Linear Planar Systems
Math 246, Fall 2009, Professor David Levermore

We now consider linear systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{1}$$

Here the entries of the coefficient matrix $\mathbf{A}$ are real constants. Such a system is called *planar* because any solution of it can be thought of as tracing out a curve $(x(t), y(t))$ in the xy-plane.

Of course, we have seen that solutions to this system can be expressed analytically as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}}\begin{pmatrix} x_I \\ y_I \end{pmatrix}, \qquad \text{where} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_I \\ y_I \end{pmatrix}, \tag{2}$$

and $e^{t\mathbf{A}}$ is given by one of the following three formulas that depend upon the roots of the characteristic polynomial $p(z) = \det(z\mathbf{I} - \mathbf{A}) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A})$.

- If $p(z)$ has simple real roots $\mu \pm \nu$ with $\nu \neq 0$ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\mathbf{I}\cosh(\nu t) + (\mathbf{A} - \mu\mathbf{I})\frac{\sinh(\nu t)}{\nu}\right]. \tag{3a}$$

- If $p(z)$ has conjugate roots $\mu \pm i\nu$ with $\nu \neq 0$ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\mathbf{I}\cos(\nu t) + (\mathbf{A} - \mu\mathbf{I})\frac{\sin(\nu t)}{\nu}\right]. \tag{3b}$$

- If $p(z)$ has a double real root μ then

$$e^{t\mathbf{A}} = e^{\mu t}\left[\mathbf{I} + (\mathbf{A} - \mu\mathbf{I})t\right]. \tag{3c}$$

While these analytic formulas are useful, you can gain insight into all solutions of system (1) by sketching a graph called its *phase-plane portrait* (or simply *phase portrait*).

As we have already observed, any solution of (1) can be thought of as tracing out a curve $(x(t), y(t))$ in the xy-plane — the so-called *phase-plane*. Each such curve is called an *orbit* or *trajectory* of the system. The existence and uniqueness theorem implies that every point in the phase-plane has exactly one orbit that passes through it. In particular, two orbits can not cross. You can gain insight into all solutions of system (1) by visualizing how their orbits fill the phase-plane.

Of course, the origin will be an orbit of system (1) for every $\mathbf{A}$. The solution that starts at the origin will stay at the origin. Points that give rise to solutions that do not move are called *stationary points*. In that case the entire orbit is a single point.

Orbits Associated with a Real Eigenpair. If the matrix $\mathbf{A}$ has a real eigenpair $(\lambda, \mathbf{v})$ then system (1) has special solutions of the form

$$\mathbf{x}(t) = ce^{\lambda t}\mathbf{v}, \tag{4}$$

where c is any nonzero real constant. These solutions all lie on the line $\mathbf{x} = c\mathbf{v}$ parametrized by c. This line is easy to plot; it is simply the line that passes through the origin and the point $\mathbf{v}$. There are three possibilities.

- If $\lambda = 0$ then every point on the line $\mathbf{x} = c\mathbf{v}$ is a stationary point, and thereby is an orbit.

- If $\lambda > 0$ then the line $\mathbf{x} = c\mathbf{v}$ consists of three orbits: the origin, corresponding to $c = 0$, plus the two remaining half-lines, corresponding to $c > 0$ and $c < 0$. Because $\lambda > 0$ all solutions on the half-lines will run away from the origin as t increases. We indicate this case by placing arrowheads on each half-line pointing away from the origin.
- If $\lambda < 0$ then the line $\mathbf{x} = c\mathbf{v}$ again consists of three orbits: the origin, corresponding to $c = 0$, plus the two remaining half-lines, corresponding to $c > 0$ and $c < 0$. Because $\lambda < 0$ all solutions on the half-lines will run approach the origin as t increases. We indicate this case by placing arrowheads on each half-line pointing towards the origin.

For every real eigenpair of $\mathbf{A}$ you should indicate the above orbits in your phase-plane portrait before doing anything else. If $\mathbf{A}$ has no real eigenpairs then you are spared this step.

Characteristic Polynomials with Two Simple Real Roots. Now consider the case when $\mathbf{A}$ has two real eigenvalues $\lambda_1 < \lambda_2$. Let $(\lambda_1, \mathbf{v}_1)$ and $(\lambda_2, \mathbf{v}_2)$ be real eigenpairs of $\mathbf{A}$. You first plot the ordits that lie on the lines $\mathbf{x} = c_1\mathbf{v}_1$ and $\mathbf{x} = c_2\mathbf{v}_2$ as described above. Every other solution of system (1) has the form

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \,, \tag{5}$$

where both c_1 and c_2 are arbitrary nonzero real numbers. There are five possibilities.

