Solutions to Mezo notes

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In the following, sometimes I will write the Weil group as $W(\overline{F}/F)$, and sometimes by W_F

Section 2:

Basic exercises in number theory.

Section 3:

1. Following the hint, want to first prove that $(O_F^*, F^{ab}/F)$ is dense in Gal(F^{ab}/F_u). So, want to show that for any $\sigma \in Gal(F^{ab}/F_u)$, and for all Gal($F^{ab}/E$) for E a finite abelian extension of F, then there exists an $x \in O_F^*$ such that $(x, F^{ab}/F) \in \sigma Gal(F^{ab}/E)$. i.e. that $(x, F^{ab}/F)|_E = x|_E$, i.e. that $(x, E/F) = x|_E$. Well, $\sigma \in Gal(F^{ab}/F_u)$, so it's clear that $\sigma|_E \in Gal(E/K)$, where K is the largest unramified extension of F contained in E. Thus, by (3) on page 5 of the notes, there is an $x \in O_F^*$ s.t. $(x, E/F) = \sigma|_E$.

Finally, since O_F^* is compact, and its image is dense under a continuous map, the image must be all of Gal(F^{ab}/F_u). □

2. We use the consistency property of the Artin symbol at the end of page 5. Note first of all, that since $Gal(F_u/F)$ is an inverse limit of $Gal(E/F)$ where E are finite unramified extensions of F, then since $X := (\omega, F^{ab}/F)|_{F_u} \in Gal(F_u/F)$, then X is determined by its restrictions to finite unramified extensions. Well, by the consistency property, let E be a finite unramified extension of F. Then $X|_E = (\omega, E/F)$. But we proved before that since E is unramified finite over F, then $(\omega, E/F)$ is the Frobenius automorphism of Gal(E/F). Thus, we know that X restricted to an unramified finite extension is the Frobenius, so since X is an element of $Gal(F_u/F)$, then we know exactly what it is. It's the Frobenius of $Gal(F_u/F)$. □

3. Following, the hint, consider a small enough open neighborhood of 1 in $\mathbb{C}^*$ that doesn't contain any subgroups of $\mathbb{C}^*$. Consider it's preimage U, which is open. Well, $Gal(\overline{F}/F)$ has a neighborhood basis of open subgroups, so U contains an open subgroup H. But the image of H under this homomorphism must be a subgroup, but by what we said before, U doesn't contain any subgroups, so H must map to 1. Thus, the kernel contains an open subgroup. Thus in particular any continuous character of $Gal(\overline{F}/F)$ must factor through a finite Galois group (for a finite extension of F). □.

Section 4:

1. We use the fact that $W(\overline{F}/F)^{ab} = \bigcup_{m \in \mathbb{Z}} Fr_q^m Gal(F^{ab}/F_u)$. Well, note first that the image of $(\cdot, F^{ab}/F)$ actually does lie in $W(\overline{F}/F)^{ab}$. The reason is that first of all the image of O_F^* under this map is $Gal(F^{ab}/F_u)$, which lies in $W(\overline{F}/F)^{ab}$. Moreover, as noted on page 8 near the bottom, $(\omega^n O_F^*, F^{ab}/F)$ acts as a power of the Frobenius automorphism on $\overline{F_q}$, which of course maps to an element of $\mathbb{Z}$ in $Gal(\overline{F_q}/F_q)$. This argument by the way showed that the map

$$(\cdot, F^{ab}/F) : F^* \to W(\overline{F}/F)^{ab}$$

is surjective. By Lemma 3.1, it is injective and continuous. Thus, by some topological argument it should also have continuous inverse, but I don't care.

By the way, is this the traditional definition of $W(\overline{F}/F)^{ab}$? You should show how you get the actual abelianization of $W(\overline{F}/F)$ from this. In particular, check out Section 4.3 of the notes.

2. This is trivial. Let $a, b \in W(\overline{F}/F)$, so $a = Fr_q^n a_1, b = Fr_q^m b_1$, where $a_1, b_1 \in I$. Then $\omega_s(ab) = (q^{m+n})^s$ and $\omega_s(a)\omega_s(b) = q^{ms}q^{ns}$. □

3.

