y_2, \cdots, Y_n](t) \neq 0\,,$$

the linear algebraic system (6.16) may be solved (by any method you choose) to find explicit expressions for $u_1'(t)$, $u_2'(t)$, $\cdots$, $u_n'(t)$. For example, when $n = 3$ you find

$$u_1'(t) = \frac{W[Y_2, Y_3](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,, \quad u_2'(t) = \frac{W[Y_3, Y_1](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,, \quad u_3'(t) = \frac{W[Y_1, Y_2](t)f(t)}{W[Y_1, Y_2, Y_3](t)}\,.$$

Find primitives $u_{1P}(t)$, $u_{2P}(t)$, $\cdots$, $u_{nP}(t)$ of these expressions. If you cannot find a primitive analytically then express that primitive in terms of a definite integral. One then has

$$u_1(t) = c_1 + u_{1P}(t)\,, \quad u_2(t) = c_2 + u_{2P}(t)\,, \quad \cdots \quad u_n(t) = c_n + u_{nP}(t)\,,$$

where $c_1, c_2, \cdots, c_n$ are the arbitrary constants of integration. The general solution given by (6.15) is therefore $y = Y_H(t) + Y_P(t)$, where

$$\begin{aligned} Y_H(t) &= c_1 Y_1(t) + c_2 Y_2(t) + \cdots + c_n Y_n(t)\,, \\ Y_P(t) &= u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t) + \cdots + u_{nP}(t)Y_n(t)\,. \end{aligned}$$

For initial-value problems you must determine $c_1, c_2, \cdots, c_n$ from the initial conditions.

6.4: General Green Functions: Second Order Case. We now derive a Green function for second order equations that are in the normal form

$$\mathrm{L}(t)y = \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_2(t)y = f(t)\,. \tag{6.17}$$

Suppose you know that $Y_1(t)$ and $Y_2(t)$ are linearly independent solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$. The starting point of our derivation will be the particular solution $Y_P(t)$ given in (6.13) — namely,

$$Y_P(t) = u_{1P}(t)Y_1(t) + u_{2P}(t)Y_2(t)\,,$$

where $u_{1P}(t)$ and $u_{2P}(t)$ are primitives that satisfy

$$u_{1P}'(t) = -\frac{Y_2(t)f(t)}{W[Y_1, Y_2](t)}\,, \qquad u_{2P}'(t) = \frac{Y_1(t)f(t)}{W[Y_1, Y_2](t)}\,.$$

If we express $u_{1P}(t)$ and $u_{2P}(t)$ as the definite integrals

$$u_{1P}(t) = -\int_{t_I}^{t} \frac{Y_2(s)f(s)}{W[Y_1, Y_2](s)}\,\mathrm{d}s\,, \qquad u_{2P}(t) = \int_{t_I}^{t} \frac{Y_1(s)f(s)}{W[Y_1, Y_2](s)}\,\mathrm{d}s\,, \tag{6.18}$$

where t_I is any initial time inside the interval (t_L, t_R), then the particular solution $Y_P(t)$ can be expressed as

$$Y_P(t) = \int_{t_I}^{t} G(t,s) f(s)\,\mathrm{d}s\,, \tag{6.19}$$

where $G(t,s)$ is given by

$$G(t,s) = \frac{Y_1(s)Y_2(t) - Y_1(t)Y_2(s)}{W[Y_1,Y_2](s)} = \frac{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1(t) & Y_2(t) \end{pmatrix}}{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(s) & Y_2'(s) \end{pmatrix}}\,. \tag{6.20}$$

The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (6.19), which by formula (6.20) is equivalent to evaluating the two definite integrals in (6.18). Of course, evaluating these integrals explicitly can be quite difficult or impossible. You may have to leave your answer in terms of one or both of these definite integrals. Formulas (6.19) and (6.20) have natural generalizations to higher order equations with variable coefficients.

We will see that (6.19) is an extension of the Green function formula (5.10) from Section 5.4 to second order equations with variable coefficients. As with that formula, (6.19) generates the unique particular solution $Y_P(t)$ of (6.17) that satifies the initial conditions

$$Y_P(t_I) = 0\,, \qquad\qquad Y_P'(t_I) = 0\,. \tag{6.21}$$

We therefore call $G(t,s)$ the Green function for the operator $\mathrm{L}(t)$.

Before justifying the foregoing claims, let us illustrate how to construct and use this Green function.