- If $\lambda_1 < \lambda_2 < 0$ then every solution will approach the origin as $t \to \infty$. Because $e^{\lambda_1 t}$ decays to zero faster than $e^{\lambda_2 t}$ it is clear that the solution (5) behaves like $c_2 e^{\lambda_2 t}\mathbf{v}_2$ as $t \to \infty$. This means that all orbits not on the line $\mathbf{x} = c_1\mathbf{v}_1$ will approach the origin tangent to the line $\mathbf{x} = c_2\mathbf{v}_2$. This portrait is called a *nodal sink*.
- If $\lambda_1 < \lambda_2 = 0$ then the line $\mathbf{x} = c_2\mathbf{v}_2$ is a line of stationary points. It is clear that as t increases the solution (5) will approach one of those stationary points as along a line that is parallel to the line $\mathbf{x} = c_1\mathbf{v}_1$. This means that all orbits not on the line of stationary points $\mathbf{x} = c_2\mathbf{v}_2$ will approach that line parallel to the line $\mathbf{x} = c_1\mathbf{v}_1$. This portrait is called a *linear sink*.
- If $\lambda_1 < 0 < \lambda_2$ then the nonzero orbits on the line $\mathbf{x} = c_1\mathbf{v}_2$ will approach the origin as t increases while the nonzero orbits on the line $\mathbf{x} = c_2\mathbf{v}_2$ will move away from the origin as t increases. It is clear that as t increases the solution (5) will approach the line $\mathbf{x} = c_2\mathbf{v}_2$ while as t decreases it will approach the line $\mathbf{x} = c_1\mathbf{v}_1$. This portrait is called a *saddle*.
- If $0 = \lambda_1 < \lambda_2$ then the line $\mathbf{x} = c_1\mathbf{v}_1$ is a line of stationary points. It is clear that as t increases the solution (5) will run away from one of those stationary points along a line that is parallel to the line $\mathbf{x} = c_2\mathbf{v}_2$. This means that all orbits not on the line of stationary points $\mathbf{x} = c_1\mathbf{v}_1$ will run away from that line parallel to the line $\mathbf{x} = c_2\mathbf{v}_2$. This portrait is called a *linear source*.
- If $0 < \lambda_1 < \lambda_2$ then every solution will run away from the origin t increases. Because $e^{\lambda_2 t}$ decays to zero faster than $e^{\lambda_1 t}$ as $t \to -\infty$, it is clear that the solution (5) behaves like $c_1 e^{\lambda_1 t}\mathbf{v}_1$ as $t \to -\infty$. This means that all orbits not on the line $\mathbf{x} = c_2\mathbf{v}_2$ will emerge from the origin tangent to the line $\mathbf{x} = c_1\mathbf{v}_1$. This portrait is called a *nodal source*.

Characteristic Polynomials with a Conjugate Pair of Roots. Now consider the case when $\mathbf{A}$ has a conjugate pair of eigenvalues $\mu \pm i\nu$ with $\nu \neq 0$. The analytic solution is

$$(6)\qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \left[\mathbf{I}\cos(\nu t) + (\mathbf{A} - \mu\mathbf{I}) \frac{\sin(\nu t)}{\nu} \right] \begin{pmatrix} x_I \\ y_I \end{pmatrix} .$$

The matrix inside the square brackets is a periodic function of t with period $2\pi/\nu$.

When $\mu = 0$ this solution will trace out an ellipse. But will it do so with a clockwise or counterclockwise rotation? You can determine the direction of rotation by considering what happens at special values of $\mathbf{x}$.

$$\text{At } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ we have } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} .$$

This vector points up if $a_{21} > 0$, which indicates counterclockwise rotation. Similarly, this vector points down if $a_{21} < 0$, which indicates clockwise rotation.

$$\text{At } \mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ we have } \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} .$$

This vector points right if $a_{12} > 0$, which indicates clockwise rotation. Similarly, this vector points left if $a_{12} < 0$, which indicates counterclockwise rotation. You can therefore read off the direction of rotation from the sign of either a_{21} or a_{12}.

It is clear from (6) that when $\mu < 0$ all solutions will approach the origin as $t \to \infty$, while when $\mu > 0$ all solutions will run away from the origin as t increases. If we put this together with the information above, we see there are six possibilities.

