AMSC 466. Spring 2020

## Homework Assignment 9. Due Thursday April 23.

1. **(5 pts)** Prove the discrete orthogonality relationships for the Chebyshev polynomials: Let

$$x_j = \cos\left(\frac{\pi(j+\frac{1}{2})}{n+1}\right), \quad j = 0, 1, \dots, n$$

be the zeros of $T_{n+1}(x)$. Then for all $0 \le r, s \le n$ we have

$$\sum_{j=0}^{n} T_r(x_j)T_s(x_j) = \begin{cases} 0, & r \ne s \\ n+1, & r = s = 0, \\ \frac{n+1}{2}, & r = s \ne 0, \end{cases} \quad j = 0, 1, \dots, n. \tag{1}$$

*Hint: There are several ways to proceed. One of them involves the formulas* $\cos(a)\cos(b) = {}^1\!/_2(\cos(a+b) + \cos(a-b))$ *and* $\cos(a) = ({}^1\!/_2)(e^{ia} + e^{-ia})$.

2. **(5 pts)** Read Section 5.2 "Setting up a system of equations for a cubic spline" in `interpolation.pdf`. Derive the additional two conditions for the case of assigned first derivatives at the end knots:

$$\frac{h_1}{3}M_0 + \frac{h_1}{6}M_1 = \frac{f_1 - f_0}{h_1} - f_0', \qquad \frac{h_n}{6}M_{n-1} + \frac{h_n}{3}M_n = f_n' - \frac{f_n - f_{n-1}}{h_n}.$$

3. **(5 pts)** Find explicit formulas for a cubic spline for the data $(x = 0, f_0 = 1)$, $(x = 1, f_0 = -1)$, and $(x = 2, f_0 = 1)$ and boundary conditions $f'(0) = 0$, $f'(2) = 0$. You can do it elegantly using the symmetry of the data and the boundary conditions. Proceed as follows.

   (a) Let $s(x)$ be the cubic polynomial satisfying

   $$s(0) = 1, \quad s'(0) = 0, \quad s(1) = -1, \quad s'(1) = 0.$$

   Define the spline function $S(x)$ by

   $$S(x) = \begin{cases} s(x), & x \in [0,1], \\ s(2-x), & x \in (1,2]. \end{cases}$$

   Check that $S$ is continuous at $x = 1$ together with its first two derivatives.

   (b) Find coefficients of $s(x)$ and then write an explicit expression for the spline function $S(x)$ on $[0, 2]$.