Homework 1. Due Feb. 11

1. Consider a particle in 1D in contact with a heat bath whose states follow the canonical distribution:

\[ \mu(x,p) = \frac{1}{Z} e^{-\beta H(x,p)}, \quad \text{where} \quad Z = \int_{\mathbb{R}^2} e^{-\beta H(x,p)} dxdp, \quad (1) \]

where \( H(x,p) = V(x) + \frac{p^2}{2} \) is its energy and \( \beta = (k_B T)^{-1} \) (\( k_B \) is Boltzmann’s constant).

(a) Show that the mean kinetic energy equals to \( k_B T/2 \), i.e., calculate the expected value of

\[ E \left[ \frac{p^2}{2} \right] = \frac{1}{Z} \int_{\mathbb{R}^2} \frac{p^2}{2} e^{-\beta (V(x) + p^2/2)} dxdp. \]

(b) Use your result to show that for a system consisting of \( n \) particles of unit mass each of which is moving in 3D, the mean kinetic energy is \( (3/2) nk_B T \). The canonical distribution of this system is given by

\[ \mu(x,p) = \frac{1}{Z} e^{-\beta H(x,p)}, \quad x,p \in \mathbb{R}^{3n}, \quad Z = \int_{\mathbb{R}^{6n}} e^{-\beta H(x,p)} dxdp, \quad (2) \]

\[ H(x,p) = V(x) + \frac{1}{2} \sum_{i=1}^{3n} p_i^2. \]

2. Suppose you are throwing two dice. Consider the random variables \( \eta = \omega_1 + \omega_2 \) (the sum of numbers on the top) and \( \theta = |\omega_1 - \omega_2| \) (the absolute value of the difference of the numbers on the top).

(a) Determine whether \( \eta \) and \( \theta \) are dependent.
(b) Calculate \( \text{Cov}(\eta, \theta) \).
(c) Calculate \( E[\eta|\theta] \) and \( \text{Var}(\eta|\theta) \).
(d) Calculate \( E[\theta|\eta] \) and \( \text{Var}(\theta|\eta) \).


References