- If $\mu < 0$ then all orbits spiral into the origin as $t \to \infty$. This portrait is called a *spiral sink*. The spiral is *counterclockwise* if $a_{21} > 0$, and is *clockwise* if $a_{21} < 0$.
- If $\mu = 0$ then all orbits are ellipses around the origin. This portrait is called a *center*. The center is *counterclockwise* if $a_{21} > 0$, and is *clockwise* if $a_{21} < 0$.
- If $\mu > 0$ then all orbits spiral away from the origin as $t \to \infty$. This portrait is called a *spiral source*. The spiral is *counterclockwise* if $a_{21} > 0$, and is *clockwise* if $a_{21} < 0$.

Characteristic Polynomials with a Double Real Root. Finally, consider the case when $\mathbf{A}$ has a single real eigenvalue μ. By (2) and (3c) the analytic solution is

$$(7)\qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \left[\mathbf{I} + (\mathbf{A} - \mu\mathbf{I})\, t \right] \begin{pmatrix} x_I \\ y_I \end{pmatrix} .$$

There are two subcases.

The simplest subcase is when $\mathbf{A} = \mu\mathbf{I}$. This subcase is easy to spot because the matrix $\mathbf{A}$ is simply a multipule of the identity matrix $\mathbf{I}$. It follows that every nonzero vector is an eigenvector of $\mathbf{A}$. In this subcase the analytic solution (7) reduces to

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\mathbf{A}} \begin{pmatrix} x_I \\ y_I \end{pmatrix} = e^{\mu t} \begin{pmatrix} x_I \\ y_I \end{pmatrix} .$$

There are three possibilities.

- If $\mu < 0$ all orbits radially approach the origin as $t \to \infty$. This portrait is called a *radial sink*.
- If $\mu = 0$ then $\mathbf{A} = 0$ and all solutions are stationary. This portrait is called *zero*.
- If $\mu > 0$ all orbits radially move away from the origin as $t \to \infty$. This portrait is called a *radial source*.

The other subcase is when $\mathbf{A} \neq \mu\mathbf{I}$. In this subcase $\mathbf{A}$ will have the eigenpair $(\mu, \mathbf{v})$ where $\mathbf{v}$ is proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. You first plot the ordits that lie on the line $\mathbf{x} = c\mathbf{v}$ as described above. There are six possibilities.

- If $\mu < 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ approach the origin as t increases. Because $e^{\mu t}$ will decay to zero faster than $e^{\mu t}t$, and because the columns of $\mathbf{A} - \mu\mathbf{I}$ are proportional to $\mathbf{v}$, it is clear from (7) that every orbit approaches the origin tangent to the line $\mathbf{x} = c\mathbf{v}$. This portrait is called a *twist sink*. The twist is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$, and is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$.
- If $\mu = 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ are stationary. All other solutions will move parallel to this line. This portrait is called a *parallel shear*. The shear is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$, and is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$.
- If $\mu > 0$ then all solutions on the line $\mathbf{x} = c\mathbf{v}$ move away from the origin as t increases. Because $e^{\mu t}$ will decay to zero faster than $e^{\mu t}t$ as $t \to -\infty$, and because the columns of $\mathbf{A} - \mu\mathbf{I}$ are proportional to $\mathbf{v}$, it is clear from (7) that every orbit emerges from the origin tangent to the line $\mathbf{x} = c\mathbf{v}$. This portrait is called a *twist source*. The twist is *counterclockwise* if either $a_{21} > 0$ or $a_{12} < 0$, and is *clockwise* if either $a_{21} < 0$ or $a_{12} > 0$.

We remark that for a twist or a shear you can have $a_{21} = 0$ or $a_{12} = 0$ but you cannot have $a_{21} = a_{12} = 0$. You will therefore always be able to determine the direction of rotation from either a_{21} or a_{12}. Alternatively, you can combine the a_{21} test and the a_{12} test into a single test on $a_{21} - a_{12}$ that works for every spiral, center, twist, or shear. Namely, if $a_{21} - a_{12} > 0$ the rotation is counterclockwise, while if $a_{21} - a_{12} > 0$ the rotation is clockwise.

The book calls radial sinks and sources *proper nodes* and twist sinks and sources *improper nodes*. This terminology is both more cumbersome and much less desciptive than that used here. Moreover, it often leaves students with the impression that improper nodes are rare. However, the opposite is true; improper nodes are much more common than proper nodes. Other books refer to radial sinks and sources as *star* sinks and sources.

The book fails to include linear sinks and sources, parallel shears, or zero in its discussion of types of phase-plane portraits. These are the types for which $\det(\mathbf{A}) = 0$. It is not made clear why these types are excluded. There is no good justification for it.