(a) This is sort of roundabout : So want to show that the inverse image of an open set is open. But the open sets in $Gal(\overline{F}/F)$ are just $Gal(\overline{F}/E)$ for some finite Galois extension E over F. Now the inverse image of a set of this type is just $W(\overline{F}/F) \bigcap Gal(\overline{F}/E)$. It's a fact (see Bushnell-Henniart's book on GL(2) page 183 Proposition 1 (a)) that this intersection is just the Weil group of E, $W(\overline{E}/E)$. But this is an open subgroup of $W(\overline{F}/F)$, so we're done. The inclusion is continuous.

(b) Let $i : W(\overline{F}/F) \to Gal(\overline{F}/F)$ be the inclusion map. Let f be a character of $Gal(\overline{F}/F)$. If $f \circ i = 0$, then since $W(\overline{F}/F)$ is dense in $Gal(\overline{F}/F)$, $f = 0$.

(c) Let ρ be of Galois type. Then continuity of ρ forces ρ to factor through some finite Galois group (cf Exercise 3.3 (3)). Thus the image of the Galois group is finite, the the image of the Weil group is finite. Conversely, suppose $Im(\rho)$ is finite. Then, by isomorphism theorem, $ker(\rho)$ is a subgroup of finite index. In fact, $ker(\rho)$ is an OPEN subgroup as well (cf Bushnell-Henniart page 10 1.6 Proposition. I guess it's because the Weil group is actually locally profinite.). But the OPEN subgroups of finite index of $W(\overline{F}/F)$ are precisely $W(\overline{E}/E)$ for some finite extension E of F. $ker(\rho)$ is normal, thus in particular, E is Galois over F (cf Bushnell-Henniart page 183 Proposition (1) (b) in the book The Local Langlands Conjecture for GL(2)). Therefore, ρ factors through $W(\overline{F}/F)/W(\overline{E}/E) \cong Gal(E/F)$ (cf same page Bushnell-Henniart). So we have a character of $Gal(E/F)$, which we can just pullback to $Gal(\overline{F}/F)$ by making it trivial on $Gal(\overline{F}/E)$.

(d) The idea is this : Again we use that $W(\overline{F}/F) = \bigcup_{m\in\mathbb{Z}} Fr_q^m I$, where I is the inertia group, Fr is the Frobenius element. Any character of the cyclic group $< Fr_q >$ is of the form ω_s for some s. Now, all you have to do is define ρ on the inertia group. But I is profinite, so ρ restricted to I factors through some finite quotient, thus ρ divided by the character ω_s is a character of $W(\overline{F}/F)$ since it's trivial on $< Fr_q >$ and again factors through some finite quotient of I. Thus, by part (c), this quotient of characters is of Galois type.

Section 5

1.

(a) This question is worded incorretly as the computation does not work out. First of all, I'm used to the normal subgroup on the left in a semi-direct product, so let's switch the Weil-Deligne group to $\mathbb{C} \rtimes W(\overline{F}/F)$ with multiplication

$$(x_1, w_1)(x_2, w_2) = (x_1 + ||w_1||x_2, w_1w_2)$$

Then the question should read : Prove that $\rho'(x, w) = exp(xN)\rho(w)$ defines a coninuous homomorphism blah blah blah. Using the two basic facts that $exp(CAC^{-1}) = Cexp(A)C^{-1}$ and that

$exp(z_1N)exp(z_2N) = exp(z_1N + z_2N)$ if $z_i \in \mathbb{C}$ (since then z_1N commutes with z_2N, the calculation just falls through. Continuity follows since ρ is assumed continuous, and exp is continuous by nature.

(b) One direction : Suppose (ρ,N) is irreducible. Then by defintion ρ is irreducible, but moreover, $N = 0$, as can be seen easily by some remarks on page 13 of the Mezo notes. Thus, $\rho' = \rho$, so ρ' is irreducible.

