Quiz 1 Solutions, Math 246, Professor David Levermore
Tuesday, 8 September 2009

(1) [4] For each of the following ordinary differential equations, give its order and state whether it is linear or nonlinear.

(a) $\dfrac{\mathrm{d}^4 w}{\mathrm{d}z^4} + z^2 \dfrac{\mathrm{d}w}{\mathrm{d}z} + e^z w = z$; **Solution:** fourth-order, linear.

(b) $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 2x\, \dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin(y)$. **Solution:** third-order, nonlinear.

(2) [2] Give the interval of definition for the solution of the initial-value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \tan(t)\, x = \frac{1}{t^2 - 4}\,, \qquad x(3) = 7\,.$$

(You do not have to solve this equation to answer this question!)

Solution: This problem is linear in x and is already in normal form. The coefficient $\tan(t)$ is continuous everywhere except where $t = \frac{\pi}{2} + n\pi$ for some integer n, while the forcing $1/(t^2 - 4)$ is continuous everywhere except at $t = \pm 2$. You can therefore read off that the interval of definition is $(2, \frac{3\pi}{2})$, the endpoints of which bracket the initial time 3 and are points where $1/(t^2 - 4)$ and $\tan(t)$ respectively do not exist.

(3) [4] Solve the initial-value problem

$$z^2 \frac{\mathrm{d}y}{\mathrm{d}z} = y\,, \qquad y(1) = e\,.$$

Solution: This is a homogeneous linear equation. Its normal form is

$$\frac{\mathrm{d}y}{\mathrm{d}z} - \frac{1}{z^2}\, y = 0\,.$$

It has a general solution $y = e^{-A(z)}c$, where $A'(z) = -1/z^2$. Let $A(z) = 1/z$, so that $y = e^{-1/z}c$. The initial condition then implies that $e = e^{-1/1}c$, which yields $c = e^2$. The solution of the initial-value problem is therefore

$$y = e^{2 - \frac{1}{z}}\,.$$

Notice that its interval of definition is $(0, \infty)$.

Alternative Solution: This is also a separable equation. Its separated differential form is

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{z^2}\,.$$

Its solutions are given implicitly by $G(y) = F(z) + c$ where $G'(y) = 1/y$ and $F'(z) = 1/z^2$. Let $G(y) = \log(|y|)$ and $F(z) = -1/z$, so that $\log(|y|) = -1/z + c$. The initial condition then implies that $\log(e) = -1/1 + c$, which yields $c = 1 + \log(e) = 1 + 1 = 2$. Hence, $\log(|y|) = 2 - \frac{1}{z}$. When you solve this for y you obtain $y = \pm e^{2 - \frac{1}{z}}$. You then take the positive solution to match the initial condition.