Mean-Discriminant Plane. You can visualize the relationships between the various types of phase-plane portraits for linear systems through the mean-discriminant plane. By completing the square of the characteristic polynomial you bring it into the form

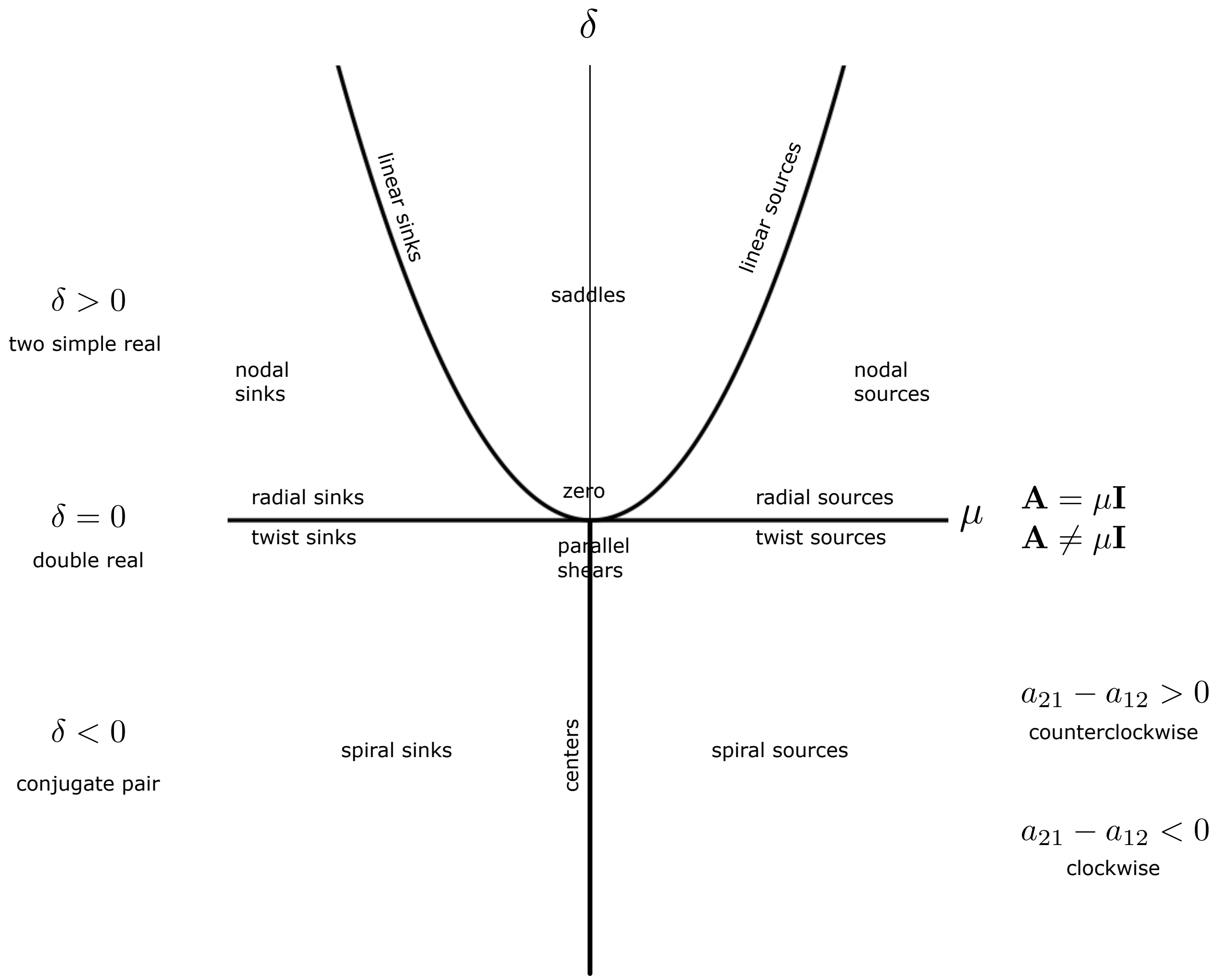
$$p(z) = z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = (z - \mu)^2 - \delta\,,$$

where the *mean* μ and *discriminant* δ are given by

$$\mu = \tfrac{1}{2}\operatorname{tr}(\mathbf{A})\,, \qquad \delta = \mu^2 - \det(\mathbf{A})\,.$$

Recall that δ is called the discriminant because it determines the root structure of the characteristic polynomial: when $\delta > 0$ there are the two simple real roots $\mu \pm \sqrt{\delta}$; when $\delta = 0$ there is the one double real root μ; when $\delta < 0$ there is the conjugate pair of roots $\mu \pm i\sqrt{|\delta|}$. In all cases μ is the average of the roots, which is why it is called the mean.