Conversely, suppose ρ' is irreducible. Again we want to split up into cases. Interestingly enough, the case $N \neq 0$ can't happen, because then $ker(N)$ is seen to be an invariant subspace. To see this, let $v \in ker(N)$. Then, $exp(xN)\rho(w)v \in ker(N) \forall x \in \mathbb{C}, w \in W(\overline{F}/F)$. THe reason is that since N is nilpotent, xN is nilpotent, so $exp(xN) = I + xN + \frac{x^2N^2}{2} + ... + \frac{x^qN^q}{q!}$ where $N^{q+1} = 0$. Then,

$$N(exp(xN)\rho(w)v) = (N + xN^2 + ... + \frac{x^qN^{q+1}}{q!})\rho(w)v$$

$$= \rho(w)\rho(w)^{-1}(...)\rho(w)v = \rho(w)(||w||N + x||w||N^2 + ... + x^q||w||N^{q+1})v = 0$$

since $Nv = 0$. Thus, we must have $N = 0$, and thus since we assumed ρ', and since now $\rho' = \rho$, we have that ρ is irreducible, hence by definition (ρ, N) is irreducible.

(c) Easy. Write up later.

2. One direction is obvious. For the other direction : Firstly, here are some nice observations. Let $\rho : G \to GL(n, \mathbb{C})$ be a homomorphism where G is a topological group that has a neighborhood basis of 1 consisting of open subgroups (like a Galois group or a locally profinite group like a Weil group). Then if we assume that ρ is continuous, we get that $ker(\rho)$ is open. Why? Well, certainly by stuff from above, $ker(\rho)$ contains an open subgroup, call it U, because GL(V) has no nontrivial small open subgroups (right? This must be a fact. For GL(1) = $\mathbb{C}^*$ it is certainly true). Ok. But now let $g \in ker(\rho)$. Then gU is also in $ker(\rho)$ since $\rho(gu) = \rho(g)\rho(u) = \rho(g) \forall u \in U$. Thus, since gU is open, $ker(\rho)$ is open. Now, let's prove what we want to prove for this problem at hand. So let ρ be continuous under the regular topology. Then $U = ker(\rho)$ is open. Now pick a point $A \in GL(n, \mathbb{C})$. WTS $\rho^{-1}(A)$ is open. Let $g \in \rho^{-1}(A)$. Then again, gU is contained in $\rho^{-1}(A)$, since $\rho(gu) = \rho(g)\rho(u) = \rho(g)$. Thus, we get that $\rho^{-1}(A)$ is open, so ρ is continuous for the discrete topology.

3.

(a) I define $(\rho_1, N_1) \oplus (\rho_2, N_2)$ to be $(\rho_1 \oplus \rho_2, N_1 \oplus N_2)$, where $N_1 \oplus N_2$ just means the block diagonal matrix. This satisfies all the requirements.

(b) I think the reason is that the matrix N from the special representation Sp(n) is already written in Jordan Canonical form, and so it can't be decomposed as a direct sum of nilpotent matrices. Since the definition of direct sum of Weil-Deligne representations involves the direct sum of the corresponding nilpotent matrices, we are done. Just draw this out for the Sp(2), Sp(3), and Sp(4) case, you'll see what I mean.

4. Whatever ρ does to the inertia group, ρ is indecomposable because of what $\rho(Fr_q)$ is: The reason is that you can check that the only invariant subspace under $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is span(1,0). Since

the complement of the invariant subspace is clearly not invariant under $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, this representation is indecomposable. For the second part : Note that if you tensor Sp(2) with a character π of $W(\overline{F}/F)$, you just get $(\pi\rho, N)$, where $Sp(2) = (\rho, N)$. The reason is that the second condition that $(\pi\rho, N)$ be a Weil-Deligne representation is satisfied since $\pi(w)\rho(w)N\rho(w)^{-1}\pi(w)^{-1} = \rho(w)N\rho(w)^{-1}$ since $\pi(w)$ is a scalar so commutes with everything. Now, in order for $\pi \otimes Sp(2)$ to be equivalent to ρ, it must be that there is a single 2 x 2 matrix A such that $A\rho(w)A^{-1} = (\pi \otimes Sp(2))(w)\forall w \in W(\overline{F}/F)$. (This is implied by the definition of equivalence of representations). But $(\pi \otimes Sp(w))(Fr_q) = \pi(w)\begin{pmatrix} 1 & 0 \\ 0 & q^s \end{pmatrix}$, so is a scalar multiple of a diagonal matrix. But this can't be similar to $\rho(Fr_q) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ since the latter is already in Jordan canonical form and Jordan Canonical form is unique up to permutation of the blocks. □