Example: Given that t and t^2-1 are a fundamental set of solutions of the associated homogeneous equation, find a particular solution of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = (1+t^2)^2 e^t\,.$$

Solution: You should first bring this equation into its normal form

$$\frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\,\frac{dy}{dt} + \frac{2}{1+t^2}\,y = (1+t^2)e^t\,.$$

Because t and t^2-1 are a fundamental set of solutions of the associated homogeneous equation, the Green function $G(t,s)$ is given by (7.20) as

$$G(t,s) = \frac{\det\begin{pmatrix} s & s^2-1 \\ t & t^2-1 \end{pmatrix}}{\det\begin{pmatrix} s & s^2-1 \\ 1 & 2s \end{pmatrix}} = \frac{(t^2-1)s - t(s^2-1)}{2s^2-(s^2-1)} = \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,.$$

Then formula (6.19) with $t_I = 1$ and $f(s) = (1+s^2)e^s$ yields

$$Y_P(t) = (t^2-1)\int_1^t se^s\,\mathrm{d}s - t\int_1^t (s^2-1)e^s\,\mathrm{d}s\,.$$

Notice that the same two integrals that arose when we treated this equation by variation of parameters on pages 29-30. As was done there, a little integration-by-parts shows that

$$\int_1^t se^s\,\mathrm{d}s = (t-1)e^t\,, \qquad \int_1^t (s^2-1)e^s\,\mathrm{d}s = (t-1)^2e^t\,.$$

The particular solution is therefore

$$Y_P(t) = (t^2-1)(t-1)e^t - t(t-1)^2e^t = (t-1)^2e^t\,.$$

It is clear that this solution satisfies $Y_P(1) = Y_P'(1) = 0$. Had we chosen a different value for the initial time t_I we would have obtained a different particular solution $Y_P(t)$.

Next we show that formula (6.19) generates the unique particular solution $Y_P(t)$ of (6.17) that satifies the initial conditions (6.21). It is clear from (6.19) that $Y_P(t_I) = 0$. To show that $Y_P'(t_I) = 0$ we will use the fact from multivariable calculus that for any continuously differentiable $K(t,s)$ one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_I}^t K(t,s)\,\mathrm{d}s = K(t,t) + \int_{t_I}^t \partial_t K(t,s)\,\mathrm{d}s\,.$$

We see from (6.20) that $G(t,t) = 0$. Upon differentiating (6.19) with respect to t and using the above calculus fact, we see that

$$Y_P'(t) = G(t,t)f(t) + \int_{t_I}^t \partial_t G(t,s)f(s)\,\mathrm{d}s = \int_{t_I}^t \partial_t G(t,s)f(s)\,\mathrm{d}s\,.$$

It follows that $Y_P'(t_I) = 0$, thereby showing that $Y_P(t)$ satifies the initial conditions (6.21).

The Green function $G(t,s)$ is defined by (6.20) whenever t and s are both in the interval (t_L, t_R) overwhich Y_1 and Y_2 exist. At first it might seem that $G(t,s)$ must depend upon the fundamental set of solutions that is used to construct it. We now show that this is not the case. Let us fix s and consider $G(t,s)$ as a function of t. It is clear from (6.20) that $G(t,s)$ is a linear combination of $Y_1(t)$ and $Y_2(t)$. Because $Y_1(t)$ and $Y_2(t)$ are solutions of the associated homogeneous equation $\mathrm{L}(t)y = 0$, it follows that $G(t,s)$ is too — namely, that $\mathrm{L}(t)G(t,s) = 0$. It is also clear from (6.20) that $G(t,s)\big|_{t=s} = 0$. By differentiating (6.20) with respect to t we obtain

$$\partial_t G(t,s) = \frac{Y_1(s)Y_2'(t) - Y_1'(t)Y_2(s)}{W[Y_1,Y_2](s)} = \frac{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(t) & Y_2'(t)\end{pmatrix}}{\det\begin{pmatrix} Y_1(s) & Y_2(s) \\ Y_1'(s) & Y_2'(s)\end{pmatrix}}\,. \tag{6.22}$$

It is clear from this that $\partial_t G(t,s)\big|_{t=s} = 1$. Collecting these facts we see for every s that $G(t,s)$ as a function of t satisfies the initial-value problem

$$\mathrm{L}(t)G(t,s) = 0\,, \qquad G(t,s)\big|_{t=s} = 0\,, \quad \partial_t G(t,s)\big|_{t=s} = 1\,. \tag{6.23}$$

This is really a family of initial-value problems — one for each s in which s plays the role of the initial time. The uniqueness theorem implies that $G(t,s)$ is uniquely determined by this family of initial-value problems. Thus, $G(t,s)$ depends only upon the operator $\mathrm{L}(t)$. In particular, it does not depend upon which fundamental set of solutions, Y_1 and Y_2, was used to construct it.

When $\mathrm{L}(t)$ has constant coefficients then it is easy to check that the family of initial-value problems (6.23) is satisfied by $G(t,s) = g(t-s)$, where $g(t)$ is the Green function that was defined by the initial-value problem (5.11) in Section 5.4. Formula (6.19) thereby extends the Green function formula (5.10) from Section 5.4 to second order equations with variable coefficients.

