Quiz 5 Solutions, Math 246, Professor David Levermore
Tuesday, 13 October 2009

(1) [5] Give a general solution of the equation

$$(\mathrm{D}-5)^2(\mathrm{D}+7)(\mathrm{D}^2+9)y = 0\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. This is a fifth-order, homogeneous, linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = (z-5)^2(z+7)(z^2+9) = (z-5)^2(z+7)(z^2+3^2)\,,$$

which has roots 5, 5, -7, $i3$, and $-i3$. A general solution of the equation is

$$y(t) = c_1 e^{5t} + c_2 t e^{5t} + c_3 e^{-7t} + c_4\cos(3t) + c_5\sin(3t)\,.$$

(2) [5] Given that e^t is a particular solution of the differential equation, solve the initial value problem

$$y'' - 6y' + 25y = 20e^t\,, \qquad y(0) = 1\,, \quad y'(0) = 9\,,$$

Solution. This is a second-order, nonhomogeneous, linear equation with constant coefficients. Its characteristic polynomial is

$$p(z) = z^2 - 6z + 25 = (z-3)^2 + 16 = (z-3)^2 + 4^2\,,$$

which has roots $z = 3 + i4$, $3 - i4$. A general solution of the associated homogeneous equation is

$$y_H(t) = c_1 e^{3t}\cos(4t) + c_2 e^{3t}\sin(4t)\,.$$

Because a particular solution is $y_P(t) = e^t$, a general solution of the equation is

$$y(t) = c_1 e^{3t}\cos(4t) + c_2 e^{3t}\sin(4t) + e^t\,.$$

The initial condition $y(0) = 1$ implies that

$$1 = y(0) = c_1 e^0\cos(0) + c_2 e^0 \sin(0) + e^0 = c_1 + 1\,.$$

Hence, $c_1 = 0$. Then because

$$y'(t) = 3c_2 e^{3t}\sin(4t) + 4c_2 e^{3t}\cos(4t) + e^t\,,$$

the initial condition $y'(0) = 9$ implies that

$$9 = 3c_2 e^0\sin(0) + 4c_2 e^0\cos(0) + e^0 = 4c_2 + 1\,.$$

Hence, $c_2 = 2$. The solution of the initial-value problem is therefore

$$y(t) = 2e^{3t}\sin(4t) + e^t\,.$$