Here's somethin worth noting actually for this problem. We don't know what it means for two representations (ρ_1, N_1) and (ρ_2, N_2) to be equivalent (What should the definition be?) Thus, you should look at the actual representation $\rho : W'(\overline{F}/F) \to GL(n, \mathbb{C})$ that you have (no N involved!). (Recall it should be true that any representation of the Weil-Deligne group itself is really equivalent to one of these (ρ, N)'s.) Anyway, whatever the actual representation of the Weil-Deligne group, the Frobenius still gets mapped to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ under one of the representations, and under the other, it's diagonal, so you're still screwed.

5. See Bushnell-Henniart p. 22 Lemma 2.7 (2), and Tate Corvallis page 20 for extra stuff on this. Anyway, one direction is obvious. Conersely, let $\rho(Fr_q)$ be semisimple. There are two cases two consider. Firstly, suppose (ρ, N) is irreducible. Again, this implies $N = 0$ since $ker(N)$ is always a subrepresentation. So we have to show that $\rho(w)$ is semisimple for all w not in the inertia group. Well, ρ restricted to the inertia group I is continuous, so by profiniteness of I, factors through a finite quotient. In particular, there's an open subgroup J of I such that ρ is trivial on J. So this representation restricted to I is the same as a representation of a finite group. If you have a representation of a finite group, then every element of the finite group gets mapped to a semisimple element because it has finite order, so just look at it's Jordan canonical form. Thus, in fact, $\rho(i)$ is semisimple for all $i \in I$. Thus, since $\rho(Fr_q)$ is also semisimple (thus its powers are), and the multiplication of semisimple elements is also semisimple, we get what we want.

To finish this off, we need to note that it is a fact that any two dimensional PHI-semisimple represntation of $W'(\overline{F}/F)$ is one of three types:

1) Irreducible 2) Completely reducible : There exist two characters χ_1, χ_2 such that ρ is equivalent to $\chi_1 \oplus \chi_2$ and $N = 0$. 3) Indecomposable: If $\rho(Fr_q) \in GL(2, \mathbb{C})$ is semisimple, then there exists a character σ such that (ρ, N) is equal to $\sigma \otimes Sp(2)$.

(cf Mezo note page 14, which I believe he gets from the first Deligne paper in Mezo's bibliography)

So we consider case 2. Well, $N = 0$, and ρ is completely reducible, so everything is semisimple.

For the final case, case 3, assume w is not in the inertia group I. We have $N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Thus, if $a \in \mathbb{C}$, then $exp(aN) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$. Moreover, by definition of the norm (page 10 Mezo notes),

$Sp(2)(w) = \begin{pmatrix} 1 & 0 \\ 0 & q^m \end{pmatrix}$, where $m \neq 0$ since we assume w is not in the inertia group. Thus, $exp(aN)\rho(w) = \begin{pmatrix} 1 & 0 \\ a & q^m \end{pmatrix}$. The eigenvalues of this matrix are 1 and $q^m \neq 0$ (since $m \neq 0$ by assumption). Distinct eigenvalues implies diagonalizable, so indeed, $exp(aN)\rho(w)$ is semisimple for all w not in the inertia group, so indeed (ρ, N) is PHI-semisimple. □