When $\mathrm{L}(t)$ has constant coefficients the fastest way to compute the Green function is to solve the single initial-value problem (5.11) from Section 5.4. When $\mathrm{L}(t)$ has variable coefficients you first have to find a fundamental set of solutions, $Y_1(t)$ and $Y_2(t)$, to the associated homogeneous equation. You can then construct the Green function either by formula (6.20) or by solving the family of initial-value problems (6.23). The later approach goes as follows. Because $\mathrm{L}(t)G(t,s) = 0$ for every s we know that there exist $C_1(s)$ and $C_2(s)$ such that

$$G(t,s) = Y_1(t)C_1(s) + Y_2(t)C_2(s)\,. \tag{6.24}$$

The initial conditions of (6.23) then imply that

$$\begin{aligned} 0 = \quad G(t,s)\big|_{t=s} &= Y_1(s)C_1(s) + Y_2(s)C_2(s)\,, \\ 1 = \partial_t G(t,s)\big|_{t=s} &= Y_1'(s)C_1(s) + Y_2'(s)C_2(s)\,. \end{aligned}$$

The solution of this linear algebraic system is

$$C_1(s) = -\frac{Y_2(s)}{Y_1(s)Y_2'(s) - Y_1'(s)Y_2(s)}\,, \quad C_2(s) = \frac{Y_1(s)}{Y_1(s)Y_2'(s) - Y_1'(s)Y_2(s)}\,,$$

which when plugged into (6.24) yields (6.20).

Example: Given that t and $t^2 - 1$ are a fundamental set of solutions of the associated homogeneous equation, find a particular solution of

$$(1+t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2y = (1+t^2)^2 e^t\,.$$

Solution: You should first bring this equation into its normal form

$$\mathrm{L}(t)y = \frac{d^2y}{dt^2} - \frac{2t}{1+t^2}\,\frac{dy}{dt} + \frac{2}{1+t^2}\,y = (1+t^2)e^t\,.$$

Because t and t^2-1 are a fundamental set of solutions of the associated homogeneous equation, by (6.24) the Green function has the form

$$G(t,s) = t\,C_1(s) + (t^2-1)\,C_2(s)\,,$$

where the initial conditions of (6.23) imply

$$\begin{aligned} 0 &= \quad G(t,s)\big|_{t=s} = s\,C_1(s) + (s^2-1)\,C_2(s)\,,\\ 1 &= \partial_t G(t,s)\big|_{t=s} = 1\,C_1(s) + 2s\,C_2(s)\,. \end{aligned}$$

These may be solved to obtain

$$C_1(s) = -\frac{s^2-1}{s^2+1}\,, \qquad C_2(s) = \frac{s}{s^2+1}\,,$$

whereby

$$G(t,s) = -t\,\frac{s^2-1}{s^2+1} + (t^2-1)\,\frac{s}{s^2+1} = \frac{(t^2-1)s - t(s^2-1)}{s^2+1}\,.$$

You then compute $y_P(t)$ by formula (6.19) as before.

6.5: General Green Functions: Higher Order Case. This method can be used to construct a particular solution of an n^{th} order nonhomogeneous linear ODE in the normal form (6.1). Specifically, a particular solution of (6.1) is given by

$$Y_P(t) = \int_{t_I}^{t} G(t,s) f(s)\,\mathrm{d}s\,, \tag{6.25}$$

where t_I is an initial time and $G(t,s)$ is given by

$$G(t,s) = \frac{1}{W[Y_1,Y_2,\cdots,Y_n](s)}\det\begin{pmatrix} Y_1(s) & Y_2(s) & \cdots & Y_n(s)\\ Y_1'(s) & Y_2'(s) & \cdots & Y_n'(s)\\ \vdots & \vdots & \vdots & \vdots\\ Y_1^{(n-2)}(s) & Y_2^{(n-2)}(s) & \cdots & Y_n^{(n-2)}(s)\\ Y_1(t) & Y_2(t) & \cdots & Y_n(t) \end{pmatrix}.$$

The function G is called the *Green function* associated with the operator $\mathrm{L}(t)$. It can also be determined as the solution of the family of initial-value problems

$$\mathrm{L}(t)G(t,s) = 0\,, \qquad G(t,s)\big|_{t=s} = \cdots = \partial_t^{n-2}G(t,s)\big|_{t=s} = 0\,, \quad \partial_t^{n-1}G(t,s)\big|_{t=s} = 1\,.$$

The method thereby reduces the problem of finding a particular solution $Y_P(t)$ for any forcing $f(t)$ to that of evaluating the integral in (6.25). However, evaluating this integral explicitly can be quite difficult or impossible. At worst, you can leave your answer in terms of definite integrals.