AMSC 466. Spring 2020

## Homework Assignment 3. Due Thursday Feb. 27.

1. **(6 pts)** Let $A = (a_{ij})$ be an $m \times n$ matrix, $m \geq n$. Show that then:

   (a) For the $l_1$-norm,
   $$\|A\|_1 = \max_j \sum_i |a_{ij}|,$$
   i.e., the maximal column sum of absolute values.

   (b) For the max-norm or $l_\infty$-norm
   $$\|A\|_{\max} = \max_i \sum_j |a_{ij}|,$$
   i.e., the maximal row sum of absolute values

2. **(8 pts)** Let $\mathcal{P}_4$ be the vector space of polynomials of degree $\leq 4$. Consider two bases in $\mathcal{P}_4$: the standard basis
   $$\mathcal{E} = \{1, x, x^2, x^3, x^4\}$$
   and the Chebyshev basis
   $$\mathcal{T} = \{T_0(x) := 1,\ T_1(x) := x,\ T_2(x) = 2x^2 - 1,\ T_3(x) := 4x^3 - 3x,\ T_4(x) = 8x^4 - 8x^2 + 1\}.$$

   (a) Write out the differentiation matrices in the bases $\mathcal{E}$ and $\mathcal{T}$, i.e., the matrices
   $$D_{\mathcal{E}} :=_{\mathcal{E}} [d/dx]_{\mathcal{E}} \quad \text{and} \quad D_{\mathcal{T}} :=_{\mathcal{T}} [d/dx]_{\mathcal{T}}$$

   (b) Let $V$ be a vector space with two bases $\mathcal{B}$ and $\mathcal{C}$, and let the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ be $P_{\mathcal{C}\leftarrow\mathcal{B}}$, i.e., if $v \in V$ is
   $$v = \sum \beta_i x_i = \sum \gamma_i y_i, \quad \text{i.e.} \quad x := [v]_{\mathcal{B}}, \quad y := [v]_{\mathcal{C}}, \quad \text{then}$$
   $$[v]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[v]_{\mathcal{B}}, \quad \text{i.e.,} \quad y = P_{\mathcal{C}\leftarrow\mathcal{B}}x.$$
   The columns of $P_{\mathcal{C}\leftarrow\mathcal{B}}$ are vectors $\beta_i$'s written in the basis $\mathcal{C}$. The inverse of the matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the transition matrix from $\mathcal{C}$ to $\mathcal{B}$ whose columns are the vectors $\gamma_i$ written in the basis $\mathcal{B}$:
   $$P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}.$$
   Let $L$ be a linear transformation: $L : V \to V$. Its matrices in bases $\mathcal{B}$ and $\mathcal{C}$ are
   $$_{\mathcal{B}}[L]_{\mathcal{B}} \quad \text{and} \quad _{\mathcal{C}}[L]_{\mathcal{C}}.$$
   Show that
   $$_{\mathcal{C}}[L]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}\, _{\mathcal{B}}[L]_{\mathcal{B}}\, P_{\mathcal{B}\leftarrow\mathcal{C}}. \tag{1}$$
   Note that while the notations I have used are somewhat long, they make the rule for matrix change as a result of basis change easy-to-remember.

(c) Find the transition matrix from the standard basis $\mathcal{E}$ to the Chebyshev basis $\mathcal{T}$ in $\mathcal{P}_4$.

(d) Verify that (1) holds for the differentiation matrices that you have found in item 2a.

3. **(5 pts)** Let $A$ be a symmetric positive definite $n \times n$ matrix, i.e., $A^\top = A$ and for all $v \in \mathbb{R}^n$, $v \neq 0$, $v^\top A v > 0$. Show that the map $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle v, w \rangle = w^\top A v$$

defines an inner product.