Section 6
Section 6

1. Actually I think I have proven this for a general representation, not just Phi-semisimple. HEre's what I do : First, assume that (ρ, N) is PHI-semisimple. If it's also indecomposable, then by page 14, it is equal to $\sigma \otimes Sp(2)$ for some character σ. Now, $exp(aN) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, so if $\rho(w) = \begin{pmatrix} \sigma(w) & 0 \\ 0 & \sigma(w)||w|| \end{pmatrix}$, then $exp(aN)\rho(w) = \begin{pmatrix} \sigma(w) & 0 \\ a\sigma(w) & \sigma(w)||w|| \end{pmatrix}$. As I explain in problem 5 of the previous section, if w is in the inertia group, then $\begin{pmatrix} \sigma(w) & 0 \\ a\sigma(w) & \sigma(w)||w|| \end{pmatrix} = \begin{pmatrix} \sigma(w) & 0 \\ a\sigma(w) & \sigma(w) \end{pmatrix}$, which is not contained in the maximal torus of GL(2), since it equals $\sigma(w)\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, which is a multiple of a matrix that is conjugate by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, which is clearly not in the standard maximal torus if $a \neq 0$. Moreover, it's not contained in any proper Levi subgroup since all proper Levi subgroups are conjugate to the standard maximal torus, but a lower triangular matrix like $\begin{pmatrix} \sigma(w) & 0 \\ a\sigma(w) & \sigma(w) \end{pmatrix}$ is not necessarily conjugate to a diagonal matrix (i.e. the argument I just made, which is the same as Jordan canonical form and the fact that conjugating by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ brings lower triangular matrices to upper triangular matrices).

Now, suppose (ρ, N) is not Φ-semisimple. That is, $\rho(Fr_q)$ is not semisimple (recall we proved earlier that Φ-semisimple is equivalent to $\rho(Fr_q)$ is semisimple). Well, suppose the image of ρ' was contained in a proper Levi subgroup. Then a conjugate of ρ' by a 2x2 matrix would be contained in the standard maximal torus. That is, there would exist a 2x2 invertible matrix A such that $A\rho'(w, x)A^{-1}$ is contained in the standard maximal torus (i.e. Levi subgroup) of GL(2) for all $w \in W(\overline{F}/F)$ and for all $x \in \mathbb{C}$. In particular, it would be true for $w = Fr_q, a = 0$. That is, a conjugate of $\rho(Fr_q)$ would be in the standard maximal torus T of GL(2). This means that $\rho(Fr_q)$ is semisimple, a contradiction.

2.

(a) This is fairly simple. Firstly, the question assumes that $\pi : W_F \to GL(2, \mathbb{C})$ is completely reducible, so that $\pi = \pi_1 \oplus \pi_2$. Thus, $det(\pi) = \pi_1 \otimes \pi_2$. Then, the way we get an induced representation is : π_i are characters of W_F, so factor through the abelianization, which by Artin reciprocity is F^*. Then, we consider the character $\pi_1 \otimes \pi_2$ of the maximal torus, pull it back to a character of the Borel, and induce it to GL(2,F). Then, let's see what the central character of this representation is. Well, the center of GL(2,F) is F^*, the scalar multiples of the identity matrix. Let $f : G \to V$ be an element in the induced representation space. Then let $t \in T$. Then $(tf)(g) = f(tg) = (\pi_1 \otimes \pi_2)(t)f(g)$, which is what we want.

By the way, in Bushnell Henniart, on page 214, they say that the central character EQUALS $det(\rho)$. What they really mean is that they correspond. The reason is that $det(\rho)$ is a character of W_F, and the central character is a character of the center of GL(2,F) which is F^*. Of course a character of W_F (via abelianization) is a character of F^*.

(b) One direction : Assume ρ is bounded. We just showed above that the central character of π is equal (or rather, corresponds) to $det(\rho)$. So if ρ is bounded, then $det(\rho)$ is a bounded subgroup of $\mathbb{C}^*$. The only bounded subgroups of $\mathbb{C}^*$ lie in S^1, thus we have unitarity. The other direction : Assume that $det(\rho)$ is unitary. By the way, what do we mean by ρ is bounded? What we mean is, using the euclidean topology on $GL(2,\mathbb{C})$. So if ρ wasn't bounded, then, well....I don't know the answer to this. I tried to use Jordan Canonical Form but got nowhere.

(c)

Section 7

1.

