Quiz 6 Solutions, Math 246, Professor David Levermore
Tuesday, 20 October 2009

(1) [3] Give the characteristic, degree, and multiplicity for each of the two forcing terms of the equation

$$\mathrm{L}y = \mathrm{D}^2 y - 6\mathrm{D}y + 13y = 5t^2 e^{3t}\sin(2t)\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The characteristic polynomial is $p(z) = z^2 - 6z + 13 = (z-3)^2 + 2^4$, which has roots $3 \pm i2$.

The forcing term $5t^2e^{3t}\sin(2t)$ has characteristic $\mu + i\nu = 3 + i2$, degree $d = 2$, and multiplicity $m = 1$.

(2) [4] Give a particular solution of the equation

$$\mathrm{L}y = \mathrm{D}^2 y - 6\mathrm{D}y + 13y = e^{2t}\,, \qquad \text{where} \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}t}\,.$$

Solution. The characteristic polynomial is $p(z) = z^2 - 6z + 13$. The forcing has characteristic $\mu + i\nu = 2$, degree $d = 0$, and multiplicity $m = 0$.

KEY Identity Evaluation. Because $d + m = 0$ you only need the KEY identity

$$\mathrm{L}(e^{zt}) = (z^2 - 6z + 13)e^{zt}\,.$$

Upon evaluating this at $z = 2$ we see that

$$\mathrm{L}(e^{2t}) = (2^2 - 6\cdot 2 + 13)e^{2t} = (4 - 12 + 13)e^{2t} = 5e^{2t}\,.$$

By dividing both sides of this by 5 you see that a particular solution is $y_P(t) = \frac{1}{5}e^{2t}$.

Undetermined Coefficients. Because $d + m = 0$, there is a particular solution of the form

$$y_P(t) = Ae^{2t}\,.$$

Because

$$y_P'(t) = 2Ae^{2t}\,, \qquad y_P''(t) = 4Ae^{2t}\,,$$

you find that

$$y_P'' - 6y_P' + 13y_P = 4Ae^{2t} - 6\cdot 2Ae^{2t} + 13Ae^{2t} = (4 - 12 + 13)Ae^{2t} = 5e^{2t}\,.$$

By setting $5A = 1$ you see that a particular solution is $y_P(t) = \frac{1}{5}e^{2t}$.

(3) [3] Compute the Green function associated with the differential operator $\mathrm{L} = \mathrm{D}^2 + 9$.

Solution. The Green function $g(t)$ satisfies the initial-value problem

$$g'' + 9g = 0\,, \qquad g(0) = 0\,, \quad g'(0) = 1\,.$$

The characteristic polynomial is $p(z) = z^2 + 9$, which has roots $\pm i3$. Therefore $g(t)$ has the form

$$g(t) = c_1\cos(3t) + c_2\sin(3t)\,.$$

Because $g(0) = c_1\cos(0) + c_2\sin(0) = c_1$, the first initial condition shows $c_1 = 0$. Then $g'(t) = 3c_2\cos(3t)$. Because $g'(0) = 3c_2\cos(0) = 3c_2$, the second initial condition shows $3c_2 = 1$, whereby $c_2 = \frac{1}{3}$. The Green function $g(t)$ is therefore given by

$$g(t) = \tfrac{1}{3}\sin(3t)\,.$$