In the (μ, δ)-plane the types of phase-plane portraits you get are shown in the figure below.



In the upper half-plane ($\delta > 0$) the matrix $\mathbf{A}$ has the two simple real eigenvalues $\mu \pm \sqrt{\delta}$. These will both be negative whenever $\mu < -\sqrt{\delta}$, which is the region labeled *nodal sinks* in the figure. These will both be positive whenever $\sqrt{\delta} < \mu$, which is the region labeled *nodal sources* in the figure. These will have opposite signs whenever $-\sqrt{\delta} < \mu < \sqrt{\delta}$, which is the region labeled *saddles* in the figure. These three regions are seperated by the parabola $\delta = \mu^2$. There is one negative and one zero eigenvalue along the branch of this parabola where $\mu = -\sqrt{\delta}$, which is labeled *linear sinks* in the figure. There is one zero and one positive eigenvalue along the branch of this parabola where $\mu = \sqrt{\delta}$, which is labeled *linear sources* in the figure.

In the lower half-plane ($\delta < 0$) the matrix $\mathbf{A}$ has the conjugate pair of eigenvalues $\mu \pm i\sqrt{|\delta|}$. These will have negative real parts whenever $\mu < 0$, which is the region labeled *spiral sinks* in the figure. These will have positive real parts whenever $\mu > 0$, which is the region labeled *spiral sources* in the figure. These will be purely imaginary whenever $\mu = 0$, which is the half-line labeled *centers* in the figure. In each case you can determine the rotation of the orbits (counterclockwise or clockwise) by the a_{21} test.

On the μ-axis ($\delta = 0$) the matrix $\mathbf{A}$ has the single real eigenvalue μ. There are two cases to consider: the case when $\mathbf{A} = \mu\mathbf{I}$ which we label above the μ-axis, and the case when $\mathbf{A} \neq \mu\mathbf{I}$ which we label below the μ-axis.

- When $\mathbf{A} = \mu\mathbf{I}$ every nonzero vector is an eigenvector associated with the eigenvalue μ. This eigenvalue is negative whenever $\mu < 0$, which is the half-line labeled *radial sink* in the figure. This eigenvalue is positive whenever $\mu > 0$, which is the half-line labeled *radial source* in the figure. This eigenvalue is zero whenever $\mu = 0$, in which case $\mathbf{A} = \mu\mathbf{I} = 0$, which is why the origin is labeled *zero* in the figure.
- When $\mathbf{A} \neq \mu\mathbf{I}$ the eigenvectors associated with the eigenvalue μ are proportional to any nonzero column of $\mathbf{A} - \mu\mathbf{I}$. This eigenvalue is negative whenever $\mu < 0$, which is the half-line labeled *twist sink* in the figure. This eigenvalue is positive whenever $\mu > 0$, which is the half-line labeled *twist source* in the figure. This eigenvalue is zero whenever $\mu = 0$, in which case the origin is labeled *parallel shears* in the figure. In each of these cases you can determine the rotation of the orbits (counterclockwise or clockwise) either by the a_{21} test or the a_{12} test or by the $a_{21} - a_{12}$ test.

Stability of the Origin. The origin of the phase-plane plays a special role for linear systems. This is because the solution $(x(t), y(t))$ of any system (1) that starts at the origin will stay at the origin. In other words, the origin is an orbit for every linear system (1).

Definition. We say that the origin is *stable* if every orbit that starts near it will stay near it. We say that the origin is *unstable* if there is at least one orbit near it that runs away from it.

The language "every orbit that starts near it will stay near it" and "at least one orbit near it that runs away from it" is not very precise. Rather than formulate a more precise mathematical definition, we will build your understanding of these notions through examples. To begin with, we hope this language makes it clear that for every system the origin is either stable or unstable.

Definition. We say that the origin is *attracting* if every orbit that starts near it will approach it as $t \to \infty$. We say that the origin is *repelling* if every orbit other than itself that starts near it will run away from it.

It should be clear that if the origin is attracting then it is stable, and that if it is repelling then it is unstable. These implications do not go the other way. Indeed, we will soon see that there are systems for which the origin is stable but not attracting, and systems which the origin is unstable but not repelling.

When we examine the different phase-plane portraits we see that the stability of the origin for each of them is as given by the following table.

attracting (and also stable)	stable (but not attracting)	unstable (but not repelling)	repelling (and also unstable)
nodal sinks	linear sinks	linear sources	nodal sources
radial sinks	zero	saddles	radial sources
twist sinks	centers	parallel shears	twist sources
spiral sinks			spiral sources

The book calls attracting *asymptotically stable* but has no terminology for repelling.