2. (a) Follows from part (b)

(b) Here we set $K = T(O_F)$, where O_F is the ring of integers of F. Then, we are set to prove that $T(F)/T(O) \cong X_*(T)$, where T is a split torus, i.e. $T \cong G_m x ... x G_m$. Now, define a map $\phi : T(F) \to X_*(T)$ as follows. Recall that $X_*(T) \cong Hom(X^*(T), \mathbb{Z})$. Let $\phi : T(F) \to Hom(X^*(T), \mathbb{Z})$ be the map $\phi(t)(\chi) = val(\chi(t))$. Then you can see that $\phi(T(O_F)) = 0$. THe reason why this is true is that T is split, so every character χ of T is of the form $\chi(t_1, ..., t_n) = t_1^{m_1} t^{m_2} ... t_n^{m_n}$, and so if $t_i \in O_F^* \forall i$, then after taking val, you get 0 (Recall that $T(O_F) = O_F^* \times ... \times O_F^*$. Thus this map passes to the quotient $T(F)/T(O_F)$. Conversely, define a map $\psi : X_*(T) \to T(F)$ by $\psi(\mu) = \mu(\pi)$, where π is the uniformizer of F. Then, you can pass this map to the quotient $T(F)/T(O_F)$, and call it ψ again. Now let's calculate :

$\phi(\psi(\mu)) = \phi(\mu(\pi))$, which is the map that sends a character χ to $val(\chi(\mu(\pi)))$, which by general theory is $val(\pi^{<\chi,\mu>}) =< \chi, \mu >$. When you identify $Hom(X^*(T), \mathbb{Z})$ with $X_*(T)$, the map $\chi \mapsto < \chi, \mu >$ is the cocharacter μ. Thus, we have verified the that composition one way is the identity.

Now let $t + T(O_F)$ be an element of $T(F)/T(O_F)$. Let's calculate $\psi(\phi(t + T(O_F)))$. Recall that ϕ was originally defined on T(F). So we can consider $\phi(t)$. Now, $\phi(t)$ is the cocharacter that sends a character χ to $val(\chi(t))$ after you identify $X_*(T)$ with $Hom(X^*(T), \mathbb{Z})$. Now let μ be this cocharacter. Then $val(\chi(t)) =< \mu, \chi >$ (cf Springer Lemma 3.2.11 (i)). Then, $\psi(\mu) = \mu(\pi)$, where π is the uniformizer of F. Well what kind of element is $\mu(\pi)$? Well it's an element of the form $(b_1, ..., b_n) \in T(F)$. (Recall T is split, so the cocharacters are defined over F, so indeed $\mu(\pi) \in T(F)$.) Let's consider the character χ_i where $\chi_i(x_1, ..., x_n) = x_i$ Then $\chi_i(\mu(\pi)) = \pi^{<\mu,\chi_i>} = \pi^{val(\chi_i(t))}$ from before. If $t = (t_1, ..., t_n) = (\pi^{m_1} u_1, ..., \pi^{m_n} u_n)$ where $u_i \in O_F^*$, then it is clear that $\chi_i(\mu(\pi)) = \pi^{m_i}$. Therefore, since χ_i picks off the i-th entry of the diagonal element, and since the i-th entry of t is $\pi^{m_i} u_i$, then you can see that the i-th entry of $\mu(\pi)$ is equal to the i-th entry of t up to an element of O_F^*. Therefore, taking all entries of the torus element, we see that $\mu(\pi) + T(O_F) = t + T(O_F)$, i.e. $\mu(\pi)$ is equal to t up to an element of $T(O_F)$. (We really should be using multiplicative notation for the cosets of $T(F)/T(O_F)$, but additive notation is easier to visualize for me at this second). So we are done! $\square$.

3. Drop the L-group notation and work back in the original group G since it doesn't matter, same argument. Now, note that $Z_G(s)^o$ is a connected reductive group, where o denotes identity component. Moreover, it is easy to check that $T \subset Z_G(s)$ and $gTg^{-1} \subset Z_G(s)$. But even moreso, they are both contained in $Z_G(s)^o$, the identity component, since T and gTg^{-1} are connected. Thus, since maximal tori are conjugate, T and gTg^{-1} are conjugate WITHIN the connected reductive group $Z_G(s)^o$. That is, there is a $z \in Z_G(s)^o$ such that $zgTg^{-1}z^{-1} = T$. Thus, $zg \in N_G(T)$. Moreover, $zgtg^{-1}z^{-1} = zsz^{-1} = s$ since $z \in Z_G(s)$. $\square$

4. This is clear.