1. This problem was inspired by works of V. Yakovenko on Econophysics.

Suppose that initially person A has $R and person B has $(2M − R). At each discrete moment of time n = 1, 2, . . . A and B attempt a transaction: they flip a fair coin, and, if head, A should give $1 to B; otherwise, B should give $1 to A. The transaction takes place if and only if whoever should give $1 has at least $1. Otherwise the transaction is canceled. Suppose that we care only about the partition of $2M and do not distinguish A and B, i.e., for us, (A has $R and B has $(2M − R)) is equivalent to (B has $R and A has $(2M − R)).

(a) Design a discrete-time Markov chain describing this process. How many states are there in this chain? What is the transition matrix \( P \)? What is the initial distribution \( \lambda \)? Write out \( P \) and \( \lambda \). Draw a graph representing this chain. Is this Markov chain irreducible?

(b) Find the invariant distribution \( \pi \) for this chain. Does the probability distribution in this chain converge to \( \pi \)?

2. (Norris, Problem 1.7.3) A particle moves on the eight vertices of a cube in the following way: at each step the particle is equally likely to move to each of the three adjacent vertices, independently of its past motion. Let \( i \) be the initial vertex occupied by the particle, \( o \) the vertex opposite to \( i \). Calculate each of the following quantities:

(a) the expected number of steps until the particle returns to \( i \);

(b) the expected number of visits of \( o \) until the first return to \( i \);

(c) the expected number of steps until first visit to \( o \);

3. Consider the following discrete-time Markov chain modeling virus propagation through a computer network. This model is a discrete-time analog of the model presented in http://vxheaven.org/lib/ajk06.html. Suppose there are \( N \) computers in the network. Each computer directly is connected to every other computer, i.e., this network is a complete graph. At each discrete moment of time \( n = 0, 1, 2, . . . \) every healthy computer becomes infected with virus with probability proportional to the number of its infected neighbors. At the same time, every infected computer recovers with some fixed probability.

We are interested in the probability distribution for the number of infected computers \( i \). Therefore, our Markov chain will have \( N + 1 \) nodes each of which corresponds to \( i = 0, 1, . . . , N \). Let at each moment of time the probability for a healthy computer
with $i$ infected neighbors to get infected be $ai/N$, while the probability for an infected computer to recover be $b$. Then the stochastic matrix $P$ can be calculated as follows. If at time $n$ there are $i$ infected computers and $N - i$ healthy computers, the probability that $k$ computers will become infected at time $n+1$ is

$$p(i, k) := \binom{N-i}{k} \left(\frac{ai}{N}\right)^k \left(1 - \frac{ai}{N}\right)^{N-i-k},$$

and the probability that $m$ infected computers will recover is

$$q(i, m) := \binom{i}{m} (b)^m (1-b)^{i-m}.$$ 

Therefore, if at time $n$, there are $i$ infected computers, then the probability that at time $n+1$ there will be $i'$ infected computers can be calculated as follows:

$$P(i, i') = \sum_{k-m=d, \ k \in \{0,1,...,N-i\}, \ m \in \{0,1,...,i\}} p(i, k)q(i, m).$$

(a) Is this Markov chain irreducible?

(b) Set $N = 40$, $a = 1$ and $b = 0.1$. Calculate $P$ using matlab (the binomial coefficients can be found using the command \texttt{nchoosek}).

(c) Using the command $[V E] = \texttt{eig}(P')$ find the equilibrium distribution. It should be $\pi = [1, 0, \ldots, 0]$, i.e., the virus disappears. What is the second largest eigenvalue?

(d) Take the initial distribution $\lambda = \frac{1}{N}[0, 1, \ldots, 1]$. Demonstrate that $\lambda P^n$ evolves toward a bell-shaped distribution $\phi$ that remains stable for long time. We will call this distribution metastable. Plot this distribution. Note that this distribution is not equilibrium. Compare this distribution with the left eigenvector $\psi$ of $P$ (the right eigenvector of $P^T$) corresponding to the second largest eigenvalue of $P$ (normalize this eigenvector so that the sum of its positive entries is 1 and superimpose the plots).

(e) Experiment with other values of $a \in (0, 1)$ and $b \in (0, 1)$. Report your observations regarding the existence and stability of the metastable distribution.

4. Let $L$ be an $N \times N$ matrix satisfying the following conditions:

- the off-diagonal entries of $L$ are nonnegative, i.e., $L_{ij} \geq 0$ for all $1 \leq i, j \leq N$, $i \neq j$.
- the row sums of $L$ are zero, i.e., $L_{ii} = -\sum_{j \neq i} L_{ij}$. 
Define the matrix $P(t)$ for $t \geq 0$ to be the matrix exponential

$$P(t) = e^{tL} := \sum_{k=0}^{\infty} \frac{(tL)^k}{k!}.$$ 

Show that

(a) $P(t)$ satisfies the semigroup property:

$$P(s + t) = P(s)P(t) \quad \text{for all} \quad s, t \geq 0;$$

(b) $P(t)$, $t \geq 0$, satisfies the forward equation

$$\frac{d}{dt}P(t) = P(t)L, \quad P(0) = I;$$

(c) $P(t)$, $t \geq 0$, satisfies the backward equation

$$\frac{d}{dt}P(t) = LP(t), \quad P(0) = I;$$

(d) for $k = 0, 1, 2, \ldots$, we have

$$\left. \left( \frac{d}{dt} \right)^k \right|_{t=0} P(t) = L^k;$$

(e) $P(t)$ is a stochastic matrix for any $t \geq 0$, i.e., its row sums are ones, and all its entries are nonnegative.